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ISOMETRIC IMMERSIONS OF MANIFOLDS OF NONNEGATIVE CONSTANT SECTIONAL CURVATURE

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Let M^d denote a C^∞ Riemannian manifold which is d-dimensional and complete. Our first result states that an isometric immersion of a flat M^d into $(d+k)$-dimensional Euclidean space, $k < d$, is n-cylindrical if the relative nullity of the immersion has constant value n. This result was obtained by O'Neill with the additional hypothesis of vanishing relative curvature. We next consider the case in which M^d and $\bar{M}^{d+k}$, $k < d$, are manifolds of the same constant positive sectional curvature. In this case we show that an isometric immersion of M^d into $\bar{M}^{d+k}$ is totally geodesic if the relative curvature of the immersion is zero on a certain subset of M^d.

Let M^d and $\bar{M}^{d+k}$ be C^∞ Riemannian manifolds of the same constant sectional curvature C, M^d being assumed complete and $k < d$. Let $\psi: M^d \to \bar{M}^{d+k}$ be an isometric immersion. The character of such immersions has been studied in [4] and [5] in terms of what Chern and Kuiper call the *index of relative nullity* of ψ [2]. This function, ν, assigns to each $m \in M$ the dimension of $\mathscr{N}(m)$, the subspace of vectors x in the tangent space M_m such that $T_x = 0$. The linear *difference operators* T_x act on $\bar{M}_{\psi(m)}$ and contain the same information as the classical second fundamental form operators S_z where z is a tangent vector to $\bar{M}$ orthogonal to $d\psi(M_m)$ [1]. In fact T_x is characterized by its skew-symmetry and the equation $T_x(z) = d\psi(S_z(x))$. Our first theorem concerns the case in which M^d is flat and $\bar{M}^{d+k} = R^{d+k}$, $d+k$ dimensional Euclidean space. It states that when ν is constant on M^d the immersion ψ is 'cylindrical'. We next investigate the corresponding situation for $C > 0$.

We use essentially the notation in [4]. In particular we identify M^d with $\psi(M^d)$ when it seems safe to do so. Let N denote the bundle of normal k-frames of M relative to ψ; that is

$$N = \{(m, E) \mid m \in M \text{ and } E \text{ is a } k\text{-frame (orthonormal set of } k \text{ vectors) of } \bar{M}_{\psi(m)} \text{ orthogonal to } d\psi(M_m)\}\,.$$

The Riemannian connection of $\bar{M}^{d+k}$ induces a natural connection on N. The curvature form of this connection is called the *relative curvature* of ψ. We say that $\psi: M^d \to R^{d+k}$ is *n-cylindrical* provided

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M and ψ can be expressed as Riemannian products $M^d = B^{d-n} \times R^n$ and $\psi = \bar{\psi} \times 1$ where $\bar{\psi}$ is an isometric immersion of B^{d-n} in R^{d+k-n} and 1 is the identity map of R^n. We can now state our first theorem precisely. This result was obtained by O'Neill as Theorem 2 of [4] but with an additional hypothesis, namely, the assumption of zero relative curvature. We shall use a similar assumption in our Theorem 3.

THEOREM 1. *Let M^d be a complete, flat, C^∞ Riemannian manifold. An isometric immersion $\psi: M^d \to R^{d+k}$ is n-cylindrical if the relative nullity has constant value n.*

We summarize some results applicable to an isometric immersion between two manifolds of constant curvature C. Let $\mathscr{N}^\perp(m)$ be the orthogonal complement of $\mathscr{N}(m)$ in M_m. From [5] we have: *If n denotes the minimum value of ν, then $n \geqq d - k$ and G, the open subset of M^d on which $\nu = n$, is foliated by complete totally geodesic subspaces (the leaves of $\mathscr{N}$) which are also totally geodesic relative to ψ. Also there exists for any $m \in G$ an $x \in \mathscr{N}^\perp(m)$ such that T_x is injective on $\mathscr{N}^\perp(m)$.* The two cases of interest to us are:

Case 1. $G = M^d$ (i.e., ν is constant), $\bar{M}^{d+k} = R^{d+k}$ $(C = 0)$ and $a = \infty$ (see below).

Case 2. $C > 0$ and $0 < a < \pi/4\sqrt{C}$.

The parameter a appears in the following lemma. *Let $\gamma: (-a, a) \to L$ be a unit speed geodesic in a leaf L of $\mathscr{N}$ in G. Then there exists a frame field $E = (E_1, \cdots, E_{d+k})$ on a neighborhood or γ in G such that:*

1. *The geodesic γ is an integral curve of E_1;*
2. *Each integral curve of E_1 is a geodesic of M;*
3. *The vector fields $E_1, \cdots, E_n$ are contained in $\mathscr{N}$, $E_{n+1}, \cdots, E_d$ in $\mathscr{N}^\perp$,* and *$E_{d+1}, \cdots, E_{d+k}$ are contained in the orthogonal complement of $\psi(M_m)$ in $\bar{M}_{\psi(m)}$;*
4. *The frame E is parallel on γ.* The construction for this lemma is contained in Lemma 1 of [5], except we use the additional fact that the leaves of $\mathscr{N}$ are R^n planes in Case 1 for $a = \infty$. We pull the connection form $\bar{\phi}$ of the frame bundle of $\bar{M}^{d+k}$ down to G by way of the frame field E. Using the following index convention,

$$1 \leqq a, b \leqq n \; ; \qquad n + 1 \leqq q, r, s \leqq d \; ;$$
$$1 \leqq i, j \leqq d \; ; \qquad d + 1 \leqq \alpha, \beta \leqq d + k \; ,$$

we get

$$\phi_{ij} = \bar{\phi}_{ij} \circ dE \quad \text{(connection forms of } M\text{)},$$
$$\tau_{i\alpha} = \bar{\phi}_{i\alpha} \circ dE \quad \text{(Codazzi forms)},$$
$$\theta_{\alpha\beta} = \bar{\phi}_{\alpha\beta} \circ dE \quad \text{(normal connection forms)}.$$

A set of linear operators on $\mathscr{N}^{\perp}$ dependent on the frame field E can be defined by

$$P_{E_a}(E_s) = \Sigma_r \phi_{ra}(E_s) E_r \, .$$

From the second structural equation and the properties of the frame field E one can show that *the matrix* $P(t)$ *of* $P_{\gamma'(t)}$ *satisfies the differential equation* $P' = -P^2 - CI$ *on* $(-a, a)$ where I denotes the $(d-n) \times (d-n)$ identity matrix. See Lemma 3 of [5]. Our proof of Theorem 1 hinges on the central result from [4] which states that *if for all* $m \in M^d$ *and* $x \in \mathscr{N}^{(m)}$ *we have that* $P_x = 0$ *then the immersion is n-cylindrical.* Theorem 1 can now be easily proved with the help of the following lemma which is applicable in both Case 1 and Case 2.

LEMMA 1. *Let* $m \in L$. *If* $x \in \mathscr{N}(m)$ *and* $y \in \mathscr{N}^{\perp}(m)$ *then* $T_{P_x(y)} = T_y \circ P_x$ *on* $\mathscr{N}^{\perp}(m)$.

Proof. Since L is complete there exists a geodesic $\gamma: (-a, a) \to L$ with $\gamma(0) = m$ and a frame field E as defined above in a neighborhood of γ. From $T_{E_i}(E_j) = \Sigma_\alpha \tau_{\alpha j}(E_i) E_\alpha$ and the definition of $\mathscr{N}$ we get that $\tau_{\alpha a} = 0$. Using this fact with the Codazzi equation for $\tau_{\alpha a}$ we have

$$0 = d\tau_{\alpha\alpha} = -\Sigma_i \phi_{ai} \wedge \tau_{i\alpha} - \Sigma_\beta \tau_{\alpha\beta} \wedge \theta_{\beta\alpha} = \Sigma_q \phi_{aq} \wedge \tau_{q\alpha} \, .$$

This implies that

$$\Sigma_{\alpha,q} \phi_{qa}(E_s) \tau_{\alpha q}(E_r) E_\alpha = \Sigma_{\alpha,q} \phi_{qa}(E_r) \tau_{\alpha q}(E_s) E_\alpha$$

or that

$$T_{E_r}(P_{E_a}(E_s)) = T_{E_s}(P_{E\alpha}(E_r)) \, .$$

Hence for $x \in \mathscr{N}(m)$ and $y, z \in \mathscr{N}^{\perp}(m)$ we have

$$T_y(P_x(z)) = T_z(P_x(y)) = T_{P_x(y)}(z) \, ,$$

the last equality above following from the symmetry of the second fundamental form operators.

2. Proof of Theorem 1. We shall show that $P_x = 0$ for $x \in \mathscr{N}(m)$, $m \in M^d$. We may assume x is a unit vector and γ is a unit speed complete geodesic of the leaf through m with $\gamma'(0) = x$. By a previous remark we may pick $y \in \mathcal{N}^{\perp}(m)$ such that T_y is injective on $\mathcal{N}^{\perp}(m)$. Then $\mathcal{N}^{\perp} + T_y(\mathcal{N}^{\perp})$ is invariant under both T_y and $T_{P_x(y)}$. Hence the $2(d-n) \times 2(d-n)$ matrix of $T_y \mid (\mathcal{N}^{\perp} + T_y(\mathcal{N}^{\perp}))$ can be represented by a $(d-n) \times (d-n)$ matrix A in the upper right hand corner, $-A^t$ in the lower left hand corner and zeros elsewhere. If B is the analogous block for $T_{P_x(y)}$ then $Q = -AB^t$ will be the matrix of $T_y \circ T_{P_x(y)} \mid \mathcal{N}^{\perp}$. The difference operators T_y and $T_{P_x(y)}$ commute on M_m since M is flat and hence we have $AB^t = BA^t$. By Lemma 1, $P_x = T_y^{-1} \circ T_{P_x(y)} \mid \mathcal{N}^{\perp}$ and hence $P(0) = (A^{-1})^t B^t$. Let

$$R = -A^{-1}Q(A^{-1})^t = B^t(A^{-1})^t .$$

Since Q is symmetric so is R and therefore $P(0)$ has the same (real) eigenvalues as R. These eigenvalues satisfy $\lambda_k' = -\lambda_k^2$ on the real line (since P satisfies this equation by a result stated above) and hence each $\lambda_k = 0$. Thus $R = 0$ and this implies $P(0) = 0$ which is the desired result.

3. Positive curvature case.

For completeness we include Corollary 1 of [5] as

THEOREM 2. *Let M^d and $\bar{M}^{d+k}$ be C^∞ manifolds with the same constant positive curvature C, M^d being assumed complete. Let ψ: $M^d \to \bar{M}^{d+k}$ be an isometric immersion with $2k \leqq d$. Then ψ is totally geodesic.*

As above let n denote the minimum value of ν and let G consist of the $m \in M^d$ for which $\nu(m) = n$.

THEOREM 3. *Let M^d and $\bar{M}^{d+k}$ be C^∞ manifolds with the same constant positive curvature C, M^d being assumed complete. Let ψ: $M^d \to \bar{M}^{d+k}$ be an isometric immersion with $k < d$. Then ψ is totally geodesic if the relative curvature of ψ is zero on G.*

Proof. The proof is by contradiction. If ψ is not totally geodesic then $n < d$. Let L be a leaf in G and let $m \in L$. We first show that for any $x \in \mathcal{N}(m)$, P_x is a symmetric operator and is independent of the frame field used in its definition. Let $y \in \mathcal{N}^{\perp}(m)$ such that T_y is injective on $\mathcal{N}^{\perp}$. Using a geodesic $\gamma: (-a, a) \to L$ with $\gamma'(0) = x$ and Lemma 1 we have as in the proof of Theorem 1 that $P(0) = (A^{-1})^t B^t$. Since the relative curvature of ψ is zero we get from the Ricci equation of the immersion that the Codazzi forms satisfy the relation $\Sigma_i \tau_{\alpha i} \wedge \tau_{i\beta} = 0$. From this we conclude that T_y and $T_{P_x(y)}$

commute on $(d\psi(M_m))^\perp$ or $A^tB = B^tA$. This equation implies that $P(0)$ is symmetric. From the first structural equation we have that

$$[E_r, E_s] = \Sigma_i(\phi_{ri}(E_s) - \phi_{si}(E_r))E_i$$

which together with the symmetry of P_x implies $[E_r, E_s] \in \mathcal{N}^\perp$; thus $\mathcal{N}^\perp$ is integrable. For $x \in \mathcal{N}$, P_x is actually a second fundamental form operator of the leaf through $\mathcal{N}^\perp$ and thus P_x is independent of the choice of frame field used in its definition.

From the completeness of L it follows that we can find a unit speed geodesic γ in L defined on the real line. Since M is of constant positive curvature, γ is a compact immersion and $P_{\gamma'}$ is a periodic function on the real line. Let λ be one of the $d - n$ real eigenvalue functions determined by the symmetric operator $P_{\gamma'}$. We may assume λ attains a maximum at $m = \gamma(0)$. Let E be a frame field as above. Then λ must satisfy $\lambda'(0) = -\lambda^2(0) - C = 0$ since P satisfies $P' = -P^2 - CI$ on an interval containing 0. This implies $\lambda(0)$ is not real, which is the desired contradiction. Hence $n \geqq d$ or ψ is totally geodesic on M.

As a Corollary we get a result of O'Neill's from [3]. Let $S^{d+1}(C)$ denote the sphere of curvature C.

COROLLARY 1. *Let M^d and $\bar{M}^{d+1}$ be C^∞ manifolds with the same constant positive curvature C, M^d being assumed complete. Then any isometric immersion $\psi: M^d \to \bar{M}^{d+1}$ is totally geodesic. In particular if $\bar{M}^{d+1} = S^{d+1}(C)$ then any such immersion is an imbedding onto a great sphere.*

Proof. The vanishing of the relative curvature of ψ is trivial in the hypersurface case. In case $\bar{M}^{d+1} = S^{d+1}(C)$ we have that $\psi(M) = S^d(C) \subset S^{d+1}(C)$. Letting $\bar{S}^d(C)$ denote the universal covering manifold of M^d and π the natural projection, we have that $\psi \circ \pi$ is a local isometry onto $\psi(M)$. Hence $\psi \circ \pi$ and therefore ψ is injective. Thus ψ is an imbedding onto $S^d(C)$.

REFERENCES

1. W. Ambrose, *The Cartan structural equations in classical Riemannian geometry*, J. Indian Math. Soc. **24** (1960), 23-76.
2. S. S. Chern and N. H. Kuiper, *Some theorems on the isometric imbedding of compact Riemann manifolds in Euclidean space*, Ann. of Math. (2) **56** (1952), 422-430.
3. B. O'Neill, *Isometric immersions which preserve curvature operators*, Proc. Amer. Math. Soc. **13** (1962), 759-763.
4. ———, *Isometric immersion of flat Riemannian manifolds in Euclidean space*, Michigan Math. J. **9** (1962), 199-205.
5. B. O'Neill and E. Stiel, *Isometric immersions of constant curvature manifolds*, Michigan Math. J. **10** (1963), 335-339.
6. E. Stiel, *Isometric Immersions of Riemannian Manifold*, Doctoral Dissertation, University of California at Los Angeles, (1963).

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