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$|\varepsilon(z)|$-CLOSENESS OF APPROXIMATION

ANNETTE SINCLAIR

For a given function $F(Q)$ defined for $Q \in S$, the connection between these questions is investigated: (1) For arbitrary $\varepsilon > 0$ (or possibly $\{\varepsilon_i\}$, where ε_i corresponds to a compoment S_i of S), does there exist a function f of a specified class $\mathscr{F}$ such that $\sup_{Q \in S} |F(Q) - f(Q)| < \varepsilon$ on S (or ε_i on S_i)?; (2) Given an admissible function $\varepsilon(Q)$, does there exist a function $f \in \mathscr{F}$ such that $|F(Q) - f(Q)| \leqq |\varepsilon(Q)|$ on S? A continuous *function* $\varepsilon(Q)$ defined on S is *admissible* if for each zero Q_β there is a positive integer n_β such that $\varepsilon(Q)/(Q - Q_\beta)^{n_\beta}$ is bounded from zero in a deleted neighborhood of Q_β. A typical result is: Corresponding to any $F(z)$ analytic on a closed bounded set S and to any admissible $\varepsilon(z)$, there exists a rational function $r(z)$ with its poles on a certain preassigned set such that $|F(z) - r(z)| \leqq |\varepsilon(z)|$ on S.

When the sup-topology is used in approximating a given function F defined on a set S by a function f in a certain class $\mathscr{F}$, it is required that, for arbitrary $\varepsilon > 0$, there exists $f \in \mathscr{F}$ such that

$$\sup |F(X) - f(X)| < \varepsilon \text{ for } X \in S .$$

In this paper the connection is investigated between existence of such an approximating function and existence of an approximating $g \in \mathscr{F}$ when for any admissible function $\varepsilon(X)$ it is required $|F(X) - g(X)| \leqq |\varepsilon(X)|$ when $X \in S$.

The latter formulation has the advantage of automatically specifying that, at any zero X_0 of $\varepsilon(X)$ on S, $g(X_0) = F(X_0)$ and at multiple zeros corresponding derivatives of F and g agree, provided F has derivatives at these points. One interesting application, in case F is continuous and is well-behaved near zeros, is that in which

$$|F(X) - f(X)| \leqq p\,|F(X)|$$

is required, where p denotes a preassigned per cent.

Approximation in the real case in which a neighborhood N_{ξ_1,ξ_2} of F consists of those f such that $\xi_1(x) \leqq F(x) - f(x) \leqq \xi_2(x)$ has been suggested by P.C. Hammer.[1] If $[\xi_2(x) - \xi_1(x)]/2$ is an "admissible" $\varepsilon(x)$, the problem reduces to the $|\varepsilon(x)|$-closeness of approximation

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considered in this paper. For $\xi_1(x) \leqq F(x) - f(x) \leqq \xi_2(x)$ if and only if

$$- [\xi_2(x) - \xi_1(x)]/2 \leqq F(x) - [\xi_1(x) + \xi_2(x)]/2 \\ - f(x) \leqq [\xi_2((x) - \xi_1(x)]/2$$

This paper is perhaps of most interest in connection with approximation in the complex plane. However, as the Weierstrass-factor Theorem, Mittag-Leffler Theorem, and Runge Theorem [2] upon which the results depend, hold also on the open Riemann surface, the theorems are stated in abstract form for the open Riemann surface: then certain specializations to the complex plane are given in the corollaries.

As is customary, "open" Riemann surface denotes a noncompact Riemann surface [1]. A point on a Riemann surface is denoted by Q, a point in the complex plane, in particular, by z, and a point on the real axis by x. For the sake of clarity the notation $f(Q)$ is frequently used to denote the function f.

When it is specified a function has *poles coinciding with* those of another function, it is to be understood that they have identical principal parts; likewise, if a function has *zeros coinciding with* those of a second function, the order of the respective zeros is the same.

For reference we state:

HYPOTHESIS H. *Suppose that S is a closed set on the open Riemann surface $\mathfrak{R}$, Let B^* consist of precisely one point of each of those components of $\mathfrak{R} - S$ whose closure is compact.*

Theorem 1 includes the case that S is compact with no interior points. For example, if $\mathfrak{R}$ is the finite complex plane, S may be a bounded closed interval on the real axis; in fact, S may be any closed bounded set with or without interior points.

THEOREM 1. *Assume Hypothesis H and suppose a function $\varepsilon(Q)$ $(\not\equiv 0)$ defined on S. Let R be an open set (which may be $\mathfrak{R}$) such that $S \subset R \subset \mathfrak{R}$ and suppose $\mathscr{S}$ is a collection of functions meromorphic on R, analytic on $R - B^*$. Then these approximation requirements* (1) *and* (2) *are equivalent.*

(1) *Corresponding to any function $M(Q)$ analytic on S^0 (the interior of S) and continuous on S, there exists $k \in \mathscr{S}$ such that $| M(Q) - k(Q) | \leqq | \varepsilon(Q) |$ when $Q \in S$.*

(2) *Corresponding to any function $m(Q)$ meromorphic on S^0 and continuous on S except at poles, there exists $f = h + k$, where $k \in \mathscr{S}$ and h is meromorphic on $\mathfrak{R}$ with its only poles coinciding with those of m on S, such that $| m(Q) - f(Q) | \leqq | \varepsilon(Q) |$ on S.*

Proof. Clearly, (2) includes (1). We proceed to prove (1) implies (2).

The set of points at which m has poles on S is an isolated set on $\mathfrak{R}$. Hence, according to the Mittag-Leffler partial fractions theorem [2, p. 591; 7] there exists a function h meromorphic on $\mathfrak{R}$ whose only poles coincide with those of m on S and have the same principal parts. (We note that, if m has only a finite number of poles on S and if $\mathfrak{R}$ is the finite complex plane, then h may be required to be a rational function.)

The function $m - h$ is analytic on S^0 and continuous on S. Hence, by the conclusion in (1), there is a function $k \in \mathscr{S}$, such that

$$|[m(Q) - h(Q)] - k(Q)| \leqq |\varepsilon(Q)|$$

when $Q \in S$, that is,

$$|m(Q) - [h(Q) + k(Q)]| \leqq |\varepsilon(Q)|$$

on S.

Thus, $h + k$, which is meromorphic on R and analytic on $R - B^*$ except for poles on S coinciding with those of m, is a function f as required.

COROLLARY 1.1. *The theorem is true if in*
(1) *$M(Q)$ is assumed analytic on S and in*
(2) *$m(Q)$ is assumed meromorphic on S.*

COROLLARY 1.2. *For $\mathfrak{R}$ the finite complex plane and S a compact set on $\mathfrak{R}$, the theorem is true if in*
(1) *k is required to be a rational function and in*
(2) *f is required to be a rational function.*

H. J. Landau [5] proved: If on the complex plane, S is a closed bounded set with no interior and if there exist cutting sets of S whose closures have arbitrarily small measure, then any function continuous on S may be uniformly approximated on S by a rational function whose poles lie in $B^* \cup \infty$. It follows from Corollary 1.2 that, if m is continuous on such a set S except for a finite number of poles, $m(z)$ can be uniformly approximated by a rational function whose poles lie in $B^* \cup \infty$ and at the poles of m on S.

By the Carleman approximation theorem [3; 4] if $w(x)$ is continuous on the real axis, then corresponding to any $\{\varepsilon_i\}$, there exists an entire function f such that $|w(x) - f(x)| < \varepsilon_i$ when $i - 1 < |x| \leqq i$, $i = 1, 2, \cdots$. Hence, Theorem 1 implies that, if $w(x)$ is continuous on the finite real axis except for a finite or a denumerable number of poles with limit point at ∞, then $w(x)$ can be approximated in the above sense by a meromorphic function f whose poles lie on the real axis and coincide with those of w. According to an extension by the author [8, Theorem 3] of the Carleman Theorem, if S consists of the union of closed circular disks S_i tangent externally on the real axis and extending to infinity and if w is analytic at interior points of S, continuous on S, then, corresponding to any $\{\varepsilon_i\}$, there exists an entire function f such that $|w(z) = f(z)| < \varepsilon_i$ on $S_i, i = 1, 2, \cdots$. By Theorem 1, w may be allowed poles on S^0 provided the approximating function f is allowed coincident poles.

An analogue of the type of generalization given in Theorem 1 for a Q-set has previously been used by the author [8; 9].

A *sequential limit point* of a set S is a limit point of a set of points chosen one from each component of S. A set S in the extended complex plane whose components $S_1, S_2, \cdots$, are compact and whose set of sequential limit points $B \subset \mathscr{C}(S)$ is called a Q-set [9]. We require, in addition, that a *Q-set on* an *open Riemann surface* $\mathfrak{R}$ be a closed set, that is, $\mathfrak{R}$ contains no sequential limit point of S. When in the complex domain $\mathfrak{R}$ is chosen as the extended plane minus B, the set of sequential limit points of S, a Q-set is closed.

A *function* $\varepsilon(Q)$ defined for $Q \in S$ is *admissible on* S if

(1) It is continuous on S;

(2) Corresponding to each of its zeros Q_β on S, there is a positive integer n_β such that $\varepsilon(Q)/(Q - Q_\beta)^{m_\beta}$ is bounded from zero in a neighborhood $N_{Q_\beta} \subset S$. The smallest positive integer n_β satisfying the condition in (2) is called the *order of* the *zero* of $\varepsilon(Q)$ at Q_β.

THEOREM 2. *Assume Hypothesis H with $S = \cup S_n$, where the S_n are compact and disjoint. Let R be an open set such that $S \subset R \subset \mathfrak{R}$. Suppose M is any function which is analytic on S^0, continuous on S. Then* (1) *below implies* (2); *also, if S is a Q-set or a compact set,* (2) *implies* (1), *and if K is any isolated interior subset of S, $f(z) = M(z)$ can be required on K.*

(1) *Corresponding to any* $\{\varepsilon_n\}$ (ε *if S is compact*), *there exists f analytic on $R - B^*$, meromorphic on R, such that* $|M(Q) - f(Q)| \leqq \varepsilon_n$ *when* $Q \in S_n, n = 1, 2, \cdots$ (*or* ε *when* $Q \in S$).

(2) *Corresponding to any* $\varepsilon(Q)$ *which is admissible on S, there exists F analytic on $R - B^*$ and meromorphic on R such that*

$$|M(Q) - F(Q)| \leqq |\varepsilon(Q)|$$

on S. If f in (1) *can be required to be a rational function and if S is compact, then F can be required to be a rational fuction.*

Proof. We first show (1) implies (2). Admissibility requirement (2) for $\varepsilon(Q)$ implies the zeros of ε on S are isolated. Hence, by the

Weierstrass-factor Theorem [2, p. 591] there exists g analytic on $\mathfrak{R}$ whose only zeros are the zeros Q_β of $\varepsilon(Q)$ and are of the respective orders n_β. Let $\varepsilon_n = \inf |\varepsilon(Q)/g(Q)|$ for Q on S_n (or $\varepsilon = \inf |\varepsilon(Q)/g(Q)|$ for Q on S). Now, by Theorem 1 with $\varepsilon(Q) = \varepsilon_n$ on S_n (or ε on S) and (1) above, there exists a function k meromorphic on R, analytic in $R - B^*$ except at zeros of g on S, such that $|M(Q)/g(Q) - k(Q)| \leqq \varepsilon_n$ (or ε on S) where defined. Then on each S_n (or S)

$$|M(Q) - g(Q)k(Q)| \leqq |g(Q)|\varepsilon_n$$

(or $|g(Q)|\varepsilon$). Now $g\cdot k$, which has removable singularities at the Q_β, satisfies the requirements for F.

Next we consider the converse, giving the proof for the case S is a Q-set. Since $\{\varepsilon_n\}$ defines an admissible $\varepsilon(Q)$, (1) is a special case of (2). We are to verify also that interpolation conditions can be assigned. The Weierstrass-factor theorem yields existence of a function g analytic on $\mathfrak{R}$ such that g has zeros on K of the same orders as the interpolation conditions. For $\varepsilon_n(Q) = \varepsilon_n[g(Q)/\max|g(Q)|]$ when $Q \in S_n$, and $\varepsilon(Q)$ defined by $\varepsilon_n(Q)$ on S_n, $\varepsilon(Q)$ is admissible on S. By hypothesis (2), there is F analytic on $R - B^*$, meromorphic on R, such that

$$|M(Q) - F(Q)| \leqq |\varepsilon(Q)|$$

on S. Since $|\varepsilon(Q)| \leqq \varepsilon_n$ on S_n and $\varepsilon(Q)$ vanishes on K, F satisfies the interpolation conditions, in addition to the requirements for f in the conclusion of (1).

COROLLARY 2.1. *If M is analytic on the closed bounded set S in the finite complex plane, then, corresponding to any admissible $\varepsilon(z)$, there exists a rational function r having its poles on B^* such that $|M(z) - r(z)| \leqq |\varepsilon(z)|$ when $z \in S$.*

Proof. This follows from the Walsh formulation of the Runge Theorem [10, p. 15] and Theorem 2 with $n = 1$ and $R = \mathfrak{R}$ defined as the finite complex plane.

The next corollary is obtained by applying a result of Mergelyan [6; 10, p. 367].

COROLLARY 2.2. *If in the complex plane M is continuous on the closed bounded set S, analytic on S^0, and if S does not separate the plane, then, corresponding to any admissible $\varepsilon(z)$, there exists a polynomial $p(z)$ such that $|M(z) - p(z)| \leqq |\varepsilon(z)|$ on S.*

COROLLARY 2.3. *Suppose S is a Q-set ($= \cup S_n$) and $\varepsilon(z)$ is admissible on $S \subset \mathfrak{R}$, the extended plane minus the set of sequential limit points of S. Then, if M is analytic on S, there exists a function*

f analytic on $\mathfrak{R} - B^*$, meromorphic on $\mathfrak{R}$, such that $|M(z) - f(z)| \leqq |\varepsilon(z)|$ *everywhere M is defined on S.*

If M is meromorphic on S, there exists f analytic on $R - B^*$, except at poles of M on S, and meromorphic on R such that $|M(z) - f(z)| \leqq |\varepsilon(z)|$ *everywhere M is defined on S.*

Proof. The first part is an immediate consequence of Theorem 2 and a previous theorem of the author [9, Theorem 3]. The latter part then follows from Corollary 1.1.

For $\varepsilon(Q)$ continuous on S, in order that (2) of Theorem 2 hold, the admissibility restriction (2) on ε is necessary at any interior zero of ε at which M is analytic. For, if $|M(Q) - F(Q)| \leqq |\varepsilon(Q)|$ on S, then, at a zero Q_β of ε, $M(Q_\beta) = F(Q_\beta)$. If (as is the case if M is analytic at Q_β and $F(Q) \not\equiv M(Q)$) $M(Q) - F(Q) = (Q - Q_\beta)^{n_\beta} g(Q)$, where, in some neighborhood $N_{Q_\beta} \subset S$, g is bounded from zero, then

$$|M(Q) - F(Q)| \leqq |\varepsilon(Q)|$$

on S implies $|(Q - Q_\beta)^{n_\beta}/\varepsilon(Q)|\,|g(Q)| \leqq 1$ on N_{Q_β}, where defined. The last inequality is possible only if the first factor is bounded on N_{Q_β}, that is, $\varepsilon(Q)/(Q - Q_\beta)^{n_\beta}$ is bounded from zero on N_{Q_β}. At an interior point of S, M is necessarily analytic if Hypothesis (1) of Theorem 2 is satisfied; hence, if the conclusion of Theorem 2 is to hold, continuous $\varepsilon(Q)$ must satisfy admissibility requirement (2) at any interior zero of ε.

An example is next given to illustrate an application of Theorem 2 for the case $n = 1$. Let $R = \mathfrak{R} = \{z/|z| < \infty\}$; $M(z) = z \sin 1/z$ for $z \neq 0$, $M(0) = 0$; $\varepsilon(z) = (z - 1)^5(z - 3/4)(z - \frac{1}{2})g(z)$, where g is any function continuous and nonvanishing on S; $S = \{x/0 \leqq x \leqq 1\} \cup_{j=1}^{3} \gamma_j$ where the γ_j are nonintersecting closed disks with centers at the zeros of $\varepsilon(z)$. Now, by a Walsh approximation theorem [10, p. 47], $M(z)$ can be uniformly approximated by a polynomial, that is, (1) in Theorem 2 is satisfied with $f(z)$ a polynomial in z. Hence, Theorem 2 implies that for any admissible $\varepsilon(z)$, in particular as defined above, there is a polynomial $F(z)$ such that $|M(z) - F(z)| \leqq |\varepsilon(z)|$ on S.

The next theorem yields degree of convergence in the $O(\varepsilon_n(Q))$-sense by setting $S = S_1 = S_2 = \cdots$, also other special results as stated in the corollaries.

Corresponding to given $\{\varepsilon_n\}$, $\{\varepsilon_n(Q)\}$ with $\varepsilon_n(Q)$, defined on S_n and nonvanishing on ∂S_n, $n = 1, 2, \cdots$, will be called ε_n-admissible on $S = \cup S_n$ if there exists $g(Q)$ analytic on $\mathfrak{R}$ such that, for each n, $\varepsilon_n(Q) = g(Q)\phi_n(Q)$ and $\varepsilon_n \leqq \inf |\phi_n(Q)|$, $n = 1, 2, \cdots$, for $Q \in S_n$.

THEOREM 3. *Assume Hypothesis H, with $S = \bigcup_{n=1}^{\infty} S_n$, where the S_n are compact, but not necessarily disjoint. Let $\mathscr{S}_n$ be a collection* *of functions each meromorphic on an open set* R_n *and analytic on* $R_n - B^*$, *where* $S_n \subset R_n \subset \mathfrak{R}$. ($R_n$ *may be* $\mathfrak{R}$.) *Suppose a certain sequence of positive constants* $\{\varepsilon_n\}$ *assigned. Then* (1) *below implies* (2).

(1) *Corresponding to any* $\{m_n\}$, *with* m_n *analytic on* S_n^0, *continuous on* S_n, *and such that* $m_n(Q) = m_j(Q)$ *on* $S_n \cap S_j$ (*if this is not the null set*), *there exists* $f_n, f_n \in \mathscr{S}_n$, *and* M (*independent of* n) *such that* $|m_n(Q) - f_n(Q)| < M\varepsilon_n$ *on* S_n.

(2) *Corresponding to any* ε_n-*admissible* $\{\varepsilon_n(Q)\}(\varepsilon_n(Q) = g(Q)\phi_n(Q))$ *and to* $\{m_n\}$ *defined as in* (1), *there exists* h *meromorphic on* $\mathfrak{R}$ *whose only poles lie on* B^* *or coincide with those of* $m_n(Q)/g(Q)$ *on* S *and there exists* $f_n \in \mathscr{S}_n$ *such that*

$$|m_n(Q) - g(Q)[h(Q) + f_n(Q)]| \leqq M_1 |\varepsilon_n(Q)|$$

on $S_n, n = 1, 2, \cdots$. *If in* (1) *the* f_n *can be chosen as the same function for all* n, *the same is true for the* f_n *in* (2). *If, in* (1), M *is independent of* $\{m_n(Q)\}$, *then, in* (2), $M_1 = M$.

Proof. By the Mittag-Leffler theorem there exists h meromorphic on $\mathfrak{R}$ whose only poles coincide with those of m_n/g on S_n, $n = 1, 2, \cdots$. Now $(m_n(z)/g(z)) - h(z)$ is analytic on S_n^0, continuous on S_n. Hence, by hypothesis (1), there exists $f_n \in \mathscr{S}_n$ such that on S_n

$$|[m_n(Q)/g(Q) - h(Q)] - f_n(Q)| < M_1\varepsilon_n \leqq M_1 |\phi_n(Q)| .$$

This yields the required result.

If in both (1) and (2) the m_n are assumed analytic on S_n, the theorem remains true.

COROLLARY 3.1. *Let* m *be analytic on the bounded closed set* S *which does not separate the complex plane. Suppose* $\{\varepsilon_n\}$ *is a certain sequence of positive constants such that there exist polynomials* $\{p_n(z)\}$ *of respective degrees* n *and some* M *such that* $|m(z) - p_n(z) < M\varepsilon_n$ *on* S. *Then, for* ε_n-*admissible* $\{\varepsilon_n(z)\}$ *with* $\varepsilon_n(z) = P_N(z)\phi_n(z)$, *where* $P_N(z)$ *is a polynomial of degree* N, *there exist polynomials* $P_{N+n}(z)$ *of degrees* $N + n$ *such that* $|m(z) - P_{n+N}(z)| \leqq M_1 |\varepsilon_n(z)|$ *on* S.

Proof. In the theorem set $S = S_1 = S_2 = \cdots$ and $m(z) = m_1(z) = m_2(z) = \cdots$, and let $\mathscr{S}_n$ denote the set of all polynomials of degree n. Since, by the hypothesis, (1) is satisfied, the conclusion of the theorem yields the result when it is noted that h can be chosen as an appropriate rational function.

EXAMPLE. If $m(z)$ is analytic on S, $|z| \leqq 1$, m is analytic in a larger region D_ρ: $|z| < \rho$ [10, p. 79]. Fix $R, 1 < R < \rho$, and set $\varepsilon_n = 1/R^n$. Let ϕ be any function which is continuous and nonvanishing on

S and let $P_N(z)$ be a polynomial of degree N, nonvanishing on ∂S. Then K can be chosen so that, for $\varepsilon_n(z)$ defined as $KP_N(z)\phi(z)/(z^n + R^n)$, and $\phi_n(z) = K\phi(z)/(z^n + R^n)$, $\{\varepsilon_n(z)\}$ is ε_n-admissible on S. There are known to be polynomials p_n of respective degrees n such that, for some M, $|m(z) - p_n(z)| < M/R^n$ on S [10, p. 79], whence, by Corollary 3.1, there exist polynomials q_{n+N} of degrees $n + N$ such that

$$| m(z) - q_{n+N}(z) | \leqq M_1 | \varepsilon_n(z) |$$

on S, for some M_1 independent of n.

The polynomials p_{n+N} in Corollary 3.1 cannot be required to be of degree less than $n + N$. For m analytic on S defined as in the Example, choose $P_N(z)$ as a polynomial whose only zeros coincide with those of $m(z)$ on S, and define $\varepsilon_n(z) = (K/R^n)P_N(z)$, $1 < R < \rho$. Suppose there exist polynomials $p_k(z)$ of degree k such that

$$| m(z) - p_k(z) | \leqq M_1K | P_n(z) | / R^n$$

on S. Without loss of generality it can be supposed the zeros of p_k coincide with those of m on S [10, p. 310]. Now $N = m/P_N$ is analytic on S, except for removable singularities, and

$$| N(z) - p_K(z)/p_N(z) | \leqq M_2/R^n$$

on S. Since $p_k(z)/p_N(z)$ is a polynomial of degree $k - N$, this would yield a degree of convergence stronger than maximal convergence if $k - N < n$ [10, p. 79].

The result stated in Corollary 2.3, which is a direct consequence of Theorem 2, is essentially that of Corollary 3.2.

COROLLARY 3.2. *Suppose $m(z)$ is analytic on $S = \cup S_n$, a Q-set with components S_n, and let B denote its set of sequential limit points. Let $\mathfrak{R}$ be the extended complex plane minus B and define B^* as in Hypothesis H. Then, corresponding to any $\varepsilon(z) = g(z)\phi(z)$ with g analytic on $\mathfrak{R}$ and ϕ bounded from zero on each S_n, there exists f analytic on $\mathfrak{R}$-B^*, meromorphic on $\mathfrak{R}$, such that*

$$| m(z) - f(z) | \leqq | \varepsilon(z) | \text{ on } S .$$

Proof. In the theorem, let $R_n = \mathfrak{R}$, $\mathscr{S} = \mathscr{S}_1 = \mathscr{S}_2 = \cdots$ be the set of functions analytic on $\mathfrak{R}$-B^*, meromorphic on $\mathfrak{R}$, and define $m_n(z) = m(z)$ on S_n, $\varepsilon_n(z) = \varepsilon(z)$ on S_n, $\phi_n(z) = \phi(z)$ on S_n, $\varepsilon_n = \inf | \phi_n(z) |$ for $z \in S_n$. We note $\{\varepsilon_n(z)\}$ is ε_n-admissible. By a theorem of the author [9], M (1) of the theorem is satisfied, with $n = 1$ and $f_1(z) = f_2(z) = \cdots$, whence the theorem implies (2), yielding the required result.

COROLLARY 3.3. *Let* $S = \bigcup_{n=1}^{\infty} S_n$, *where the* S_n *are closed circular disks of radii one-half tangent externally along the positive real axis and ordered by increasing distance from the origin. Suppose* m *is analytic on each* S_n^0, *continuous on* S. *Then, for* $\varepsilon(z) = g(z)\phi(z)$, *where* g *is an entire function (nonvanishing on* ∂S) *and* ϕ *is bounded from zero on each* S_n, *there exists an entire function* F *such that* $|m(z) - F(z)| \leqq |\varepsilon(z)|$ *on* S.

Proof. Let $R = \mathfrak{R}$ be the finite complex plane, B^* the null set, and $\mathscr{S} = \mathscr{S}_1 = \mathscr{S}_2 = \cdots$ the class of entire functions. Define $m_n(z) = m(z)$ on S_n, $n = 1, 2, \cdots$, and set $\varepsilon_n(z) = \varepsilon(z)$ on S_n. Then define $\phi_n(z) = \phi(z)$ on S_n and $\varepsilon_n = \inf|\phi_n(z)|$ for $z \in S_n$. By a previous result [8, Theorem 3], corresponding to any $\{\varepsilon_n\}$, there exists $f(z) = f_1(z) = f_2(z) = \cdots$, $f \in \mathscr{S}$, such that $|m(z) - f(z)| < \varepsilon_n$ on S_n. Then (2) of the theorem with $F(z) = g(z)[h(z) + f(z)]$ yields the required result.

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PURDUE UNIVERSITY