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ON SMALL MAPS OF MANIFOLDS

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A result announced by R. F. Brown in 1963, and completed by Brown and Fadell, generalizing classical results of H. Hopf for differentiable manifolds, is the following:

THEOREM: Let M be a compact connected topological manifold; then

(a) M admits arbitrarily small maps with a single fixed point;

(b) If the Euler characteristic χ_M of M is zero, then M admits arbitrarily small maps without fixed points (and conversely). Here a map is small if it is close to the identity map. We propose to give a short proof of this theorem.

We will use the recent result of J. Kister (also Mazur and Stallings) that any microbundle over a complex is a bundle [4]. We note that according to [2] the result (b) holds also for manifolds with boundary.

2. **Characteristic class.** We consider the tangent microbundle $\tau_M: M \xrightarrow{d} M \times M \xrightarrow{p_1}$; here d is the diagonal map, and p_1 the first projection (cf. [5]). Attached to τ_M is the Thom class u, a well-defined element of $H^n(M \times M, M \times M - d(M))$ (here $n = \dim M$); the coefficients used are the integers $\mathbf{Z}$, if M is orientable, and twisted integers, determined by the orientations of the horizontal factor M at the points of $M \times M$, in the nonorientable case. (Cf. [6] for details in the orientable case.) We write $\tilde{u}$ for the image of u in the absolute group $H^n(M \times M)$; the Euler class e_M is the image of $\tilde{u}$ in $H^n(M)$ under the diagonal map d^* (twisted coefficients in the nonorientable case). Furthermore, M has a fundamental cycle μ (again twisted coefficients for nonorientable M). It is a well-known fact that the value $\langle e_M, \mu \rangle$ of e_M on μ equals the Euler-Poincaré characteristic χ_M of M.

[Since this is not easy to find in the literature, we sketch a proof: First assume M orientable. Let $\{x_i\}$ be a basis for $H^*(M)$ modulo torsion, and let $\{\alpha_i\}$ be the basis of $H_*(M)$ modulo torsion, dual to $\{x_i\}$ under $\langle\ ,\ \rangle$; put $r_i = \dim \alpha_i$. Define $\{x'_i\}$ by $\mathfrak{d}x'_i = \alpha_i$, where $\mathfrak{d}$ is the Poincaré duality operator $\mathfrak{d}x = x \cap \mu$; then $\{x'_j\}$ is again a basis for $H^*(M)$ modulo torsion. Finally let $\{\alpha'_j\}$ be dual to $\{x'_j\}$ under $\langle\ ,\ \rangle$. One verifies that $d_*\mu = \Sigma \alpha_i \times \alpha'_i$ modulo torsion (use $\langle x \times y, d_*\mu \rangle = \langle x \cup y, \mu \rangle$). Now $\tilde{u}$ satisfies the relation $\langle x, \alpha \rangle = (-1)^{n-r} \langle \tilde{u}, \mathfrak{d}x \times \alpha \rangle$ for $x \in H^r(M)$ (cf. [6]). Therefore we have $\langle e_M, \mu \rangle = \langle \tilde{u}, d_*\mu \rangle =$

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$\langle\tilde{u}, \Sigma\alpha_i \times \alpha_i'\rangle = \Sigma(-1)^{r_i}\langle x_i', \alpha_i'\rangle = \Sigma(-1)^{r_i} = \chi_M$. For nonorientable M let $\hat{M}$ be the orientable double covering, and use the facts that the Thom class is preserved under the covering map, that the fundamental cycle of $\hat{M}$ maps onto twice the (twisted) fundamental cycle of M, and that $\chi_{\hat{M}} = 2\chi_M$ (as one can see, e.g., from the Smith sequence).]

In particular, if $\chi_M = 0$, then also the Euler class e_M vanishes. Furthermore, in all this discussion we may, by Kister's result, replace the tangent microbundle by an actual bundle (in the local product sense) whose fibre is $\mathbf{R}^n$ with a well-defined origin and which therefore has a well-defined 0-section. We denote this bundle by $\bar{\tau}_M$.

3. Proof of theorem. We begin with part (b); thus assume $\chi_M = 0$. Embed M in a number space $\mathbf{R}^k$ with $k \geqq 2n + 1$, and let V be a (closed) polyhedral neighborhood of M that retracts onto M, via the map r. We consider the bundle $r^*\bar{\tau}_M$, induced from the bundle $\bar{\tau}_M$ (see end of § 2) by r. By naturality the Euler class of $r^*\bar{\tau}_M$ vanishes. Therefore, if K is any polyhedron of dimension $\leqq n$ contained in V, the restriction of $r^*\bar{\tau}_M$ to K admits a nonvanishing section (i.e., one that does not meet the 0-section of $r^*\bar{\tau}_M$); to prove this one uses the interpretation of the Euler class as obstruction. Let $\mathscr{A}$ be a finite, open covering of M, of dimension n, such that (a) the nerve $N_{\mathscr{A}}$ can be realized in V and (b) an associated barycentric map $f\colon M \rightarrow N_{\mathscr{A}}$ (cf. [3], p. 69) is homotopic to the identity $\mathbf{1}_M$ of M in V; this exists of course. Let s be a nonvanishing section of $r^*\bar{\tau}_M \mid N_{\mathscr{A}}$. Applying the covering homotopy theorem to the map $s \circ f$ of M into the bundle formed by the complement of the 0-section of $r^*\bar{\tau}_M$ and to the homotopy between f and $\mathbf{1}_M$, one gets a nonvanishing section of $r^*\bar{\tau}_M \mid M$, i.e. of $\bar{\tau}_M$. This section amounts of course to a fixed-point-free map of M into itself. Again according to Kister, $\bar{\tau}_M$ can be assumed to lie in any preassigned neighborhood of the diagonal of $M \times M$, which means that the map can be constructed as close to the identity as one pleases.

The converse is classical (Lefschetz fixed point theorem).

4. Proof of theorem continued. We come to part (a). As before we imbed M in a Euclidean space $\mathbf{R}^k$, and r is a retraction of some neighborhood of M onto M. Let A be a coordinate system in M (i.e., an open subset homeomorphic to R^n), and let B, respectively C, be the subsets of A corresponding to the set of points in R^n of norm <1, respectively $<\frac{1}{2}$. There exists a polyhedral neighborhood W of $M - B$ in R^k, whose r-image lies in $M - C$. Since $H^n(M - C)$ (twisted coefficients if needed) vanishes ($M - C$ being a manifold with nonempty boundary), the characteristic class of $r^*\bar{\tau}_M \mid W$ is zero. By the same argument as before, the bundle $\bar{\tau}_M \mid M - B$ has a nonvanishing section, which can be interpreted as a map f of $M - B$ into M, without fixed points. We may assume that the f-image of the bounday of $M - B$ lies in A (by taking $\bar{\tau}_M$ small enough), and it is then clear, using $A \approx \boldsymbol{R}^n$, how to extend f to a map of M into itself whose only fixed point is the point of A corresponding to the origin of $\boldsymbol{R}^n$.

If f is homotopic to the identity map of M (as it will be if it is small enough: apply r to the linear homotopy in $\boldsymbol{R}^k$), then the index of the fixed point is χ_M: the index equals $\pm$ the intersection number of the graph of f in $M \times M$ and the diagonal, and it is well known that this is χ_M under the present circumstances. In fact, this last remark yields another version of the proof of (a): if $\chi_M = 0$, one can extend f over B without any fixed point.

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