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SOME TOPOLOGICAL PROPERTIES OF CERTAIN SPACES OF DIFFERENTIABLE HOMEOMORPHISMS OF DISKS AND SPHERES

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Let $D_n = \{x \in E_n : |x| \leqq 1\}$, and $S^n = \{x \in E_{n+1} : |x| = 1\}$. We denote by H_n the space of C^∞ homeomorphisms of D_n onto itself leaving a neighborhood of the boundary fixed. Let K_n be the space of C^∞ orientation preserving homeomorphisms of S^n onto itself. It is not required that maps in the two spaces have differentiable inverses. In both space the C^k topology is used.

The purpose of this paper is to establish the following two theorems:

THEOREM 1. *H_n is contractible to a point for any n.*

THEOREM 2. *K_n is arcwise connected for any n.*

NOTATION. $f(x) = (f_1(x_1, \cdots, x_n), \cdots, f_n(x_1, \cdots, x_n))$ where $x = (x_1, \cdots, x_n)$, or simply $f(x)$ will denote mappings of E_n into E_n. The shorter form will be used where the meaning is clear.

The topological analog of Theorem 1 is established by a mapping described by Alexander (1923) [1]. Smale (1959) [4] proved the corresponding result for $n = 2$ in the space of diffeomorphisms on D_n leaving a neighborhood of the boundary fixed. Kneser (1926) [3] proved that the space of all orientation preserving homeomorphisms of S^2 onto S^2 has the rotation group as a deformation retract, while Smale gave the corresponding result for the space of orientation preserving diffeomorphisms on S^2 in the paper referred to above. Fisher's work (1960) [2] gives the analog of Theorem 2 in the topological case for $n = 3$.

II. Proof of Theorem 1. Let $m(v)$ be a mapping on I (the unit interval $[0, 1]$) with the following properties:

(a) $m(v) \in C^\infty$;

(b) $m'(v) > 0$ on $\left(0, \dfrac{3}{4}\right)$;

(c) $m(v) = 1$ on $\left(\dfrac{3}{4}, 1\right]$;

(d) $m(v) = e^{-(1/r)}$ on $\left(0, \dfrac{1}{4}\right)$;

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(e) $m(0) = 0$.

Now define $k(v, t) = \begin{cases} 1 - (1 - e^{-(1/t)+1})\ (1 - m(v)) & t \neq 0, \\ m(v) & t = 0, \end{cases}$

on $I \times I$. We see that:

(a′) $k(v, t) \in C^\infty$ on $I \times I$;

(b′) $k(v, t)$ is monotonic in v for each $t \in I$;

(c′) $k(v, t) = 1$ for $v \geqq \dfrac{3}{4}$ for all $t \in I$;

(d′) $k(v, 1) = 1$ for all $v \in I$;

(e′) $k(v, 0) = m(v)$;

(f′) $0 \leqq k(v, t) \leqq 1$ on $I \times I$.

The mapping

$$x \longrightarrow k(|\,x\,|^2, t)x \tag{1}$$

is in H_n for each t. At $t = 1$ the mapping is the identity, while at $t = 0$ the mapping has all partial derivatives of all orders zero at the origin.

The mapping given by Alexander was defined as follows:

$$f_t(x) = \begin{cases} tf\left(\dfrac{x}{t}\right), & t \neq 0 \ (f \text{ extended to be the identity outside } D_n)\ , \\ x, & t = 0. \end{cases}$$

In the C^k topology the mapping of $H_n \times I \longrightarrow H_n$ defined by $(f, t) \longrightarrow f_t$ (the Alexander map) will not be continuous for $k \geqq 1$. In general, $\lim_{t \to 0} f_t \neq f_0$ because at the origin the derivatives of f_t do not converge to the derivatives of the identity mapping. However, by composing the Alexander mapping with (1), we obtain the mapping required in Theorem 1. Thus define

$$h : H_n \times I \longrightarrow H_n$$

by

$$h(f, t) = kf_t$$

where

$$kf_t(x) = k(|\,f_t(x)\,|^2, t)\ f_t(x)\ .$$

In particular $h(f, 1) = f$ for all $f \in H_n$, while $h(f, 0)$ is the mapping given by (1). Because of the form of map (1) at the origin, all derivatives of all orders of kf_t approach zero there and the problem mentioned above is removed. The argument that h is continuous is tedious but straightforward.

III. Local straightening of mappings in E_n. The proof of Theorem 2 requires some local straightening procedures for maps in E_n which we now give. For this purpose let L be the space of C^∞ orientation preserving homeomorphisms mapping $U_r = \{x \in E_n : |x| \leqq r\}$ into E_n, leaving the origin fixed and topologized by the C^k topology. We will use $J(f)_p$ to represent the Jacobian matrix of f evaluated at $p \in U_r$, and $|J(f)_p|$ the corresponding determinant.

LEMMA 1. *Suppose $f \in L$ with $J(f)_p = (a_{ij})$, p the origin and (a_{ij}) nonsingular. Then there is a path $f_t \in L$ from f to g, where g agrees with f in a neighborhood of the boundary of U_r and is the linear map with Jacobian (a_{ij}) in a neighborhood of the origin. Also for all t, f_t agrees with f in a neighborhood of the boundary of U_r.*

Proof. Let $\sigma(v)$ be a mapping on $[0, \infty)$ with the following properties:

(a) $\sigma(v) \in C^\infty$;

(b) $\sigma(v) = 1$ on $[0, \alpha)$, $\alpha > 0$;

(c) $\sigma(v) = 0$ for $v \geqq 1$;

(d) $\sigma'(v) \leqq 0$ for $v \in [0, \infty)$.

We see that $|\sigma'(v)| < M$ for some M. Let $c < r$ be chosen so that for $x \in U_c$,

(i) $\left| a_{ij} - \dfrac{\partial f_i(x_1, \cdots, x_n)}{\partial x_j} \right| < \varepsilon, \varepsilon > 0; i = 1, \cdots, n; j = 1, 2, \cdots, n.$

Then for $x \in U_c$,

(ii) $|a_{i1}x_1 + \cdots + a_{in}x_n - f_i(x_1, \cdots, x_n)| < n\varepsilon c$ for $i = 1, 2, \cdots, n$.

Now define

$$\begin{aligned} f_t(x_1, \cdots, x_n) = \Big(& f_1(x_1, \cdots, x_n) \\ & + t\sigma\left(\frac{x_1^2 + \cdots + x_n^2}{c^2}\right)(a_{11}x_1 + \cdots + a_{1n}x_n \\ & - f_1(x_1, \cdots, x_n)), \cdots, f_n(x_1, \cdots, x_n) \\ & + t\sigma\left(\frac{x_1^2 + \cdots + x_n^2}{c^2}\right)(a_{n1}x_1 + \cdots + a_{nn}x_n \\ & - f_n(x_1, \cdots, x_n))\Big) . \end{aligned}$$

At $t = 0$, $f_t = f$; at $t = 1$, f_t is linear with Jacobian (a_{ij}) inside a neighborhood of the origin; for all t, f_t agrees with f outside U_c. The element in the (i, j)th position of $J(f_t)$ differs from a_{ij} by at most

$$\left| a_{ij} - \frac{\partial f_i(x_1, \cdots, x_n)}{\partial x_j} \right|$$

$$+ \left| \frac{2x_j t}{c^2} \sigma' \left(\frac{x_1^2 + \cdots + x_n^2}{c^2} \right) (a_{i1}x_1 + \cdots + a_{in}x_n - f_i(x_1, \cdots, x_n)) \right|$$

$$+ \left| t\sigma \left(\frac{x_1^2 + \cdots + x_n^2}{c^2} \right) \left(a_{ij} - \frac{\partial f_i(x_1, \cdots, x_n)}{\partial x_j} \right) \right| .$$

On U_c, $|x_j| \leqq c$ so that the expression is bounded by $\varepsilon + (2/c)Mn\,\varepsilon\,c + \varepsilon = (2 + 2Mn)\varepsilon$. Hence by choosing ε sufficiently small, $|J(f_t)|$ will remain positive on U_c for all t so that f_t will be a homeomorphism on U_r. Continuity of the path f_t in L is immediate from the definition of f_t.

LEMMA 2. *Let $f(x_1, \cdots, x_n) = (a_{11}x_1 + \cdots + a_{1n}x_n, a_{22}x_2 + \cdots + a_{2n}x_n, \cdots, a_{nn}x_n) \in L$ with $a_{11} \cdots a_{nn} > 0$. Then there is a path $f_t \in L$ such that $a \leqq t \leqq b$, $f_t(x_1, \cdots, x_n) = f(x_1, \cdots, x_n)$ at $t = a$, $f_t(x_1, \cdots, x_n) = (a_{11}x_1, \cdots, a_{nn}x_n)$ in a neighborhood of the origin at $t = b$, and $f_t(x_1, \cdots, x_n) = f(x_1, \cdots, x_n)$ in a neighborhood of the boundary of U_r for each t.*

Proof. We construct the path in $n - 1$ arcs as follows. Choose a positive c_1 less than r. Let $k_1 > 1$ be sufficiently large so that whenever

$$x_1^2 + k_1^2 x_2^2 + \cdots + k_1^2 x_n^2 \leqq c_1^2 ,$$

we have $|x_i| < \varepsilon$, $i = 2, 3, \cdots, n$
Now define

$$f_t(x_1, \cdots, x_n) = (a_{11}x_1 + \cdots + a_{1n}x_n - t\sigma \left(\frac{x_1^2 + k_1^2 x_2^2 + \cdots + k_1^2 x_n^2}{c_1^2} \right) \cdot$$
$$(a_{12}x_2 + \cdots + a_{2n}x_n), a_{22}x_2 + \cdots + a_{2n}x_n, \cdots, a_{nn}x_n) .$$

Then $f_t = f$ when $t = 0$ and f_t at $t = 1$ is the mapping

$$(a_{11}x_1, a_{22}x_2 + \cdots + a_{2n}x_n, \cdots, a_{nn}x_n)$$

in a neighborhood of the origin. For each t, $f_t = f$ outside

$$x_1^2 + k_1^2 x_2^2 + \cdots + k_1^2 x_n^2 = c_1^2$$

so that $f_t = f$ outside U_{c_1}. Also $J(f_t)$ in the (1, 1) position differs from a_{11} by

$$\left| t \frac{2x_1}{c_1^2} \sigma' \left(\frac{x_1^2 + k_1^2 x_2^2 + \cdots + k_1^2 x_n^2}{c_1^2} \right) (a_{12}x_2 + \cdots + a_{1n}x_n) \right| .$$

This expression is zero outside the ellipsoid $x_1^2 + k_1^2 x_2^2 + \cdots + k_1^2 x_n^2 = c_1^2$. Inside this, $|x_1| < c_1$ so if $|a_{ij}| < M_1$, $j = 2, \cdots, n$, and M is a bound on the derivative of $\sigma(v)$, the expression is at most $1 \cdot (2/c_1) \cdot M(n - 1) \cdot M_1 \cdot \varepsilon$. This expression is small whenever ε is small (nothing that ε

can be chosen independent of c_1). Thus by choosing ε small, $|J(f_t)|$ will remain positive inside the ellipsoid and f_t will be a homeomorphism for each t.

Thus we assume $f \in L$ and for $c_i > 0$ with $|x| \leqq c_i < r$ the mapping is given by

$$(a_{11}x_1, \cdots, a_{i-1,i-1}x_{i-1}, a_{ii}x_i + a_{i,i+1}x_{i+1} + \cdots + a_{in}x_n, \\ a_{i+1,i+1}x_{i+1} + \cdots + a_{i+1,n}x_n, \cdots, a_{nn}x_n).$$

Let $k_i > 1$ be sufficiently large so that whenever $x_1^2 + \cdots + x_i^2 + k_i^2x_{i+1}^2 + \cdots + k_i^2x_n^2 \leqq c_i^2$, it follows that $|x_j| < \varepsilon$, $j = i + 1, \cdots, n$. Define for $x \in U_r$

$$\begin{aligned} f_t(x_1, \cdots, x_n) &= \Big(f_1(x_1, \cdots, x_n), \cdots, f_{i-1}(x_1, \cdots, x_n), f_i(x_1, \cdots, x_n) \\ &- t\sigma\Big(\frac{x_1^2 + \cdots + x_i^2 + k_i^2x_{i+1}^2 + \cdots + k_i^2x_n^2}{c_i^2}\Big)(a_{i,i+1}x_{i+1} + \cdots + a_{in}x_n)\,, \\ &f_{i+1}(x_1, \cdots, x_n), \cdots, f_n(x_1, \cdots, x_n)\Big)\,. \end{aligned}$$

For the proper choice of ε, we can repeat the argument given above.

LEMMA 3. *Suppose $f(x_1, \cdots, x_n) = (a_1x_1, \cdots, a_nx_n) \in L$, $a_i > 0$ for all i. There is a path f_t in L from f to a mapping which is the identity in a neighborhood of the origin, and $f_t = f$ for all t in a neighborhood of the boundary of U_r.*

Proof. First, if $a > 0$ let $p(x)$ be a function on $(-\infty, \infty)$ with:
(a) $p(x) \in C^\infty$;
(b) $p'(x) > 0$ on $(-\infty, \infty)$;
(c) $p(x) = x$ in a neighborhood of the origin;
(d) $p(x) = ax$ outside $(-s + \alpha, s - \alpha)$, $\alpha > 0$.
We again construct the arc in segments. Choose $s_1 < r$ and define

$$\begin{aligned} f_t(x_1, \cdots, x_n) \\ = \Big(a_1x_1 + t\sigma\Big(\frac{x_2^2 + \cdots + x_n^2}{s_1^2}\Big)(p_1(x_1) - a_1x_1), a_2x_2, \cdots, a_nx_n\Big)\,, \end{aligned}$$

where σ is defined in Lemma 1 and $p_1(x_1)$ satisfies properties (*a*) – (d) above for $s = s_1$. At $t = 0$, $f_t = f$; at $t = 1$ in a neighborhood of the origin f_t is the mapping $(x_1, a_2x_2, \cdots, a_nx_n)$. Also for all $t \in I$, $f_t = f$ outside the cylinder $x_2^2 + \cdots + x_n^2 \leqq s_1^2$, $-s_1 \leqq x_1 \leqq s_1$. $J(f_t)$ in the (1, 1) position is

$$\left[1 - t\sigma\left(\frac{x_2^2 + \cdots + x_n^2}{s_1^2}\right)\right] a_1 + t\sigma\left(\frac{x_2^2 + \cdots + x_n^2}{s_1^2}\right) p_1'(x_1) ,$$

which is positive for all t on the cylinder given above. Hence f_t is a homeomorphism for each t.

Now there is an s_2 with $0 < s_2 < s_1$ so that on the cylinder $x_1^2 + x_3^2 + \cdots + x_n^2 \leqq s_2^2$, $-s_2 \leqq x_2 \leqq s_2$ the mapping is given by $(x_1\, a_2x_2, \cdots, a_nx_n)$. On this cylinder define

$$f_t(x_1, \cdots, x_n) = \Big(x_1, a_2x_2 + t\sigma\left(\frac{x_1^2 + x_3^2 + \cdots + x_n^2}{s_2^2}\right) \times (p_2(x_2) - a_2x_2), a_3x_3, \cdots, a_nx_n\Big) .$$

Here $p_2(x_2)$ satisfies conditions (a)-(d) given above with $s = s_2$. Repeating the process we complete the desired path.

IV. Proof of Theorem 2. The proof now consists of fitting together properly the mappings already constructed.

Let $f \in K_n$. Then there is a point p on S^n so that f has nonsingular Jacobian at that point. Let $(0_1, P_1)$ be a coordinate neighborhood where $0_1 = S^n - p_1(p_1$ antipodal to $p)$ and P_1 an associated stereographic projection. Now there is a path e_t, $t \in I$, in the rotation group on S^n so that e_0 is the identity map, $e_1 f = g$ leaves p fixed and $P_1(e_1 f)P_1^{-1} = P_1 g P_1^{-1}$ has a triangular Jacobian with positive diagonal elements at the origin. Let C be a closed disk on S^n so that for some $r > 0$, $U_r \subset P_1(C)$. Applying Lemmas $1-3$ there is a path $(P_1gP_1^{-1})_t$, $t \in I$, in the space of mappings on U_r from $P_1gP_1^{-1}$ to a mapping which is the identity in a neighborhood of the origin. Furthermore, for all t, $(P_1gP_1^{-1})_t$ agrees with $P_1gP_1^{-1}$ for all $x \in P_1(C)$ except on an interior set of U_r. Define $g_t \in K_n$ by

$$g_t = \begin{cases} P_1^{-1}(P_1gP_1^{-1})_t P_1 & \text{on } C \\ g & \text{outside } C. \end{cases}$$

Then $g_0 = g$ and g_1 is the identity in a neighborhood of p.

Next let C_1 and C_2 be two closed sets covering S^n where C_1 is a circular disk on S^n with p the center of the disk, and so that C_1 is in an open set left pointwise fixed by g_1. We further assume $p \notin C_2$. Let $(0_2, P_2)$ be a coordinate neighborhood with $C_2 \subset 0_2 = S^n - p$ and P_2 an associated stereographic projection. Then except for a trivial dilation $P_2g_1P_2^{-1}$ is an element of the space H_n. By Theorem 1 there is a path $(P_2g_1P_2^{-1})_t$ from $P_2g_1P_2^{-1}$ to the identity map on $P_2(C_2)$. We now define $h_t \in K_n'$ by

$$h_t = \begin{cases} P_2^{-1}(P_2 g_1 P_2^{-1})_t P_2 \text{ on } C_2 \\ g_1 \text{ outside } C_2. \end{cases}$$

The path from f to the identity map is now complete and Theorem 2 is established.

The spaces H_n and K_n are intermediate spaces to the topological spaces of Alexander and Kneser, and the diffeomorphism spaces treated by Smale. It is interesting to note that methods used in this paper are related to methods used in the larger nondifferentiable spaces and the smaller differomorphism spaces. Alexander's mapping is altered to give Theorem 1, while Theorem 2 parallels Smale's work.

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