PACIFIC JOURNAL OF MATHEMATICS
Vol. 15, No. 4, 1965

ON n-ORDERED SETS AND ORDER COMPLETENESS

LINO GUTIERREZ NOVOA

In this paper, the notion of an n-ordered set is introduced as a natural generalization of that of a totally ordered set (chain). Two axioms suffice to describe an n-order on a set, which induces three associated structures called respectively: the incidence, the convexity, and the topological structures generated by the order. Some properties of these structures are proved as they are needed for the final theorems. In particular, the existence of natural k-orders in the "flats" of an n-ordered set and the fact that (as it happens for chains) the topological structure is Hausdorff.

The idea of Dedekind cut is extended to n-ordered sets and the notions of strong-completeness, completeness, and conditional completeness are introduced. It is shown that the S^n sphere is s-complete when considered as an n-ordered set. It is also proved that E^n, the n-dimensional euclidean space, fails to be s-complete or complete, but that it is conditionally complete. It is also proved that every s-complete set is compact in its order topology but that the converse is not true. These results generalize classical ones about the structure of chains and lattices.

II. n-Ordered sets. An element of the cartesian product X^{n+1} of a set X will be called an n-simplex and denoted by $\sigma^n = (s_0, s_1, \cdots, s_n)$ where $s_i \in X$ for every i. The class of even permutations of this sequence is called an oriented n-simplex and denoted by $|\sigma^n| = |s_0, s_1, \cdots, s_n|$. The class of odd permutations is another oriented n-simplex denoted by $|-\sigma^n| = |-(s_0, s_1, \cdots, s_n)|$. The set of all oriented n-simplexes of X will be denoted by $|X^n|$. In what follows n-simplex will mean oriented n-simplex.

The *join* of two simplexes $|\sigma^h| = |s_0, s_1, \cdots, s_h|$ and $|\tau^k| = |t_0, t_1, \cdots, t_k|$ is the $h + k - 1$ — simplex $|s_0, s_1, \cdots, s_h, t_0, t_1, \cdots, t_k|$ and will be denoted by $|\sigma^h, \tau^k|$.

An *n-ordered set* is a pair (X, φ_n), where X is a set and φ_n is a function from $|X^n|$ to the set $\{-1, 0, 1\}$ and which satisfies A_1 and A_2.

A_1.—*For every* $|\sigma^n| \in |X^n|$; $\varphi_n|-\sigma^n| = -\varphi_n|\sigma^n|$.

Before stating A_2 we introduce the following notation:

$$\Phi_i(\sigma^n, \tau^n) = \varphi_n|t_i, s_1, s_2, \cdots, s_n|\varphi_n|t_0, t_1, \cdots, t_{i-1}, s_0, t_{1+1}, \cdots, t_n|$$

Received March 8, 1964.

A_2.—*If* $\Phi_i(\sigma^n, \tau^n) \geqq 0$ *for* $i = 0, 1, \cdots n$; *then* $\varphi_n \mid \sigma^n \mid \varphi_n \mid \tau^n \mid \geqq 0$.

D_1.—The simplex $\mid \pi^{n-1} \mid$ is said to be an *upper bound* for the set $\{x_\alpha; \alpha \in I\} \subset X$ if $\varphi_n \mid x_\alpha, \pi^{n-1} \mid \geqq 0$ for every $\alpha \in I$. If all the relations are strictly $>$ then $\mid \pi^{n-1} \mid$ is a *proper upper bound*. Similar definitions for *lower bounds* using $\leqq$ and $<$.

D_2.—The n-order φ_n is *open from above* (*from below*) if every finite subset of X has a proper upper bound (lower bound).

T_1.—*If* φ_n *is an open from above* (*or from below*) *n-order of* X *then the following transitive property holds*:

If $\varphi_n \mid s_0, s_1, \cdots, s_{i-1}, x, s_{i+1}, \cdots, s_n \mid \geqq 0$ *for all* i *and some* $x \in X$ *then*: $\varphi_n \mid \sigma^n \mid \geqq 0$.

Proof. Apply A_2 to the pair $\mid x, \pi^{n-1} \mid, \mid \sigma^n \mid$ where $\mid \pi^{n-1} \mid$ is a proper bound for $\{s_i\} \cup \{x\}$

EXAMPLES.

(a) In the vector space V^n over the reals define:

$$\varphi_{n-1} \mid v_0, v_1, \cdots, v_{n-1} \mid = \text{sign of det.} \mid v_0, v_1, \cdots, v_{n-1} \mid$$

The function φ_{n-1} is an $n-1$-order of V^n.

(b) In the same space define:
$\varphi_n \mid v_0, v_1, \cdots, v_n \mid = \text{sign of det.} \mid v_i - v_0 \mid$, $i = 1, 2, \cdots, n$. φ_n is an n-order of V^n.

(c) The function of example (a) restricted to the sphere $\mid V \mid = 1$ gives an n-1 order of the n-1-sphere.

(d) Any 1-order satisfying the transitive property of T_1 is equivalent to a chain if we define: $\varphi_1 \mid a, b \mid$ to be $-1, 0$ or 1 according to $a > b$, $a = b$ and $a < b$ respectively.

(e) A field G is said to be n-ordered if it is also an n-ordered set and the mappings: $f_a: x \to ax$ and $g_a: x \to a + x$ are order-automorphisms for any $a \neq 0$.
If we call: $\mid \sigma^n \mid = \mid S_0, S_1, \cdots, S_n \mid$; $\mid a\sigma^n \mid = \mid aS_0, aS_1 \cdots aS_n \mid$, and $\mid a + \sigma^n \mid = \mid a + S_0, a + S_1, \cdots, a + S_n \mid$, then the definition means exactly that $\varphi_n \mid \sigma^n \mid \varphi_n \mid a\sigma^n \mid$ and $\varphi_n \mid \sigma^n \mid \varphi_n \mid a + \sigma^n \mid$ depend only on a. The following examples can be given:

(e_1) The real numbers field is a 1-ordered (open) field. (This is a well known result).

(e_2) The complex numbers field is a 2-ordered (open) field if we define for any $\mid \sigma^2 \mid = \mid \alpha_0, \alpha_1, \alpha_2 \mid$:

$$\varphi_2(\sigma^2) = \frac{i\Delta(\sigma^2)}{|\Delta(\sigma^2)|} \qquad \text{where } \Delta(\sigma^2) = \begin{vmatrix} 1 & 1 & 1 \\ \alpha_0 & \alpha_1 & \alpha_2 \\ \bar{\alpha}_0 & \bar{\alpha}_1 & \bar{\alpha}_2 \end{vmatrix};$$

$\bar{\alpha}$ being the complex conjugate of α, $|\alpha|$ the modulus of α.

(e_3) The field of quaternions, considered as a 4-dimensional vector space over R and with the 4-order of example (b) above becomes a 4-ordered (noncommutative) field.

(f) The n-order of V^n given in example (b) makes an n-ordered vector space out of V^n in the sense that the mappings f_a: $x \longrightarrow ax$ and g_y: $x \longrightarrow x + y$ are order-isomorphisms for any $a \in R$, $a \neq 0$ and any $y \in V^n$. This example can be generalized as follows:

(g) Let V be any linear space over the ordered commutative field K, and $B \subset V$ any Hamel base for V. If $N = \{b_1, b_2, \cdots, b_n\}$ is any finite subset of B: we can make V into an n-ordered vector space by defining $\varphi_n(V_0, V_1, \cdots, V_n) = +1, -1$ or 0 whenever det $(V_i^j - V_0^j)$ is $>$, $<$ or $= 0$ in $K(V_i^j$ is the coefficient of b_j in the expression of V_i in terms of the base B)

The independence of the axioms follows from the following examples:

In the set $\{a, b, c\}$ define: $\varphi_2 | a, b, c | = \varphi_2 | b, a, c | = 1$ and $\varphi_2 = 0$ elsewhere. This system satisfies A_2 but not A_1.

In the set $\{a, b, c, d, e\}$ define:

$$\begin{aligned} &\varphi_2 | e, c, d | = \varphi_2 | e, c, a | = \varphi_2 | e, c, b | = \varphi_2 | d, a, b | \\ &= \varphi_2 | d, b, c | = \varphi_2 | d, c, a | = \varphi_2 | a, c, b | = 1 \end{aligned}$$

and define φ_2 on the remaining simplexes according to A_1. This system satisfies A_1 but not A_2.

III. Consequences of the Axioms.

D_3.—Two elements x, y of W are said to be *equivalent* if for every $|\pi^{n-1}| \in |X^{n-1}|$ we have: $\varphi_n | x, \pi^{n-1} | = \varphi_n | y, \pi^{n-1} |$. They are *conjugate* if $\varphi_n | x, \pi^{n-1} | = - \varphi_n | y, \pi^{n-1} |$. The relation between equivalent elements is an equivalence relation and the set of equivalence classes can be n-ordered in the usual way. For this set the following axiom holds.

A_3.—*There are no distinct equivalent points.*

From now on we assume (X, φ_n) satisfies A_1, A_2 and A_3 and call (X, φ_n) a reduced n-ordered system. An easy consequence of A_3 is:

C_3.—An element $x \in X$ has at most one conjugate x^*.

D_4.—A simplex $|\sigma^k|$, $k \leqq n$, is said to be *singular* if for every $|\pi^{n-k-1}|$ we have: $\varphi_n | \sigma^k, \pi^{n-k-1} | = 0$. In particular $|\sigma^n|$ is singular if: $\varphi_n | \sigma^n | = 0$.

The following theorems follow easily and are stated without proof:

T_2.—*x^* nonsingular, is the conjugate of x, if and only if $|x, x^*|$ is singular.*

T_3.—*Any simplex with repeated elements is singular.*

T_4.—*There is at most one singular 0-simplex.*

T_5.—*If $x \neq y$, for some $|\pi^{n-1}|$: $\varphi_n|x, \pi^{n-1}| \neq \varphi_n|y, \pi^{n-1}|$.*

We have also:

T_6.—*If $\Phi_i(\sigma^n, \tau^n) \leqq 0$ for $i = 0, 1, 2, \cdots, n$ then $\varphi_n|\sigma^n|\varphi_n|\tau^n| \leqq 0$.* (Compare A_2)

C_6.—If $\Phi_i(\sigma^n, \tau^n) = 0$ for $i = 0, 1, 2, \cdots, n$ then $\varphi_n|\sigma^n|\varphi_n|\tau^n| = 0$.

T_7.—*If $\Phi_i(\sigma^n, \tau^n) \geqq 0$ for $i = 0, 1, 2, \cdots n$ and $\varphi_n|\sigma^n|\varphi_n|\tau^n| = 0$ then: $\Phi_i(\sigma^n, \tau^n) = 0$ for every i.*

IV. Flats and relative orders.

D_5.—Given a nonsingular k-simplex $|\pi^k|$, $k < n$, the set $F|\pi^k| = \{x; |x, \pi^k| \text{ is singular}\}$ will be called the *flat* determined by π^k

T_8.—*If $s_i \in F|\pi^{n-1}|$, $i = 0, 1 \cdots n$ then $|\sigma^n|$ is singular.*

Proof. Apply C_6 to the pair $|\sigma^n|$, $|x, \pi^{n-1}|$ where the last simplex is nonsingular (Such an x exists by D_4)

C_8.—If $|\sigma^n|$ and $|\tau^n|$ are both nonsingular, then for some i: $|t_i, s_1, s_2, \cdots, s_n|$ is not singular.

T_9.—*If $|\mu^k, \pi^k|$, $h + k = n - 1$, is nonsingular, the function $\varphi_h|\sigma^h| = \varphi_n|\sigma^h, \pi^k|$ is a reduced h-order defined on the h-simplexes $|\sigma^h|$ of the set $F|\mu^h|$, φ_h is called the order of $F|\mu^h|$ relative to $|\pi^k|$.* The proof is straightforward.

T_{10}.—*(Invariance of the relative order).—If φ_h and ψ_h are the relative orders of $F|\mu^h|$ by $|\pi^k|$ and $|\tau^k|$ respectively, then:*

$$\varphi_h|\sigma^h| = \psi_h|\mu^h|\varphi_h|\mu^h|\psi_h|\sigma^h| \quad \textit{for any } |\sigma^h| \subset F|\mu^h|.$$

Proof. We consider first the case where $|\pi^k|$ and $|\tau^k|$ differ by only one element. Let $|\pi^k| = |a, \xi^{k-1}|$ and $|\tau^k| = |\pm(b, \xi^{k-1})|$ and apply A_2 to the pair: $|b, \xi^{k-1}, \mu^h|$ and $|a, \xi^{k-1}, \sigma^h|$. It is easily seen that the only Φ_i different from 0 is:

$$\varphi_n|a, \xi^{k-1}, \mu^h|\varphi_n|b, \xi^{k-1}, \sigma^h|.$$

Hence: $\varphi_n|\tau^k, \mu^h|\varphi_n|\pi^k, \sigma^h| = \varphi_n|\pi^k, \mu^h|\varphi_n|\tau^k, \sigma^h|$ and the theorem follows since: $\varphi_n|\tau^k, \mu^h| \neq 0$.

For the general case we construct inductively, using C_8, the sequence: $|\pi^k_{-1}| = |\pi^k|$; $|\pi^k_j| = |t_{i_0}, t_{i_1}, \cdots, t_{i_j}, p_{j+1}, p_{j+2}, \cdots, p_n|$ where the t_i are elements of $|\tau^k|$ and apply the previous result several times to the h-orders relative to $|\pi^k_j|$ and $|\pi^k_{j+1}|$ for $j = -1, 0, 1, \cdots, n$.

Since the previous result is independent of $|\tau^k|$ and $|\pi^k|$ we have:

C_{10}.—*The orders induced in* $F|\mu^h|$ *by* $|\pi^k|$ *and* $|\tau^k|$ *are either identical or opposite and we may speak of the two "natural" orders in any flat* $F|\mu^h|$.

V. Convexity theorems.

D_6.—The element x is said to be *contained* in the nonsingular simplex $|\pi^h|$ if for some natural order of $F|\pi^h|$ we have:

$$a_i = \varphi_h|p_0, p_1, \cdots, p_{i-1}, x, p_{i+1}, \cdots, p_n|\varphi_h|\pi^h| \geqq 0$$

for every $\qquad 0 \leqq i \leqq h$.

If every $a_i > 0$ we say that x is *interior* to $|\pi^h|$.

D_7.—The *segment* $(\bar{a}, \bar{b})$ is the set of interior points of the non-singular 1-simplex $|a, b|$.

D_8.—A set $C \subset X$ is said to be *convex* if for every $a, b \in C$, such that $|a, b|$ is not singular we have: $(\bar{a}, \bar{b}) \subset C$.

From the definitions follows:

T_{11}.—*If x is contained in (interior to) $|\sigma^h|$ it is also contained in (interior to) $|-(\sigma^h)|$*

T_{12}.—*If x is contained in (interior to) $|\sigma^n|$ and every s_i satisfies: $\varphi_n|s_i, \pi^{n-1}| \geqq 0$ for some $|\pi^{n-1}|$, then $\varphi_n|x, \pi^{n-1}| \geqq 0$ (> 0).*

Proof. We assume $\varphi_n|\sigma^n| > 0$ and apply A_2 to the pair:

$$|x, \pi^{n-1}|, |\sigma^n| \text{ to get } \varphi_n|x, \varphi^{n-1}| \geqq 0 .$$

Now if x is interior to $|\sigma^n|$, $\varphi_n|x, \pi^{n-1}|$ cannot be 0, otherwise by C_6 and T_8 we would have $\varphi_n|\sigma^n| = 0$ which contradicts our assumption.

D_9.—We say that $|\sigma^k|$ is contained in (interior to) $|\pi^h|$ if every s_i is contained in (interior to) $|\pi^h|$.

Using the previous theorem we now can prove:

T_{13}.—*If x is contained in $|\sigma^n|$ and $|\sigma^n|$ is contained in (interior to) $|\pi^n|$ then x is contained in (interior to) $|\pi^n|$.*

This theorem can be extended in a natural way to the case of two simplexes $|\sigma^h|$ and $|\pi^k|$ where h and k can be different from n. We omit the details. As a corollary of these theorems we have:

T_{14}—*The sets $Ct|\sigma^h|$ and Int $|\sigma^h|$ formed by the elements which are contained in and interior to $|\sigma^h|$ respectively, are convex.*

VI. The induced structures. Given an n-ordered set (X, φ_n) the following structures are said to be induced by the order:

(a) The *incidence structure*, $(X, \mathscr{R})$ where $\mathscr{R}$ is the family of flats of (X, φ_n).

(b) The *convexity structure* $(X, \mathscr{G})$ where $\mathscr{G}$ is the family of convex subsets of (X, φ_n).

(c) The *topological structure* $(X, \mathscr{T})$ where $\mathscr{T}$ is the family of closed sets generated by the sub-base $\mathscr{B}$. The elements of $\mathscr{B}$ are the sets $\bar{B}^+_{\pi^{n-1}} = \{x; \varphi_n \mid x, \pi^{n-1} \mid \geqq 0\}$ for any nonsingular $\mid \pi^{n-1} \mid$, together with the $\bar{B}^-_{\pi^{n-1}} = \{x, \varphi_n \mid x, \pi^{n-1} \mid \leqq 0\}$. We prove the following theorem concerning the topological structure $(X, \mathscr{T})$

T_{15}.—*The topological space* $(X, \mathscr{T})$ *is Hausdorff, provided* (X, φ_n) *contains no singular point.*

Proof. If $\mid x, y \mid$ is singular then by T_2, $x = y^*$. Since x is not singular, for some $\mid \pi^{n-1} \mid$ we have $\varphi_n \mid x, \pi^{n-1} \mid > 0$, and therefore $\varphi_n \mid y, \pi^{n-1} \mid < 0$. The sets $B^+_{\pi^{n-1}} = \{z; \varphi_n \mid z, \pi^{n-1} \mid > 0\}$ and $B^-_{\pi^{n-1}} = \{z; \varphi_n \mid z, \pi^{n-1} \mid < 0\}$ are disjoint (open) neighborhoods of x and y respectively.

If $\mid x, y \mid$ is not singular, for some π^{n-2}, $\varphi_n \mid x, y, \pi^{n-2} \mid > 0$. Assume first that for some z, we have: $0 \neq \varphi_n \mid z, x, \pi^{n-2} \mid \neq \varphi_n \mid z, y, \pi^{n-2} \mid \neq 0$. To be precise let $\varphi_n \mid z, x, \pi^{n-2} \mid < 0$ and $\varphi_n \mid z, y, \pi^{n-2} \mid > 0$ and call $\mid \pi^{n-1} \mid = \mid z, \pi^{n-2} \mid$. Then $B^+_{\pi^{n-1}}$ and $B^-_{\pi^{n-1}}$ are the required neighborhoods. If such a z does not exist, call $\mid \tau^{n-1} \mid = \mid x, \pi^{n-2} \mid$ and $\mid \sigma^{n-1} \mid = \mid y, \pi^{n-2} \mid$. It is easily verified that $B^-_{\tau^{n-1}}$ and $B^+_{\sigma^{n-1}}$ satisfy the requirement. The above theorem is an extension of a well known result in the topology of chains. (See [1] p. 39)

The following result is important and will be needed in the sequel:

T_{16}.—*If* x, y *are contained in* $\mid \sigma^n \mid$ *then* x *is contained in some* $\mid \sigma^n_i \mid = \mid s_0, s_1, \cdots, s_{i-1}, y, s_{i+1}, \cdots, s_n \mid$.

Proof. Call $P_i = \varphi^n \mid \sigma^n_i \mid$ and $P_{ij} = \varphi^n \mid \sigma^n_{ij} \mid = \varphi^n \mid s_0, s_1, \cdots, s_{j-1}, x, s_{j+1}, \cdots, s_{i-1}, y, s_{i+1}, \cdots, s_n \mid$ for $i \neq j$. Clearly $P_{ij} = -P_{ji}$. We put $P_{ii} = \varphi^n \mid s_0, s_1, \cdots, s_{i-1}, x, s_{i+1}, \cdots, s_n \mid$.

Applying A_2 to the pair:

$$\mid \sigma^n_i \mid \text{ and } \mid \sigma^n_{jk} \mid \text{ we get:}$$

If P_jP_{ki} and P_kP_{ij} are both $\geqq 0$ then $P_iP_{kj} \geqq 0$. We may assume $\varphi^n \mid \sigma^n \mid > 0$. Then by D_6 all P_r are $\geqq 0$. Hence we have transitively: $P_{ki} \geqq 0$ and $P_{ij} \geqq 0$ imply $P_{kj} \geqq 0$. Using this, we can prove easily, by induction on n, that for a certain value of K, say $k = k_0$ all $P_{k_0j} \geqq 0$, $j = 0, 1, 2, \cdots, n$, and this means that x is contained in $\mid \sigma^n_{k_0} \mid$.

VI. ***n*-Order completeness.** In the theory of ordered sets a lattice is said to be complete if every subset of it has a L.U.B. and a G.L.B. This notion is equivalent to that of compactness of the associated topological space (interval topology) when applied to chains. (See [3]) In this sense the lattice of real numbers fails to be complete.

(See [3], p. 51) On the other hand it is *conditionally complete* because every *bounded* subset has a L.U.B. and a G.L.B. This property is equivalent to the fact that every Dedekind cut has a separation element. We proceed to extend these ideas to n-ordered sets. Let (x, φ_n) be a reduced n-ordered set. Every element $x \in X$ determines an n-1-order in X by defining: $\varphi_{n-1} | \pi^{n-1} | = \varphi_n | x, \pi^{n-1} |$. Consider now the subsets of $| X^{n-1} |$ defined by: $C_x^+ = \{| \pi^{n-1} | ; \varphi_{n-1} | \pi^{n-1} | \geqq 0\}$ and $C_x^- = \{| \pi^{n-1} | ; \varphi_{n-1} | \pi^{n-1} | \leqq 0\}$ It is clear that $C_x^+ \cup C_x^- = | X^{n-1} |$ and x is called the separation element of the pair (C_x^+, C_x^-). We also have for every nonsingular $| \pi^{n-1} | : | \pi^{n-1} | \in C_x^+ \cap C_x^-$ if and only if: $x \in F | \pi^{n-1} |$. We now extend the notion of "cut" to n-ordered sets. Let C^+ and C^- be two subsets of $| X^{n-1} |$ such that $C^+ \cup C^- = | X^{n-1} |$ and γ any object not in X. Let X^* be the set $X \cup \{\gamma\}$. We extend the function φ_n to the set $| X^{*n} |$ by defining: $\varphi_n^* | \gamma, \pi^{n-1} | = +1, -1$ or 0 whenever $| \pi^{n-1} |$ is in $C^+ - C^-, C^- - C^+$ or in $C^+ \cap C^-$, resp. Then $\varphi_n^* | \pi^n | = \varphi_n | \pi^n |$ for $| \pi^n | \in | X^n |$. We call γ the ideal element defined by (C^+, C^-).

D_{10}.—A pair (C^+, C^-) of subsets of $| X^{n-1} |$ is said to be a *cut* if the following properties are satisfied:

(a) $C^+ \cup C^- = | X^{n-1} |$

(b) (X^*, φ_n^*) is an n-ordered set. (Satisfies A_1 and A_2)

D_{11}.—A cut (C^+, C^-) is said to be *interior* or a *Dedekind cut* if the ideal element γ defined by the cut is interior to some $| \sigma^n |$ of X. This means that for some $| \sigma^n |$ and every i we have:

$$\varphi_n^* | s_0, s_1, \cdots, s_{i-1}, \gamma, s_i, \cdots, s_n | \varphi_n^* | \sigma^n | > 0 .$$

D_{12}.—An n-ordered set (X, φ_n) is said to be *strongly complete* (s-complete) if every cut has a separation element in X. It is *conditionally s-complete* if every interior cut has a separation element. It is *order complete* if the topological space $(X, \mathscr{T})$ is compact.

T_{17}.—*If (X, φ_n) is s-complete, then every element has a conjugate.*

Proof. For every $x \in X$ the sets C_x^+ and C_x^- obviously form a cut (C_x^+, C_x^-). It is also clear that the pair (C_x^-, C_x^+) is also a cut defining x^*.

T_{18}.—*The S^n sphere with the n-order defined in II, example c, is strongly complete.*

We give only an idea of the proof: For any nonsingular n-simplex $| \pi^n |$ in S^n and taking antipodal points, we have a decomposition of S^n into 2^{n+1} simplexes. Given a cut (C^+, C^-), the ideal element, γ is order-contained in one of them say $| \pi_0^n |$. The repeated barycentric subdivisions of $| \pi_0^n |$ furnish, (because of T_{16}) a sequence of simplexes

$|\pi_i^n|$, $i = 0, 1, 2 \cdots$ such that γ is interior to all of them and their diameters tend to 0. There is also a unique point p of S^n common to all the $|\pi_i^n|$. It is easily shown that p is the separation element of the cut.

It follows from T_{17} that E^n, the euclidean n-space with the n-order of example II(b) is not *s-complete* and from D_{11} that it is not *order complete.* This is not surprising if we recall the initial remark of this section. But we can prove:

T_{19}.—*E^n with the n-order of example* II(b) *is conditionally s-complete.*

We omit the proof since it is entirely similar to that of T_{18}. The relationship between order-completeness, s-completeness, and compactness is established in the following theorem which is similar to the classical result for partially ordered sets and chains. (See [3] and [2])

T_{20}.—*If the ordered set (X, φ_n) is s-complete, then it is order complete i.e. the space $(X, \mathscr{F})$ is compact.*

Proof. Let $\mathfrak{G}$ be a collection of closed sets of $(X, \mathscr{F})$ with the finite intersection property. It follows from a well known theorem of Alexander that we may restrict ourselves to the case where $\mathscr{G}$ consists of elements from the sub-base $\mathscr{B}$. (See VI a) Let $\mathscr{M}$ be a maximal extension of $\mathscr{G}$ in $\mathscr{F}$ with respect to the property. Then an element of $\mathscr{F}$ belongs to $\mathscr{M}$ if and only if it meets every element of $\mathscr{M}$. (See [4])
Using the notation of T_{15} we now define (C^+, C^-):

$$|\pi^{n-1}| \in C^+ \text{ if } \bar{B}^+_{\pi^{n-1}} \in \mathscr{M} \quad \text{and} \quad |\pi^{n-1}| \in C^- \text{ if } \bar{B}^-_{\pi^{n-1}} \in \mathscr{M} \ .$$

We shall prove that (C^+, C^-) satisfies D_{10} and is therefore a cut. If $|\pi^{n-1}|$ is not in C^+ for some $M_0 \in \mathscr{M}$ we have:

$$M_0 \subset X - \bar{B}^+_{\pi^{n-1}} = B^-_{\pi^{n-1}} \subset \bar{B}^-_{\pi^{n-1}} \ .$$

It follows that every $M \in \mathscr{M}$ meets $\bar{B}^-_{\pi^{n-1}}$ since it meets M_0. Or $|\pi^{n-1}| \in C^-$. Therefore $C^+ \cup C^- = |X^{n-1}|$. In order to show that D_{10}(b) holds, we first prove the following result:

If γ is the ideal element defined by (C^+, C^-) and γ satisfies a finite system of equalities: $\varphi_n|\gamma, \sigma_i^{n-1}| = e_i$; $i = 1, 2, \cdots, l$, then there is some $z \in X$ which, when substituted for γ, also satisfies the equalities.

Proof. If $e_i = 1$, then $|\sigma_i^{n-1}|$ is in C^+ but not in C^- and therefore $\bar{B}^-_{\sigma_i{}^{n-1}}$ fails to meet at least one element of $\mathscr{M}$. Denote it by M_i. Similarly if $e_j = -1$, $\bar{B}^+_{\sigma_j{}^{n-1}}$ does not meet $M_j \in \mathscr{M}$. And if $e_k = 0$, both $\bar{B}^+_{\sigma_k{}^{n-1}}$ and $\bar{B}^-_{\sigma_k{}^{n-1}}$ belong to $\mathscr{M}$. We call $M_k = \bar{B}^+_{\sigma_k{}^{n-1}} \cap \bar{B}^-_{\sigma_k{}^{n-1}}$;

clearly $M_k \in \mathscr{M}$.

We consider now $I = \bigcap_r M_r$, $r = 1, 2, \cdots, l$. Since I is not empty, we take any $z \in I$. It can be readily seen that z satisfies all the equalities. To show now that (C^+, C^-) is a cut, it suffices to check A_2 since A_1 is obviously satisfied. But if some pair $|\sigma^n|, |\tau^n|$ of n-simplexes of X^* fails to satisfy A_2, by the previous result the same is true when we put $z \in X$ instead of γ, and this leads to a contradiction. Let s be the separation element of the cut (C^+, C^-) and G any element of $\mathscr{G}$. Since G belongs to the sub-base $\mathscr{B}$ and T_{17} holds, it can be written $G = \bar{B}^+_{\tau n-1}$ for some $|\tau^{n-1}|$: This means $|\tau^{n-1}| \in C^+$ and $\varphi_n |s, \tau^{n-1}| \geqq 0$, or equivalently, $s \in \bar{B}^+_{\tau n-1} = G$. This completes the proof.

That the converse of the above theorem is not true, can be seen by means of the following example:

Let (S^2, φ_2) be the 2-sphere with the 2-order of Example II(c) and K the finite subset of six elements $(\pm i, \pm j, \pm k)$, 2-ordered by the restriction of φ_2 to K. Then K is compact in the induced topology but the cut generated by the elements $\pm(1/3)(i + j + k)$ of S^2 in (S^2, φ_2), restricted to (K, φ_2), have no separation elements in (K, φ_2) and therefore it is not s-complete.

References

1. G. Birkhoff, *Lattice Theory*, Rev, Ed. 1948.
2. O. Frink, *Topology in lattices*, Trans. Amer. Math. Soc. **51** (1942).
3. Haar and Konig, *Über einfach geordnete Mengen*, Crelle's Jour. 139 (1910).
4. J. L. Kelley, *General Topology*, 1955.

UNIVERSITY OF ALABAMA