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Φ-BOUNDED HARMONIC FUNCTIONS AND CLASSIFICATION OF RIEMANN SURFACES

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Let $\Phi(t)$ be a nonnegative real valued function defined for t in $[0, \infty)$ such that $\Phi(t)$ is unbounded in $[0, \infty)$ and bounded in a neighborhood of a point in $[0, \infty)$. A harmonic function u on a Riemann surface R is said to be Φ-bounded if the composite function $\Phi(|u|)$ has a harmonic majorant on R. Denote by $O_{H\Phi}$ the class of all Riemann surfaces on which every Φ-bounded harmonic function reduces to a constant. The main result in this paper is the following: $O_{H\Phi} = O_{HP}$ (resp. O_{HB}) if and only if $d(\Phi) < \infty$ (resp. $d(\Phi) = \infty$), where $d(\Phi) = \limsup_{t\to\infty} \Phi(t)/t$. This is the best possible improvement of a result of M. Parreau.

We also prove a similar theorem for the classification of subsurfaces of Riemann surfaces using Φ-bounded harmonic functions vanishing on the relative boundaries of subsurfaces.

The chief tool of our proof is the theory of Wiener compactifications of Riemann surfaces.

Consider a nonnegative real valued function $\Phi(t)$ defined for all real numbers t in $[0, \infty)$. A harmonic function u on a Riemann surface R is said to be Φ-bounded if the composite function $\Phi(|u|)$ has a harmonic majorant on R. The totality of Φ-bounded harmonic functions on R is denoted by $H\Phi(R)$, or simply $H\Phi$. We denote by $O_{H\Phi}$ the class of all Riemann surfaces R on which every Φ-bounded harmonic function reduces to a constant. Our problem is to determine $O_{H\Phi}$ for every Φ.

First assume that $\Phi(t)$ is bounded on $[0, \infty)$. Then every harmonic function is Φ-bounded. Hence R belongs to $O_{H\Phi}$ if and only if there exists no nonconstant harmonic function on R. Thus *the class $O_{H\Phi}$ consists of all closed Riemann surfaces if Φ is bounded.* Soon we see that the converse is also valid. Hence, hereafter, we always assume that

(1) *$\Phi(t)$ is unbounded on* $[0, \infty)$.

We say that $\Phi(t)$ is bounded at a point t_0 in $[0, \infty)$ if there exists a neighborhood of t_0 relative to $[0, \infty)$ in which $\Phi(t)$ is bounded. Now assume that $\Phi(t)$ is not bounded at any point of $[0, \infty)$. Let u be a nonconstant harmonic function on R. Then $\Phi(|u|)$ is not bound at any neighborhood of any point of R and so u is not Φ-bounded. Thus *the class $O_{H\Phi}$ consists of all Reimann surfaces if $\Phi(t)$ is not bounded at*

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any point of $[0, \infty)$. Soon we see that the converse is also true. Hence, hereafter, we always assume that

(2) *$\Phi(t)$ is bounded at least at one point in* $[0, \infty)$.

Now our problem which is left is to determine $O_{H\Phi}$ for functions Φ satisfying the two conditions (1) and (2). For the aim, we put

$$d(\Phi) = \limsup_{t\to\infty} \Phi(t)/t \, .$$

Clearly $0 \leqq d(\Phi) \leqq \infty$. Our result is stated as follows:

THEOREM 1. *Assume that Φ satisfies* (1) *and* (2). *If $d(\Phi)$ is finite (resp. infinite), then $O_{H\Phi} = O_{HP}$ (resp. O_{HB}).*

Since the restrictions on Φ are exclusive each other, we also see that $O_{H\Phi} = O_{HP}$ (resp. O_{HB}) implies that Φ satisfies (1) and (2) and $d(\Phi)$ is finite (resp. infinite). This theorem is proved by Parreau [3] for the special Φ which is increasing and convex (and so continuous) (see also Ahlfors-Sario's book [1], pp. 216–219). Parreau's proof keenly uses the increasingness and convexity of Φ and one might suspect that these assumptions are inevitable. We are interested in the fact that for the validity of Parreau's result, no assumption is needed for Φ except the inevitable conditions (1) and (2). Thus our Theorem 1 is the best possible generalization of Parreau's result at least in the above formulation.

2. Before entering the proof of Theorem 1, for convenience, we explain an outline of the *Wiener compactification* of a Riemann surface and its some properties which we use in the proof of Theorem 1. For details, consult Constantinescu-Cornea's book [2], § 6, 8 and 9.

Let F be a Riemann surface not belonging to O_G and f be a real valued function on F. Let $\overline{W}_f^F$ (resp. $\underline{W}_f^F$) be the totality of superharmonic (resp. subharmonic) functions s on F such that there exists a compact subset K_s of F with the property that $f \leqq s$ (resp. $f \geqq s$) on $F - K_s$. If $\overline{W}_f^F$ and $\underline{W}_f^F$ are nonvoid, then $\overline{W}_f^F$ and $\underline{W}_f^F$ are Perron's families and so

$$\bar{h}_f^F(p) = \inf\,(s(p);\, s \in \overline{W}_f^F) \text{ and } \underline{h}_f^F(p) = \sup\,(s(p);\, s \in \underline{W}_f^F)$$

are harmonic and $\bar{h}_f^F \geqq \underline{h}_f^F$. If $\bar{h}_f^F = \underline{h}_f^F$ on F, then we write $h_f^F = \bar{h}_f^F = \underline{h}_f^F$ and we call f to be harmonizable on F.

Let R be an arbitrary Riemann surface. A real-valued function f on R is said to be a continuous Wiener function if (a) for any subsurface F of R with $F \notin O_G$ as a Riemann surface, the restriction of f on F is harmonizable on F and the restriction of $|f|$ on F has a superharmonic majorant on F; and if (b) f is finitely continuous on R. We denote by $WC = WC(R)$ the totality of continuous Wiener functions on R. We also denote by $WB = WB(R)$ the totality of bounded members in WC. Observe that WC (resp. WB) is a vector space and closed under max and min operations. Any continuous superharmonic function on R which has a harmonic majorant clearly belongs to WC. Hence $HP \subset WC$ and $HB \subset WB$.

There exists a unique compact Hausdorff space R^* containing R as its open and dense subset such that $C(R^*)| R = WB(R)$, where $C(R^*)$ is the totality of finitely continuous functions on R^* and $C(R^*)|R$ is the totality of restrictions of functions in $C(R^*)$ to R. We call R^* the Wiener compactification of R. By the obvious identification, we may simply write as $C(R^*) = WB(R)$. It is clear that any function in $WC(R)$ is (not necessarily finitely) continuous on R^*, or more accurately, is continuously extended to R^*. Hereafter, we use topological notions relative to R^* only. For example, $\bar{A}$ for $A \subset R$ means the closure of A in R^*. But the notation ∂A for $A \subset R^*$ is the only exceptional. ∂A means the boundary of $A \cap R$ relative to R.

Let $W_0C(R) = (f \in WC;\ h_f^R = 0)$ if $R \notin O_G$ and $W_0C(R) = WC$ if $R \in O_G$. We set $\varDelta = (p \in R^*;\ f(p) = 0$ for any f in $W_0C)$. This is a compact subset of $\varGamma = R^* - R$ and called the (Wiener) harmonic boundary of R. It is seen that $W_0C = (f \in WC;\ f = 0$ on $\varDelta)$. From the definition, it is obvious that $R \in O_G$ if and only if $\varDelta = \varphi$. Moreover,

LEMMA 1. *$R \in O_{HB} - O_G$ if and only if $\varDelta$ consists of only one point.*

Let F be an open subset of R each boundary point of which is regular for Dirichlet problem and $\partial F \neq \phi$. Such an F is called a regular open subset of R. We say that $F \in SO_{HB}$ if any connected component of F does not carry any nonconstant bounded harmonic functions vanishing continuously at ∂F. The most important is the following

LEMMA 2. *$F \notin SO_{HB}$ if and only if $\bar{F} - \overline{\partial F}$ contains a point of $\varDelta$.*

As an corollary of this, we can easily see the following useful

LEMMA 3. *Let F be a regular open subset of R and s be a superharmonic function on F bounded from below. If*

$$\lim_{F \ni p \to q} \inf s(p) \geqq 0$$

for any q in $\partial F \cup (\bar{F} \cap \varDelta)$, then $s \geqq 0$ on F.

3. Proof of Theorem 1 for $d(\varPhi) < \infty$. Since $d(\varPhi) < \infty$, we can find a positive number c and a point t_0 in $[0, \infty)$ such that $\varPhi(t) \leqq ct$ for any $t \geqq t_0$. Assume that there exists a nonconstant HP-function

u_1 on R. Then $u = u_1 + t_0$ is also a nonconstant harmonic function on R with $u \geqq t_0 \geqq 0$ on R. Thus $\Phi(|u|) \leqq c|u| = cu$ and cu is an HP-function on R. Hence $O_{H\Phi} \subset O_{HP}$.

Conversely, assume that there exists a nonconstant $H\Phi$-function u on R. We have to prove the existence of a nonconstant HP-function on R. By the definition, there exists an HP-function v on R with $\Phi(|u|) \leqq v$ on R. If v is not a constant or u is bounded, then nothing is left to prove and so we assume that v is a constant and u is not bounded. Then the connected open set $D = (|u(p)|;\ p \in R)$ in $[0, \infty)$ does not contain 0. Contrary to the assertion, assume that $D \ni 0$. Then $D = [0, \infty)$ and so $(\Phi(|u(p)|);\ p \in R) = (\Phi(t);\ t \in [0, \infty))$ is unbounded in $[0, \infty)$ by the assumption (1) for Φ. But this is impossible, since $\Phi(|u|) \leqq v$(constant) on R. Thus $0 \notin D$. This shows that u does not change sign on R. Hence u or $-u$ is a nonconstant HP-function on R. Therefore, $O_{H\Phi} \supset O_{HP}$. Thus $O_{H\Phi} = O_{HP}$ for Φ with $d(\Phi) < \infty$.

4. Proof of Theorem 1 for $d(\Phi) = \infty$. First assume that there exists a nonconstant HB-function u on R. By the assumption (2) for Φ, there exists an interval $(a, b) \subset [0, \infty)$ in which $\Phi(t) \leqq c$ (constant). By choosing a suitable constants A and B, the range of $v = Au + B$ is contained in (a, b). Then $\Phi(|v|) = \Phi(v) \leqq c$ on R. Thus v is a nonconstant $H\Phi$-function on R. Hence $O_{HB} \supset O_{H\Phi}$.

Next we prove the converse inclusion $O_{HB} \subset O_{H\Phi}$, or equivalently, $R \notin O_{H\Phi}$ implies $R \notin O_{HB}$. Assume that there exists a nonconstant $H\Phi$-function u on R. We have to prove that $R \notin O_{HB}$. Contrary to the assertion, assume that $R \in O_{HB}$. By the definition, there exists an HP-function v such that $\Phi(|u|) \leqq v$ on R. From this, we see that $R \notin O_{HP}$. For, if $R \in O_{HP}$, then $\Phi(|u|) \leqq v$ (constant) and since $d(\Phi) = \infty$, $|u|$ is bounded. This contradicts $R \in O_{HB}$. Hence $R \notin O_{HP}$ and a fortiori $R \notin O_G$. Thus $R \in O_{HB} - O_G$ and so by Lemma 1, the harmonic boundary Δ of R consists of only one point δ, i.e. $\Delta = (\delta)$. By $d(\Phi) = \infty$, we can find a stricly increasing sequence $(r_n)_{n=1}^{\infty}$ of positive numbers such that

$$\lim_{n\to\infty} \Phi(r_n)/r_n = \infty \text{ and } \lim_{n=\infty} r_n = \infty .$$

Let $G_n = (p \in R;\ |u(p)| < r_n)$. Since u is not a constant and u is unbounded by $R \in O_{HB}$, G_n is a regular open subset of R with $\partial G_n \neq \phi$ and $G_n \nearrow R$. We see that $G_n \notin SO_{HB}$ for some n. For, if this is not the case, then $G_n \in SO_{HB}$ for all $n = 1, 2, \cdots$. Let $a_n = r_n/\Phi(r_n)$. Then $a_n \searrow 0 (n \to \infty)$. Consider the function $a_n v - |u|$, which is superharmonic and bounded from below on G_n and continuous in $G_n \cup \partial G_n$. If $q \in \partial G_n$, then

$$|u(q)| = r_n = (r_n/\Phi(r_n))\,\Phi(r_n) = a_n\Phi(|u(q)|) \leqq a_n v(q)\ .$$

Thus $a_n v - |u| \geqq 0$ on ∂G_n. Hence $a_n v - |u| \geqq 0$ in G_n. For, if $a_n v(p_0) - |u(p_0)| < d < 0$ for some p_0 in G_n, then $G_n' = (p \in G_n;\ a_n v(p) - |u(p)| < d)$ is a nonempty regular open subset with $G_n' \cup \partial G_n' \subset G_n$. The function $d - (a_n v - |u|)$ is a positive and bounded (with bound $d + r_n$) subharmonic function in G_n' vanishing continuously at $\partial G_n'$. So $G_n' \notin SO_{HB}$. But this is a contradiction, since $G_n \supset G_n' \cup \partial G_n'$ and $G_n \in SO_{HB}$. Hence $a_n v - |u| \geqq 0$ in G_n. Now let p be an arbitrary point in R. There exists an n_0 such that $p \in G_n$ for all $n \geqq n_0$. Then $|u(p)| \leqq a_n v(p)$ for all $n \geqq n_0$. Thus by making $n \nearrow \infty$, $|u(p)| = 0$, i.e. $u \equiv 0$ on R, which is a contradiction. Hence $G_{n_1} \notin SO_{HB}$ for some n_1 and so $G_n \notin SO_{HB}$ for all $n \geqq n_1$ and so without loss of generality, we may assume that $G_n \notin SO_{HB}$ for all $n = 1, 2, \cdots$. In particular, $G_1 \notin SO_{HB}$ implies that $\bar{G}_1 - \overline{\partial G_1}$ contains δ by Lemma 2 (recall that $\Delta = (\delta)$), i.e. $\bar{G}_1$ is a neighborhood of δ in the Wiener compactification R^* of R. Hence in the topology of R^*,

$$(*)\qquad \limsup_{R \ni p \to \delta} |u(p)| = \limsup_{G_1 \ni p \to \delta} |u(p)| \leqq r_1\ .$$

Now consider the function $f_n = a_n v + r_1 - |u|$, which is superharmonic and bounded from below on G_n and continuous in $G_n \cup \partial G_n$. If $q \in \partial G_n$, then as before,

$$|u(q)| = r_n = (r_n/\Phi(r_n))\,\Phi(r_n) = a_n\Phi(|u(q)|) \leqq a_n v(q) \leqq a_n v(q) + r_1$$

and so $f_n(q) \geqq 0$ on ∂G_n. This with (*) gives that

$$\liminf_{G_n \ni p \to q} f_n(p) \geqq 0$$

for any q in $\partial G_n \cup (\delta) = \partial G_n \cup (\bar{G}_n \cap \Delta)$. Hence by Lemma 3, $f_n \geqq 0$ in G_n, or

$$|u| \leqq a_n v + r_1$$

in G_n. Let p be an arbitrary point in R. There exists an n_0 such that $p \in G_n$ for all $n \geqq n_0$. Thus $|u(p)| \leqq a_n v(p) + r_1$ for all $n \geqq n_0$. Hence by making $n \nearrow \infty$, $|u(p)| \leqq r_1$, i.e. $|u| \leqq r_1$ on R. Hence $R \notin O_{HB}$. This is a contradiction, since we assumed that $R \in O_{HB}$. Thus $R \in O_{HB}$.

5. Finally we make a few remark to the classification of Riemann surfaces with regular boundaries. Let $\Phi(t)$ be a non-negative real-valued function defined in $[0, \infty)$. Let R be a Riemann surface and F be a regular open subset of R. We denote by $H_0\Phi = H_0\Phi(R, F)$ the totality of harmonic functions u in F vanishing continuously at ∂F such that $\Phi(|u|)$ admits a harmonic majorant in F. We say that

$F \in SO_{H\Phi}$ if $H_0\Phi$ contains only zero. We want to determine $SO_{H\Phi}$ for every Φ. As before, unless Φ satisfies (1), then $F \in SO_{H\Phi}$ if and only if F does not carry any nonzero harmonic function in F vanishing continuously at ∂F. Thus *$SO_{H\Phi}$ consists of all relatively compact regular open subsets of Riemann surfaces if $\Phi(t)$ is bounded in* $[0, \infty)$. Similarly as before, *$SO_{H\Phi}$ consists of all regular open subsets of Riemann surfaces if $\Phi(t)$ is not bounded at $t = 0$.* Hence we have only to consider the problem of determining $SO_{H\Phi}$ under the condition

(3) *$\Phi(t)$ is bounded at $t = 0$ and unbounded in* $[0, \infty)$.

As before $d(\Phi) = \limsup_{t \to \infty} \Phi(t)/t$. By (3), $SO_{H\Phi} \subset SO_{HB}$ is always valid. Without assuming (3), we can show $SO_{H\Phi} \supset SO_{HB}$ if $d(\Phi) = \infty$ (see the proof of Theorem 2 below). If $d(\Phi) < \infty$, then we cannot get any definite conclusion in general. So we prove only the following

THEOREM 2. *Assume that Φ satisfies* (3) *and* $d(\Phi) = \infty$. *Then* $SO_{H\Phi} = SO_{HB}$.

Proof. Assume that there exists a nonconstant $H_0\Phi$-function u in F. Then $\Phi(|u|) \leqq v$ in F for some harmonic function v in F. We want to show that $F \notin SO_{HB}$. Contrary to the assertion, assume that $F \in SO_{HB}$. By $d(\Phi) = \infty$, there exists an increasing sequence $(r_n)_{n=1}^{\infty}$ of positive numbers such that $a_n = r_n/\Phi(r_n) \searrow 0$ and $r_n \nearrow \infty$ as $n \nearrow \infty$. Let $F_n = (p \in F;\ |u(p)| < r_n)$. Clearly $F_n \nearrow F$ and $F_n \in SO_{HB}$. As in the proof of Theorem 1 for $d(\Phi) = \infty$, $a_n v - |u| \geqq 0$ on ∂F_n and $a_n v - |u|$ is lower bounded superharmonic function in F_n and so $F_n \in SO_{HB}$ implies that $a_n v \geqq |u|$ in F_n and finally $u = 0$ in F. This is a contradiction and so $F \notin SO_{HB}$, or $SO_{H\Phi} \supset SO_{HB}$.

Now we change the definition of $H_0\Phi = H_0\Phi(R, F)$ as follows: $H_0\Phi$ is the totality of harmonic functions u in F vanishing continuously at ∂F such that $\Phi(|u|)$ admits a harmonic majorant in R, where we define $u = 0$ in $R - F$. Under this new definition, Theorem 2 is again valid. In fact, $SO_{H\Phi} \subset SO_{HB}$ is clear by (3) and the above proof for $SO_{H\Phi} \supset SO_{HB}$ for $d(\Phi) = \infty$ can be applied with an obvious modification to the present case. Moreover, we can show the following

THEOREM 3. *Assume that Φ satisfies* (3). *If F is a regular open subset of R with the compact complement in R, then $F \in SO_{H\Phi}$ if and only if $F \in SO_{HB}$, or equivalently, $R \in O_G$.*

Proof. Clearly $F \in SO_{H\Phi}$ implies $F \in SO_{HB}$ by the condition (3). Hence we have to show that $F \notin SO_{H\Phi}$ implies $F \notin SO_{HB}$. Evidently, $F \notin SO_{HB}$ is equivalent to $R \notin O_G$. Let u be a nonconstant $H_0\Phi$-function in F. Then there exists an HP-function v in R such that $\Phi(|u|) \leqq v$ on R, where we define $u = 0$ in $R - F$. Contrary to the assertion, assume that $F \in SO_{HB}$, or equivalently $R \in O_G$. Then the inclusion $O_G \subset O_{HP}$ implies that v is a constant, i.e. $\Phi(|u|)$ is a bounded function on R. Let $D = (|u(p)|;\ p \in R)$. Since D is connected and $|u|$ is not bounded, $D = [0, \infty)$. Thus $(\Phi(|u(p)|);\ p \in R) = (\Phi(t);\ t \in [0, \infty))$. From this, the boundedness of $\Phi(|u|)$ implies the boundedness of $\Phi(t)$, which contradicts the assumption (3).

References

1. L. V. Ahlfors and L. Sario, *Riemann Surfaces*, Princeton, 1960.

2. C. Constantinescu and A. Cornea, *Ideale Räder Riemannscher Flächen*, Springer, 1963.

3. M. Parreau, *Sur les moyennes des fonctions harmoniques et analytiques et la classification des surfaces de Riemann*, Ann. Inst. Fourier, **3** (1952), 103-197.

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