PACIFIC JOURNAL OF MATHEMATICS
Vo. 15, No. 4, 1965

ON RELATIVE COIMMUNITY

T. G. McLAUGHLIN

The paper relates to questions raised by A. A. Muchnik in a 1956 Doklady abstract, namely, whether a noncreative r.e. set can be simple in a creative one, and whether a creative r.e. set can be simple in a noncreative one. We furnish a negative answer to the second question, and give a variety of partial results having to do with the first. Thus, we show that no universal set can have immune relative complement inside a noncreative r.e. set and that any r.e. set which is hyperhypersimple in a creative set must itself be creative; whereas, there exist three sets α, β, γ, $\alpha \subseteqq \beta \subseteqq \gamma$, such that β is creative, α and γ are nonuniversal, and both $\beta - \alpha$ and $\gamma - \beta$ are hyperhyperimmune.

In addition, we answer two questions of J. P. Cleave regarding the comparison of effectively inseparable (e.i.) and "almost effectively inseparable" (almost e.i.) sequences of r.e. sets. Thus: a sequence can be almost e.i. without being e.i.; and an almost e.i. sequence of disjoint r.e. sets may have a noncreative union.

1. In [7], Muchnik formulated (in slightly different language) the following two problems: given two r.e. sets Δ, Σ, with $\Delta \subseteqq \Sigma$ and $\Sigma - \Delta$ immune, can we have

(1) Δ creative and Σ mesoic?

(2) Δ mesoic and Σ creative?

In the present paper, we consider these questions relative to not-necessarily-r.e. universal sets; and we make two or three applications of our results to matters considered in [7] and [1]. We are indebted to J. P. Cleave for providing us with a draft copy of [1], which has since been supplanted by a (forthcoming) joint paper of Cleave and C. E. M. Yates. (For an abstract of the Cleave-Yates paper, see [2].)

2. **Definitions and preliminary lemmas.** Basic terminology is essentially as in [3]. Notational departures from [3]: we use 'W_x' in place of 'ω_x', 'ϕ', in place of '0' for the null set, '$\cup$' for union, '$\cap$' for intersection, and '$-$' instead of a prime symbol for complementation. A set Δ of natural numbers is said to be *immune* just in case Δ is infinite and, for all i, if $W_i \subseteqq \Delta$ then W_i is finite. If Δ,

Received June 6, 1963, and in revised form July 16, 1964. Preparation of this paper has been carried out, in part, under fellowship support from the National Science Foundation. The material in the paper is based on some results given in the writer's Ph. D. dissertation, submitted to the Graduate Division of the University of California at Los Angeles in May, 1963. We are indebted to the referee for pointing out some expository flaws in the original version of section 2.

Σ are sets of numbers such that $\Delta \subseteqq \Sigma$ and $\Sigma - \Delta$ is immune, we say that Δ is *coimmune in* Σ. (In case $\Delta = W_j$, $\Sigma = W_k$, for some j and k, we say instead that Δ is *simple in* Σ.) Similarly, if $\Delta \subseteqq \Sigma$ and $\Sigma - \Delta$ is hyperhyperimmune, we say that Δ is *cohyperhyperimmune* in Σ, or that Δ is *hyperhypersimple in* Σ, in case both Δ and Σ are r.e. (For definition and discussion of the notion of hyperhyperimmunity, the reader may consult [9] or [10]; the existence of hyperhypersimple sets is known from [5].)

LEMMA 1. *There exists a set of numbers, α, such that both α and its complement, $\bar{\alpha}$, are hyperhyperimmune.*

Proof. This follows from the definition of hyperhyperimmunity ([10]) by a straightforward diagonal argument, since there are only countably many recursive sequences of pairwisedisjoint nonempty finite sets.

The terms 'creative', 'productive' 'contraproductive' 'mesoic' and 'simple', as applied to number sets, have their customary significance (see [3]). A mesoic set Δ is said to be *pseudosimple* just in case, for some number j, $W_j \subseteqq \bar{\Delta}$ and $\Delta \cup W_j$ is simple. We will make use of the (more or less) standard notations '$\leqq_{m-1}$' and '$\leqq_{1-1}$' for the relations of (recursive) many-one and one-to-one reducibility, respectively. By a *universal* set is meant a set Δ of numbers such that $W_j \leqq_{1-1} \Delta$ for all j (or, equivalently as it happens, $W_j \leqq_{m-1} \Delta$ for all j).

Lemma 2. ([11, *Chapter* 5, *Proposition* 2 *and Theorem* 6], *noting that g and t can be one-to-one in the cited Theorem* 6; *see also* [4, *Proposition* 1.12]). *Δ is universal if and only if $\bar{\Delta}$ is productive.*

Let an infinite set Δ be given. Suppose there is a partial recursive function p such that, for all j, if $W_j \subseteqq \Delta$ then j is in the domain of p and $(\forall i)(i \in W_j \Rightarrow p(j) > i)$. Then (and only then), we say that Δ is *strongly effectively immune.* An r.e. set with a strongly effectively immune complement is called *strongly effectively simple.* An example of a strongly effectively simple set: the simple-but-not-hypersimple set of Post [9]. The following fact is easy to establish, using a trick due to Myhill ([4]):

LEMMA 3. *If Δ is strongly effectively immune, then there is a recursive function, r, such that* $(\forall_j)(W_j \subseteqq \Delta \Rightarrow (\forall i)(i \in W_j \Rightarrow r(j) > i))$.

In [1] and [6], it has been noted that Friedberg's procedure ([5]) for decomposing a nonrecursive r.e. set into two nonrecursive, r.e., disjoint subsets can be extended to provide K(r.e.) components, for any K such that $2 \leqq K \leqq \aleph_0$, in such a way that, in the case $K = \aleph_0$, the components are presented *in a recursive sequence* (i.e., in a sequence indexed by a recursive function). In [1], extending an important observation of Yates, Cleave shows that if decomposition of a nonrecursive r.e. set Σ into K r.e. components ($2 \leqq K \leqq \aleph_0$) is carried out according to this extension of Friedberg's construction, then any two of the resulting components are recursively inseparable in a remarkably strong sense: namely, if W_j is any one of the components, then, for arbitrary k, $W_k \subseteqq \bar{W}_j \Rightarrow W_k - \Sigma$ is r.e.

In general, suppose $\{W_r\}_{r \in R}$ is an indexing of the set of components in a K-component decomposition of the nonrecursive r.e. set Σ into r.e. subsets ($2 \leqq K \leqq \aleph_0$), where R is understood to be r.e. in case $K = \aleph_0$. Then, we shall say that the decomposition in question is a *$CFY(K)$-decomposition* just in case, for any such index set R, $r \in R \Rightarrow (\forall_j)(W_j \subseteqq \bar{W}_r \Rightarrow W_j - \Sigma$ is r.e.).

Suppose that, in fact, there is a partial recursive function p such that $r \in R \Rightarrow (\forall_j)(W_j \subseteqq \bar{W}_r \Rightarrow p(\mathrm{r}, j)$ is defined and $W_{p(r,j)} = (W_j - \Sigma) \cup$ (a finite subset of Σ). We shall, under these circumstances, say that the $CFY(K)$-decomposition of Σ whose components are given by the set $\{W_r\}_{r \in R}$ is a *strong* $CFY(K)$-decomposition of Σ.

The fundamental observation of Cleave and Yates is then just this:

LEMMA 4. *Let Σ be a nonrecursive r.e. set, and suppose $2 \leqq K \leqq \aleph_0$. Then Σ admits a strong $CFY(K)$-decomposition.*

The next two lemmas express simple but useful properties of $CFY(K)$-decompositions.

LEMMA 5. *Let Σ be an r.e., nonrecursive set, and W_j a component in a $CFY(K)$-decomposition of Σ. Then W_j is not simple in any r.e. set.*

Proof. Suppose, to the contrary, that $W_j \subseteqq W_k$, where $W_k - W_j$ is immune. The union of the components other than W_j is an r.e. set, say W_e; hence, since $W_k - W_j$ is immune and $W_j \cap W_e = \phi$, we have $W_e \cap W_k =$ a finite set. Therefore, $W_k - (W_e \cap W_k)$ is r.e., includes W_j, and misses W_e (and hence misses each of the components going to make up W_e). Thus, $(W_k - (W_e \cap W_k)) - W_j$ must be an r.e. set. But here is an absurdity, since $(W_k - (W_e \cap W_k)) - W_j$ must be immune. The lemma follows.

LEMMA 6. *Let Δ be either a creative set or a nonpseudosimple mesoic set. Let Δ_1, Δ_2 be the components in a $CFY(2)$-decomposition* *of Δ. Then, at least one of Δ_1, Δ_2 has the property of being neither pseudosimple nor many-one reducible to a simple set.*

Proof. Let Σ_1, Σ_2 be pseudosimple mesoic sets. Suppose $\Sigma_1 \cup \Sigma_2$ is neither simple nor recursive. Let Σ_1', Σ_2' be r.e. sets such that $\Sigma_1' \subseteqq \bar{\Sigma}_1$, $\Sigma_2' \subseteqq \bar{\Sigma}_2$, and $\Sigma_1 \cup \Sigma_1'$, $\Sigma_2 \cup \Sigma_2'$ are simple. Now, $(\Sigma_1 \cup \Sigma_1') \cap (\Sigma_2 \cup \Sigma_2')$ is simple, and is a subset of $(\Sigma_1 \cup \Sigma_2) \cup (\Sigma_1' \cap \Sigma_2')$. Hence, since $\Sigma_1 \cup \Sigma_2$ is not recursive, $\overline{(\Sigma_1 \cup \Sigma_2) \cup (\Sigma_1' \cap \Sigma_2')}$ is infinite and so $(\Sigma_1 \cup \Sigma_2) \cup (\Sigma_1' \cap \Sigma_2')$ is simple. Therefore $\Sigma_1 \cup \Sigma_2$ is pseudosimple. Thus, we see that either Δ_1 or Δ_2 must be nonpseudosimple. It is an evident feature of $CFY(K)$-decompositions that the components are pairwise recursively inseparable; and from this it follows that neither Δ_1 nor Δ_2 can be many-one reducible to a simple set. The lemma follows. (We will see later, in Theorem 6, that Δ_1, Δ_2 must be mesoic when Δ is creative, as well as when Δ is noncreative.)

Recall that W_i, W_j are termed *effectively inseparable* just in case $W_i \cap W_j = \phi$ and there is a partial recursive function p such that, for all k and m, if $W_i \subseteqq W_k$, $W_j \subseteqq W_m$, and $W_k \cap W_m = \phi$, then $p(k, m)$ is defined and lies outside $W_k \cup W_m$. In [1], Cleave considers the following two sequential variations on this concept:

Let $\{W_{r(i)}\}$ be a recursive sequence (i.e., the indexing function r is recursive) of pairwise-disjoint, nonrecursive r.e. sets. $\{W_{r(i)}\}$ is *e.i.* (Cleave) just in case, for $i \neq j$, there is a partial recursive function $P_{i,j}$ such that, if $W_{r(i)} \subseteqq W_k$, $W_{r(j)} \subseteqq W_m$, and $W_k \cap W_m = \phi$, then $P_{i,j}(k, m)$ exists and lies outside $W_k \cup W_m \cup (\cup_n W_{r(n)})$. Again, Cleave calls $\{W_{r(i)}\}$ *almost e.i.* just in case, whenever $i \neq j$, there is a partial recursive function $p_{i,j}$ such that, if $W_{r(i)} \subseteqq W_k$, $W_{r(j)} \subseteqq W_m$, and $W_k \cap W_m = \phi$, then $p_{i,j}(k, m)$ is defined and $W_{p_{i,j}(k,m)}$ is an infinite recursive set whose intersection with $W_k \cup W_m \cup (\cup_n W_{r(n)})$ is finite. Cleave shows, in [1], that the $CFY(\aleph_0)$-decomposition of the creative set $\{x \mid x \in W_x\}$ given by the extended Friedberg construction presents an almost e.i. sequence; his argument, in fact, is valid for *any strong* $CFY(\aleph_0)$-decomposition of a creative set[1]. He then asks:

(1) Do there exist almost e.i., non-e.i. sequences?

(2) Must the union of the terms of an almost e.i. sequence be creative?

In § 3 we shall provide pleasantly straight-forward proofs that the answers to these two questions are, respectively, "yes" and "no".

One other concept, of Muchnik's ([7]), will receive a little of our attention in § 3: the notion of "sets-of-a-pair in an r.e. set". We

[1]This proof of Cleave's, showing that any strong $CFY(\aleph_0)$-decomposition of a creative set presents an almost e.i. sequence, will, presumably, appear in the paper corresponding to [2].

rephrase Muchnik's original definition as follows: Disjoint r.e. sets Δ_0, Δ_1 are said to be *sets-of-a-pair in the r.e. set* Σ just in case $\Delta_0 \cup \Delta_1 \subseteqq \Sigma$, $\Sigma - (\Delta_0 \cup \Delta_1)$ is infinite, Σ is indeed r.e., and, for all i and for $j = 0, 1$, $\Delta_j \subseteqq W_i \subseteqq \Sigma \Rightarrow [(W_i - \Delta_j$ is finite) or $(W_i \cap \Delta_{1-j} \neq \phi)]$. We shall say that an r.e. set Δ is *SOPRE* just in case there exist two other r.e. sets, Σ_1 and Σ_2, such that Δ, Σ_1 are sets-of-a-pair in the r.e. set Σ_2.

REMARK. It is not hard to show that any creative set is SOPRE. In Theorem 9 we will put forward an additional bit of information on SOPREness.

One further lemma will prove handy in § 3.

LEMMA 7. *The question whether a mesoic set can be coimmune in a universal set reduces to the question whether a mesoic set can be simple in a creative set.*

Proof. Suppose Δ is universal, Σ mesoic, and Σ is coimmune in Δ. Let Σ_1, Σ_2 be an effectively inseparable pair of disjoint creative sets. Since Δ is universal, there is a one-to-one recursive function f such that $f(\Sigma_1) \subseteqq \Delta$, $f(\Sigma_2) \subseteqq \bar{\Delta}$. Now, $f(\Sigma_1)$, $f(\Sigma_2)$ are themselves effectively inseparable ([11, Chapter 5, *Proposition* 4]). Hence, since $\Sigma \subseteqq \overline{f(\Sigma_2)}$, the sets $f(\Sigma_1) \cup \Sigma$, $f(\Sigma_2)$ are effectively inseparable. Therefore, $f(\Sigma_1) \cup \Sigma$ is creative. Hence $(f(\Sigma_1) \cup \Sigma) - \Sigma$ must be infinite; and so Σ is simple in the creative set $f(\Sigma_1) \cup \Sigma$, proving the lemma.

3. Theorems.

THEOREM 1. *A universal set cannot be coimmune in a mesoic set.*

Proof. Suppose that Δ is universal, Σ mesoic, $\Delta \subseteqq \Sigma$, and $\Sigma - \Delta$ is immune. Let Σ_1, Σ_2 be disjoint, effectively inseparable r.e. sets. Let f be a one-to-one recursive function reducing Σ_1 to Δ. Then $f(\Sigma_2) \subseteqq \bar{\Delta}$; hence, since $\Sigma - \Delta$ is immune, $f(\Sigma_2) \cap \Sigma$ must be finite. Therefore, $\Sigma - (f(\Sigma_2) \cap \Sigma) = \Sigma_3$ is a mesoic superset of $f(\Sigma_1)$ which is disjoint from $f(\Sigma_2)$. *B*ut $f(\Sigma_1)$, $f(\Sigma_2)$ are effectively inseparable; and so also Σ_3, $f(\Sigma_2)$ are effectively inseparable. But this is impossible, since Σ_3 is mesoic. The theorem follows.

THEOREM 2. *If Δ is a universal set, then there are two non-universal sets Σ_1 and Σ_2 such that Σ_1 is cohyperhyperimmune in Δ and Δ is cohyperhyperimmune in Σ_2.*

Proof. Applying Lemma 1, let Δ_1 be a hyperhyperimmune set whose complement is likewise hyperhyperimmune. Since both a universal set and its complement have infinite r.e. subsets, the sets $\varDelta \cap \varDelta_1$, $\bar{\varDelta} \cap \varDelta_1$ must be infinite and therefore immune (indeed, hyperhyperimmune); and we have, clearly, $\Sigma_1 = \varDelta \cap \varDelta_1$ cohyperhyperimmune in $\varDelta$ cohyperhyperimmune in $\Sigma_2 = \varDelta \cup (\bar{\varDelta} \cap \varDelta_1)$. It remains to see that $\bar{\Sigma}_1$, $\bar{\Sigma}_2$ are not productive. Now, $\bar{\Sigma}_2 = \bar{\varDelta}_1 \cap \bar{\varDelta}$ cannot be productive, since it is immune. If $\bar{\Sigma}_1$ were productive, it would be contraproductive (Myhill); hence, since a contraproductive set has a nonimmune complement, $\bar{\Sigma}_1$ is not productive, and the proof is complete.

THEOREM 3. (i) *A pseudosimple set cannot be coimmune in a universal set.*

(ii) *If $\varDelta$ is a mesoic set such that $\varDelta \leqq_{m-1} \Sigma$ for some simple set Σ, then $\varDelta$ cannot be coimmune in a universal set.*

(iii) *There are mesoic sets $\varDelta$, neither pseudosimple nor many-one reducible to a simple set, such that $\varDelta$ is not coimmune in any universal set.*

Proof. It follows from Lemma 7 that we need only prove (i), (ii), and (iii) with 'universal' replaced by 'creative'. Then (i) becomes evident, since a simple set cannot have a creative superset; (ii) is an easy consequence of the (easily proved) *Theorem* 5 of [7] together with the fact that any creative set is recursively inseparable from some r.e. subset of its complement; and (iii) results at once from Lemmas 5 and 6.

THEOREM 4. *If an r.e. set $\varDelta$ is hyperhypersimple in Σ, where Σ is creative, then $\varDelta$ must also be creative.*

Proof. It was pointed out by Yates, in [12], that an r.e. set $\varDelta$, with infinite complement, is hyperhypersimple if and only if there is no recursive sequence $\{W_{r(i)}\}$, of pairwise-disjoint r.e. sets (*finite or infinite*), such that $W_{r(i)} \cap \bar{\varDelta} \neq \phi$ holds for all i. It readily follows from consideration of inverse images of r.e. sets under one-to-one recursive functions that, for r.e. sets $\varDelta$ and Σ, $\varDelta$ is hyperhypersimple in Σ if and only if $\varDelta \subseteqq \Sigma$, $\Sigma - \varDelta$ is infinite, and there is no recursive sequence $\{W_{r(i)}\}$ of pairwise-disjoint r.e. subsets (finite or infinite) of Σ such that $(\forall i)(W_{r(i)} \cap (\Sigma - \varDelta) \neq \phi)$. Now, it follows straightforwardly from Myhill's isomorphism theorem ([8]) that if Σ is creative, then there is a recursive sequence $\{W_{r(i)}\}$ of pairwise-disjoint *creative* sets such that $\Sigma = \bigcup_i W_{r(i)}$. Let $\{W_{r(i)}\}$ be such a sequence, relative to the given creative set Σ; and suppose $\varDelta$ is an r.e. set hyperhypersimple in Σ. It follows that there is at least one i such that $W_{r(i)} \cap (\Sigma - \varDelta) = \phi$; i.e., $W_{r(i)} \subseteqq \varDelta$. But then $\varDelta$ is the disjoint union of the r.e. sets $W_{r(i)}$ and $\varDelta \cap (\bigcup_{j \neq i} W_{r(j)})$; and hence, since $W_{r(i)}$ is creative, $\varDelta$ is creative.

THEOREM 5. *If an r.e. set Δ is strongly effectively simple in Σ, where Σ is creative, then Δ must also be creative.*

Proof. Applying Lemma 3, Let s be a recursive function such that, for all numbers i, $W_i \subseteqq \Sigma - \Delta \Rightarrow (\forall x)(x \in W_i \Rightarrow s(i) > x)$. (It is not really essential to our purposes to have a *total* function s, but the proof is just a bit less cumbersome if we do.) Now, there exist a recursive function r, and a *strictly increasing* recursive function q, such that, for all i, $W_{r(i)} = W_i \cap \Sigma$ and $W_{q(i)} = W_i - \{x \mid x < sr(i)\}$. Let p be productive for $\bar{\Sigma}$: by results of Myhill, we may assume p to be strictly increasing and recursive. Let h be a 2-place recursive function such that $W_{h(i,j)} = W_i$ and $h(i, j) > j$, for all i and j. Then, the function p^* defined by $p^*(x) = p(h(x, sr(x)))$ is productive for $\bar{\Sigma}$, and has the property that $p^*(i) > sr(i)$, for all i. Since p^* and q are strictly increasing, then, we see that $p^*q(i) > sr(i)$, for all i. We now claim, and the reader will easily check, that the function p^*q is productive for $\bar{\Delta}$. This completes the proof.

THEOREM 6. *Each component of a $CFY(K)$-decomposition of a creative set, $2 \leqq K \leqq \aleph_0$, is mesoic.*

Proof. Suppose, to the contrary, that Σ is a component in a $CFY(K)$-decomposition of a creative set Δ, and that Σ is creative. Now, it is easily verified that if f is a one-to-one recursive function generating Σ, and Σ_1 is a hyperhypersimple (strongly effectively simple) set, then $f(\Sigma_1)$ is hyperhypersimple (strongly effectively simple) in Σ. Hence, by Theorem 4 or Theorem 5, $f(\Sigma_1)$ is creative. It follows from the Myhill isomorphism theorem that Σ itself is simple in a creative set (consider a recursive permutation mapping $f(\Sigma_1)$ onto Σ). But, by Lemma 5, Σ cannot be simple in *any* r.e. set; and from this contradiction, the theorem follows.

THEOREM 7. *Let $\{W_{r(i)}\}$ be a recursive sequencing of the components of a strong $CFY(\aleph_0)$-decomposition of Σ, Σ a creative set. Then, the sequence $\{W_{r(i)}\}$ is almost e.i. but not e.i.*

Proof. $W_{r(0)}, W_{r(1)}, \cdots$ is an almost e.i. sequence by the result of Cleave ([1]) cited in §2. It is clear that if $W_{r(0)}, W_{r(1)}, \cdots$ were an e.i. sequence, then, for $i \neq j$, the terms $W_{r(i)}, W_{r(j)}$ would be effectively inseparable. But hence, $W_{r(i)}, W_{r(j)}$ would be creative; whereas, by Theorem 6, they must be mesoic. From this contradiction in the subjunctive mood, we conclude to Theorem 7.

THEOREM 8. *The union of the terms of an almost e.i. recursive sequence of pairwise-disjoint r.e. sets need not be creative. Indeed: if Σ is a creative set, then Σ is the disjoint union of two mesoic* *sets Δ_1, Δ_2, each of which is the union of the terms of such a sequence.*

Proof. Again, let $\{W_{r(i)}\}$ be a recursive indexing of the components of a strong $CFY(K)$-decomposition of Σ, so that the sequence $W_{r(0)}$, $W_{r(1)}, \cdots$ is almost e.i. Now, it is easy to see that each of the subsequences $W_{r(0)}, W_{r(2)}, W_{r(4)}, \cdots, W_{r(1)}, W_{r(3)}, W_{r(5)}, \cdots$ is likewise almost e.i. Since $\{W_{r(i)}\}$ is a $CFY(\aleph_0)$-decomposition of Σ, the pair of sets $\bigcup_{2|i} W_{r(i)}$, $\bigcup_{2\nmid i} W_{r(i)}$ are the components of a $CFY(2)$-decomposition of Σ. Hence, by Theorem 6, each of $\bigcup_{2|i} W_{r(i)}$, $\bigcup_{2\nmid i} W_{r(i)}$ is mesoic, and the theorem is proved.

THEOREM 9. *Suppose Δ_1, Δ_2 are sets-of-a-pair in the r.e. set Σ. If Δ_1 is creative, then Σ is creative. On the other hand, if Σ is creative, there exist two mesoic sets Δ_1, Δ_2 such that Δ_1, Δ_2 are sets-of-a-pair in Σ.*

Proof. For the first assertion: if Δ_1 is creative, so is $\Delta_1 \cup \Delta_2$. But $\Delta_1 \cup \Delta_2$ is simple in Σ. Hence, by Theorem 1, Σ must be creative. For the second assertion: Let Σ' be any r.e. set which is simple in Σ, and let Δ_1, Δ_2 be the components of any $CFY(2)$-decomposition of Σ'. It is then easily checked that Δ_1, Δ_2 are sets-of-a-pair in Σ; and, by Theorem 6, Δ_1 and Δ_2 must be mesoic.

REMARK. The first assertion of Theorem 9 extends and completes *Theorem* 8 of [7].

Notice that, in the proof of the second part of Theorem 9, we proceeded in such a way that at least one of Δ_1, Δ_2 must be nonpseudosimple; this follows from Theorem 3(*i*) and Lemma 6. It is not hard to insure that *both* Δ_1, Δ_2, be nonpseudosimple mesoic sets. For choose Σ' to be a creative set, and apply the following general result.

THEOREM 10. *Let Σ be a creative set, and Δ_1, Δ_2 two mesoic sets (not necessarily disjoint) such that $\Delta_1 \cup \Delta_2 = \Sigma$. Then neither Δ_1 nor Δ_2 can be pseudosimple.*

Proof. Suppose, to the contrary, that (say) Δ_1 is pseudosimple: let j be a number such that $W_j \subseteqq \bar{\Delta}_1$, $\Delta_1 \cup W_j = \Delta_3$ is simple. Let f be a one-to-one recursive function generating Δ_3. Now, creative sets intersect simple sets creatively ([3, *Theorem* T2.6(2)]); so, $\Sigma \cap \Delta_3 = \Delta_1 \cup (\Delta_2 \cap \Delta_3) =$ a creative set. Again, as is easy to verify, mesoic sets intersect simple sets mesoically; thus, $\Delta_2 \cap \Delta_3$ is mesoic. Hence, by [3, *Theorem* T2.6(2)] and the fact that removal of any recursive subset from a creative set leaves a creative residue, we see that $\Delta_2 \cap W_j$ is mesoic. Now, $f^{-1}(\Sigma \cap \Delta_3) =$ a creative set. This follows from

[3, *Theorem* $T2.6(1)$] and the (easily verified) fact that if $r(\Delta)$ is productive, r a 1-to-one recursive function, then also Δ is productive. But $f^{-1}(\Sigma \cap \Delta_3) = f^{-1}(\Delta_1) \cup f^{-1}(\Delta_2 \cap W_j)$. Furthermore, $f^{-1}(\Delta_1)$ is recursive, since its complement is $f^{-1}(W_j)$. Hence, $f^{-1}(\Delta_2 \cap W_j)$ must be creative. But therefore, since f is one-to-one recursive, $ff^{-1}(\Delta_2 \cap W_j) = \Delta_2 \cap W_j =$ a creative set: contradiction. The theorem follows.

REMARK. Theorem 10 can also be proved with the word 'pseudosimple' replaced by the words 'many-one reducible to a simple set'; however, the latter result does not interest us here.

The following two assertions, related to Theorem 3, may also be proven: (i) a mesoic set Δ which is many-one reducible to a *pseudosimple* set cannot be coimmune in a universal set; and (ii) if Δ is r.e. and is coimmune in a creative set, then Δ is almost effectively inseparable from some creative set.

REFERENCES

1. J. P. Cleave, *Infinite sequences of pairwise recursively inseparable, recursively enumerable sets* (unpublished) Southampton, February, 1963.
2. J. P. Cleave and C. E. M. Yates, *Sequences of pairwise recursively inseparable, recursively enumerable sets* (abstract), J. Symbolic logic **28** (1963), 268.
3. J. C. E. Dekker, *Productive sets,* Trans. Amer. Math. Soc. **78** (1955), 129-149.
4. J. C. E. Dekker and J. R. Myhill, *Some theorems on classes of recursively enumerable sets,* Trans. Amer. Math. Soc. **89** (1958), 25-59.
5. R. Friedberg, *Three theorems on recursive enumeration,* J. Symbolic Logic **23** (1958), 309-316.
6. T. G. McLaughlin, *On an extension of a theorem of Friedberg,* Notre Dame J. of Formal Logic, III, No. **4** (1962), 270-273.
7. A. A. Muchnik, *On separability of recursively enumerable sets* (Russian), Dokl. Akad. Nauk SSSR **109** (1956), 29-32.
8. J. R. Myhill, *Creative sets,* Zeitschrift f. math. Logik und Grundl. der. Math. **1** (1955), 97-108.
9. E. L. Post., *Recursively enumerable sets of positive integers and their decision problems,* Bull. Amer. Math. Soc. **50** (1944), 284-316.
10. Hartley Rogers, *Theory of Recursive functions and effective computability,* McGraw-Hill, forthcoming.
11. R. M. Smullyan, *Theory of formal systems,* Princeton University Press, 1961 and 1962.
12. C. E. M. Yates, *Recursively enumerable sets and retracing functions,* Zeitschrift f. Math. Logik und Grundl. der Math. **8** (1962), 331-345.