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GENERALIZED CHARACTER SEMIGROUPS: THE SCHWARZ DECOMPOSITION

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The author's résumé: A structure theorem due to Š. Schwarz asserts that if S is a finite abelian or a compact abelian semigroup admitting relative inverses, than the character semigroup of S is decomposed into a disjoint union of character groups of certain maximal subgroups of S. In this note, among other things, we generalize this Schwarz Decomposition Theorem to a broader class of semigroups, the so-called pseudo-invertible semigroups. We also relax the range of the characters from the semigroup of complex numbers to a more general semigroup.

For notations and terms not defined here see A. D. Wallace [11].

Throughout this paper, let S be always a compact commutative semigroup, unless otherwise stated. By a *character* of S is meant a continuous homomorphism of S into the multiplicative semigroup C of the complex numbers endowed with the usual Euclidean topology. The collection of all characters of S, with the value-wise multiplication of functions, endowed with the compact-open topology, forms a semigroup which will be denoted by $(S, C)^\wedge$ or simply $S^\wedge$, and will be called the *character semigroup* of S. Hewitt and Zuckerman [4] use the term *semicharacter*, in the discrete case, for not identically zero characters. Here we use $(\hat{S}, C)$ or simply $\hat{S}$, as distinguished from $S^\wedge$, to denote the collection of semicharacters of S. We note that $\hat{S}$, in general, need not be a semigroup. We first draw attention to the fact that if χ is a character of S, then $|\chi(x)| \leqq 1$ for every x in S. For, otherwise $\chi(S)$ would not be compact. Thus, in the study of characters, only the unit disc $\{z: |z| \leqq 1\}$ of the complex numbers is used. Let us write D for this unit disc. The set D itself forms an important semigroup which is compact, connected, commutative, cancellable,[1] has zero 0 and unit 1; moreover the circumference $\{z: |z| = 1\}$ of D is the maximal subgroup $H(1)$ and $D\backslash H(1)$ is an ideal. However, only some of these are needed as we shall see below.

Throughout the rest of this paper, let T be an arbitrary, but fixed, compact commutative cancellable semigroup with zero z and unit u[2] such that $T\backslash H(u)$ is a subsemigroup of T. By a *generalized character*

[1] A semigroup S is cancellable if and only if for any nonzero elements a, b, c in S such that $ab = ac$ or $ba = ca$, then $b = c$.

[2] It is to be understood that $z \neq u$.

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of S is meant a continuous homomorphism of S to T. As in the case of character semigroup, the collection $(S, T)^{\wedge}$ of all generalized characters of S, with value-wise multiplication of mappings and the compact-open topology, forms a commutative (topological) semigroup which will be called the *generalized character semigroup* of S. We write $(\hat{S}, T)$ for the collection of all not identically zero elements in $(S, T)^{\wedge}$. It is quite easy to see that if S is a group, then $(\hat{S}, T) = (\hat{S}, H(u))$ and $(\hat{S}, T)$ is a group.

THEOREM 1. *If S is discrete, then $(S, T)^{\wedge}$ is compact.*

Proof. Since S is discrete, the compact-open topology on $(S, T)^{\wedge}$ is the relative topology, on the set (not topologized) $(S, T)^{\wedge}$, of the Tychonoff product topology on the product $P\{T : s \in S\}$, which is compact by the Tychonoff theorem. The compactness of $(S, T)^{\wedge}$ now follows from the fact that $(S, T)^{\wedge}$ is a closed subset of $P\{T : s \in S\}$.

DEFINITION 1. For any χ in $(S, T)^{\wedge}$, the *support* of χ, $sp(\chi)$, is the set $\{x : x \in S, \chi(s) \neq z\}$.

We have immediately $sp(\chi_1 \cdot \chi_2) = sp(\chi_1) \cap sp(\chi_2)$ for any χ_1, χ_2 in $(S, T)^{\wedge}$. It is clear that if χ in $(S, T)^{\wedge}$ is not identically zero, $sp(\chi)$ is an open subsemigroup of S; such an open subsemigroup will be called a *supporting subsemigroup*. Since the support of the zero generalized character is the void set $\square$, as a convenience we also call $\square$ a supporting subsemigroup. *We write henceforth, $\mathscr{P}(S)$ or $\mathscr{P}$ for the collection of all supporting subsemigronps of S.*

DEFINITION 2 [8]. The *Rees partial-ordering* $\leqq$ on the set E of idempotents in S is the subset $\{(e_1, e_2) : (e_1, e_2) \in E \times E, e_1 e_2 = e_1\}$. If (e_1, e_2) is in $\leqq$ we write, equivalently, $e_1 \leqq e_2$.

LEMMA 0 [13]. *Let $(X, \leqq)$ be a nonvoid compact topological space endowed with a quasi-ordering $\leqq$ such that for each t in X the set $\{x : x \in X, x \leqq t\}$ is closed in X. Then $(X, \leqq)$ has a minimal element.*

Proof. See Ward [13, Theorem 1].

LEMMA 1. *If S_0 is a compact subsemigroup of S, then $E(S_0)$ has a unique minimal element with respect to the Rees partial-ordering $\leqq$.*

Proof. Since S_0 is a compact semigroup, the set $E(S_0)$ of all idempotents in S_0 is a nonvoid closed subset of S_0. It is fairly easy to see that $\{x: x \in E(S_0), x \leqq t\}$ is closed for each t in $E(S_0)$. It then follows from Lemma 0 that $(E(S_0), \leqq)$ has a minimal element, say e_0. If there were another minimal element e_1 in $E(S_0)$, then one concludes $e_1 = e_0e_1 = e_0$. This proves the uniqueness.

DEFINITION 3. If $P \in \mathscr{P}(S)$, then $\sigma(P)$ is the set

$$\{\chi : \chi \in (S, T)^\wedge, sp(\chi) = P\} .$$

For each P in $\mathscr{P}(S), \sigma(P)$ is easily seen to be a cancellable subsemigroup of $(S, T)^\wedge$. In general, $(S, T)^\wedge$ may be decomposed into the union of the disjoint family $\{\sigma(P): P \in \mathscr{P}(S)\}$ of cancellable subsemigroups of $(S, T)^\wedge$.

DEFINITION 4 [1]. *A semigroup S admitting relative inverses* is a semigroup such that to each x in S there is a pair (e, x') in $E \times S$ such that $xe = x = ex$ and $xx' = e = x'x$.

A well-known result of A. H. Clifford [1] says that a semigroup is a semigroup admitting relative inverses if and only if it is the disjoint union of its maximal subgroups. The more general class of semigroups that we are interested in is the following.

DEFINITION 5 [3]. A semigroup S is *pseudo-invertible* if and only if, to each element x in S there is an $\bar{x}$ in S such that
 (i) $x\bar{x} = \bar{x}$;
 (ii) $\bar{x}x^{n+1} = x^n$ for some positive integer n, and
 (iii) $\bar{x}^2x = \bar{x}$.

The element $\bar{x}$ satisfying conditions (i), (ii) and (iii) above turns out to be unique if it exists [3], in which case it is called the *pseudo-inverse* of x. A semigroup S is pseudo-invertible if and only if, to every x in S there is an integer $n > 0$ such that x^n is in some subgroup of S [3], [5], [6]. From this, one sees that the class of pseudo-invertible semigroups includes all semigroups admitting relative inverses, all periodic semigroups; all semigroups of matrices; all finite dimensional affine semigroups (for definition of an affine semigroup, see [2]) and many others.

LEMMA 2. *Let S be a compact commutative pseudo-invertible semigroup. Then each supporting subsemigroup P of S is open and closed. Therefore, if $P \neq \square$ then P has a unique minimal idempotent e_P with respect to the Rees partial-ordering on $E(P)$.*

Proof. To show each P in $\mathscr{P}$ is open and closed, we may consider only $P \neq \square$. For any nonvoid P in $\mathscr{P}$, there is a χ in $(S, T)^{\wedge}$ such that $P = \{x : x \in S, \chi(x) \neq z\}$. We show, for pseudo-invertible S, also, $P = \{x : x \in S, \chi(x) \in H(u)\}$ and consequently P is open and closed. To this end, let x be an arbitrary element of the nonvoid set $P = \{x : x \in S, \chi(x) \neq z\}$. Then since S is pseudo-invertible, there is a positive integer n such that $x^n \in H(e)$ for some e in E; thus, since $E(T) = \{z, u\}$, we must have $\chi(x)^n = \chi(x^n) \in H(u)$. Consequently, since $T \backslash H(u)$ is a subsemigroup, we obtain $\chi(x) \in H(u)$. This proves

$$\{x : x \in S, \chi(x) \neq z\} = P = \{x : x \in S, \chi(x) \in H(u)\} ,$$

so that P is open and closed

The set P being a closed subsemigroup of the compact semigroup S, by Lemma 1, $E(P)$ has a unique minimal element e_P with respect to the Rees partial ordering.

In the following, for $Q \subset S$, e_Q will be the least idempotent in Q if it exists.

THEOREM 2. *Let S be a compact commutative pseudo-invertible semigroup. Then the generalized character semigroup $(S, T)^{\wedge}$ of S may be decomposed into the union of the disjoint family $\{(\widehat{H(e_P)}, T) : P \in \mathscr{P}\}$ of groups, where we agree that $(\hat{H}_{\square}, T) = \{0\}$.*

The proof of this theorem is contained in the following two lemmas. It should be noted that when $T \subset C$, $(\widehat{H(e)}, T)$ is the familiar character group $\widehat{H(e)}$.

LEMMA 3. *Under the hypothesis of Theorem 2, for any nonvoid P in $\mathscr{P}(S)$, the maximal subgroup $H(e_P)$ of S is the kernel of P; and the mapping $r_P : P \rightarrow H(e_P)$ which takes every x in P to xe_P is a (continuous) retraction of P onto $H(e_P)$.*

Proof. Let x be an arbitrary element of S. Let $\Gamma(x) = \{x^n : n \geqq 1\}^-$ and $N(x) = \cap \{x^n \Gamma(x) : n \geqq 1\}$, then $N(x)$ is the kernel of $\Gamma(x)$ as well as, since S is compact, a closed subgroup of S. Thus $N(x)$ contains a unique idempotent which is designated simply by e_x instead of the rather complicated symbol $e_{N(x)}$. If x is in P we have $e_x e_P = e_P$ and hence the unique idempotent in $N(xe_P)$ is e_P. Therefore, $xe_P = (xe_P)e_P \in N(xe_P) \subset H(e_P)$ for all x in P. To show $H(e_P)$ is an ideal of P, we first show that $H(e_P) \subset P$. This is true since there is a χ_P in $(S, T)^{\wedge}$ such that an element x of S is in P if and only if $\chi_P(x) \neq z$;

so $\chi_P(e_P) \neq z$ (consequently $\chi_P(e_P) = u$) and thus $\chi_P(x) \neq z$ for all x in $H(e_P)$. Now $PH(e_P) = PH(e_P)e_P \subset Pe_P \subset H(e_P)$ shows that $H(e_P)$ is an ideal of P; since it is a group it must be the minimal ideal of P. The fact that r_P: $P \rightarrow H(e_P)$ is a continuous retraction onto is then evident.

LEMMA 4. *Under the hypothesis of Theorem 2, for any nonvoid P in $\mathscr{P}(S)$, $\sigma(P) = \{\chi : \chi \in (S, T)^\wedge, sp(\chi) = P\}$ is isomorphic to the generalized character group $(\widehat{H(e_P)}, T)$ of the maximal subgroup $H(e_P)$ of S. In particular, if $T \subset C$, $\sigma(P)$ is also homeomorphic with the character group $\widehat{H(e_P)}$ of $H(e_P)$.*

Proof. Let h: $\sigma(P) \rightarrow (\widehat{H(e_P)}, T)$ be the mapping which takes each χ in $\sigma(P)$ to $\chi | H(e_P)$. Clearly $\chi \in \sigma(P)$ implies $\chi | H(e_P) \in (\widehat{H(e_P)}, T)$. We have $h(\chi_1 \cdot \chi_2) = (\chi_1 \cdot \chi_2) | H(e_P) = (\chi_1 | H(e_P)) \cdot (\chi_2 | H(e_P)) = h(\chi_1) \cdot h(\chi_2)$ so that h is a homomorphism. To show h is an isomorphism, we show each φ in $(\widehat{H(e_P)}, T)$ may be extended, uniquely, to a χ in $\sigma(P)$. To this end, we define, for φ in $(\widehat{H(e_P)}, T)$.

$$\chi = \begin{cases} z, & \text{on } S \backslash P\,, \\ \varphi \circ r_p & \text{on } P\,. \end{cases}$$

This is continuous because r_P is continuous and P is open and closed. A routine verification shows that χ is an extension of φ to an element in $\sigma(P)$. Such an extension is unique as we shall now see. If χ' is any element in $\sigma(P)$ with $\chi' | H(e_P) = \varphi$, then $\chi'(x) = z = \chi(x)$ for all x in $S \backslash P$ and $\chi'(x) = \chi'(x) \cdot u = \chi'(x) \cdot \chi'(e_P) = \chi'(xe_P) = \varphi(xe_P) = \varphi \circ r_{p\ P}(x) = \chi(x)$, for all x in P. Therefore, h is an isomorphism of $\sigma(P)$ onto $(\widehat{H(e_P)}, T)$. It remains to show, in the case $T \subset C$, that h: $\sigma(P) \rightarrow \widehat{H(e_P)}$ is also a homeomorphism. This follows from the fact that h is one-to-one, continuous and that $\widehat{H(e_P)}$ is discrete [7].

COROLLARY. *If S satisfies the hypotheses of Theorem 2, and if S is connected, then $\mathscr{P} = \{S, \Box\}$ and hence $(S, T)^\wedge = (\widehat{H(e_s)}, T) \cup \{0\} = (H(e_s), T)^\wedge$.*

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