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ALMOST INVARIANT MEASURES

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Let μ be a regular complex-valued Borel measure on a locally compact topological (LC) group G which is finite on compact sets; and for each $s \in G$ define the measure $T_s\mu$ by $T_s\mu(E) = \mu(E + s)$, $E \in B_c(G)$ the collection of all Borel subsets of G with compact closure. If f is a function on G then for each $s \in G$ we set $T_sf(t) = f(t + s)$, $t \in G$. Let X be a translation invariant subspace of $C_0(G)$, the space of continuous complex-valued functions on G which vanish at infinity, i.e., a subspace such that $f \in X$ implies $T_{-s}f \in X$, $s \in G$; and let U be an open symmetric neighborhood of zero in G. Then we shall say μ *acts U-almost invariantly* on X if $\int_G | h(t) | d | \mu | (t) < \infty$, $h \in X$, and

$$\int_G h(t) dT_s\mu(t) = \sum_{i=1}^{n} \alpha_i(s) \int_G h(t) dT_{s_i}\mu(t) \qquad (s \in U,\ h \in X),$$

where $s_1, s_2, \cdots, s_n$ are fixed elements of U. We shall say μ is a *U-almost invariant measure* on G if $\{T_s\mu \mid s \in U\}$ spans a finite dimensional space of measures. When $U = G$ we shall say μ *acts almost invariantly* and μ is an *almost invariant measure*, respectively. The main results of this paper show that if μ acts U-almost invariantly on X then there exists some continuous function f such that

$$\int_G h(t) d\mu(t) = \int_G h(t) f(t) dm(t)\,, \qquad h \in X,$$

where dm is right invariant Haar measure on G; and that μ is a U-almost invariant measure if and only if there exists a continuous f such that $d\mu(t) = f(t)dm(t)$ and $\{T_sf \mid s \in U\}$ spans a finite dimensional space of functions.

We shall also establish the equivalence for connected groups of the two notions of acting almost invariantly and of the two notions of almost invariance, and shall say something about the uniqueness of measures which act U-almost invariantly.

We shall denote by $V(G)$ the linear space of all regular complex valued Borel measures on a LC group G, and by $C_c(G)$ the subspace of $C_0(G)$ consisting of those functions with compact support. Throughout the paper we shall use m and dm to denote right invariant Haar measure on the LC group G, i.e., $m(E + s) = m(E)$.

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REMARKS. (a) The concept of a measure which acts U-almost invariantly is o generalization of the notion of a measure acting invariantly, i.e. a measure μ such that $\int_G h(t)dT_s\mu(t) = \int_G h(t)d\mu(t)$, $s \in G$, $h \in X$. For abelian LC groups measures which act invariantly were considered in [1].

(b) Since, in general, we shall consider nonabelian groups it would perhaps be better to speak of measures which "act right U-almost invariantly" or are "right U-almost invariant". However, in the interests of notational simplicity we choose the terminology given above. It is easy to see that a similar development can be made using left invariant Haar measure, $T_s(E) = \mu(s + E)$ and $T_sf(t) = f(s + t)$.

(c) The restriction in the definitions that U be an open symmetric neighborhood of zero in G is mainly one of convenience. Indeed, it is not difficult to see that if W is a Borel subset of G with finite positive Haar measure for which

$$\int_G h(t)dT_s\mu(t) = \sum_{i=1}^{n} \alpha_i(s) \int_G h(t)dT_{s_i}\mu(t) \qquad (s \in W,\ h \in X)$$

then some left translate of $W + W$ contains an open symmetric neighborhood U of zero in G for which a similar relation holds. However, in the proofs which follow it is necessary that the Haar measure of U be positive.

2. Measures which act almost invariantly. For G a LC group and X a translation invariant subspace of $C_0(G)$ we shall denote by $L(X)$ the topological linear space of all linear complex-valued functionals on X with the topology given by pointwise convergence; i.e. a net of functionals $< F_\alpha > \subset L(X)$ converges to $F_0 \in L(X)$ if and only if $\lim F_\alpha(h) = F_0(h)$, $h \in X$. If $\mu \in V(G)$ acts U-almost invariantly on X, then for each $s \in G$ we define the functional $F_s \in L(X)$ by

$$F_s(h) = \int_G h(t)dT_s\mu(t) \qquad (h \in X)\,.$$

This notation for the functionals F_s will be used consistently in the remainder of the paper. It should be noted that the functionals F_s need not be continuous.

The main result of this section is the following theorem which is comparable to Theorem 2 in [1].

THEOREM 1. *Let G be a LC group, X a translation invariant subspace of $C_0(G)$, $\mu \in V(G)$ and U an open symmetric neighborhood of zero* *in G. If μ acts U-almost invariantly on X then there exists a continuous function f such that*

$$\int_G h(t)d\mu(t) = \int_G h(t)f(t)dm(t) \qquad (h \in X)\ .$$

PROOF. Since μ acts U-almost invariantly on X it is clear from the definition 1 that, without loss of generality, we may write

$$F_s = \sum_{i=1}^{n} \alpha_i(s)F_{s_i} \qquad (s \in U)$$

where $F_{s_1}, F_{s_2}, \cdots, F_{s_n}$, are assumed to form a linearly independent subset of $L(X)$. Let C be the subspace of $L(X)$ spanned by $F_{s_1}, F_{s_2}, \cdots, F_{s_n}$.

It is easy to verify that the mapping $\varphi: G \to L(X)$ defined by $\varphi(s) = F_s$, $s \in G$, is continuous; and hence the mapping $\psi = \varphi\mid_U$ is a continuous mapping on U to C in the relative topology inherited from $L(X)$. Thus, since C is a finite dimensional subspace of $L(X)$, the mapping ψ is also continuous if we put on C the topology given by the norm, $\| \sum_{i=1}^{n} b_i F_{s_i} \| = \sum_{i=1}^{n} | b_i |$.

Furthermore, in this norm topology it is clear that the projection mappings $P_k : C \to C$ defined by

$$P_k\Big(\sum_{i=1}^{n} b_i F_{s_i}\Big) = b_k F_{s_k}\ , \qquad k = 1, 2, \cdots, n$$

are also continuous.

But then from the continuity of the composite mappings $P_k \circ \psi$, $k = 1, 2, \cdots, n$, it is immediate that $\alpha_1, \alpha_2, \cdots, \alpha_n$ are continuous functions on U.

Let A be the set of all functions in $C_c(G)$ with support contained in U. For each $g \in A$ we define the linear functional $F_g \in L(X)$ by

$$F_g = \int_G g(s)F_s dm(s)\ .$$

This vector valued integral makes sense since the support of g lies in U and so

$$(1) \qquad F_g = \int_G g(s)\Big[\sum_{i=1}^{n} \alpha_i(s)F_{s_i}\Big]dm(s) = \sum_{i=1}^{n} \int_G g(s)\alpha_i(s)dm(s)F_{s_i}\ ,$$

where the coefficients in the last expression exist since $\alpha_1, \alpha_2, \cdots, \alpha_n$ are continuous on U.

Set $B = \{F_g \mid g \in A\}$. From (1) it is clear that $B \subset C$, and simple verification shows that B is linear space. Hence B is a closed subspace of C.

Let $< g_\beta > \subset A$ be a net of functions such that

(i) $g_\beta \geqq 0$, all β;

(ii) $\int_G g_\beta(t)dm(t) = 1$, all β;

(iii) for any open symmetric neighborhood W of zero in G there is a β_0 such that for $\beta \succ \beta_0$ the support of g_β is contained in W. (We shall call such a net of functions a *compact approximate identity.*) Then since $\alpha_1, \alpha_2, \cdots, \alpha_n$ are continuous on U, using (1), we obtain:

$$\lim_\beta F_{g_\beta} = \lim_\beta \sum_{i=1}^n \int_G g_\beta(s)\alpha_i(s)dm(s)F_{s_i}$$
$$= \sum_{i=1}^n \alpha_i(0)F_{s_i} = F_0 .$$

Therefore $F_0 \in B$, and so there exists some $k \in A$ such that $F_0 = F_k = \int_G k(s)F_s dm(s)$.

But then for each $h \in X$,

$$\begin{aligned}\int_G h(t)d\mu(t) &= F_0(h)\\ &= F_k(h)\\ &= \int_G k(s)F_s(h)dm(s)\\ &= \int_G k(s)\int_G h(t-s)d\mu(t)dm(s)\\ &= \int_G\int_G k(s+t)h(-s)dm(s)d\mu(t)\\ &= \int_G\int_G k(-s+t)h(s)\Delta(-s)dm(s)d\mu(t)\\ &= \int_G h(s)\Delta(-s)\int_G k(-s+t)d\mu(t)dm(s)\\ &= \int_G h(s)f(s)dm(s) ,\end{aligned}$$

where Δ is the modular function of G and f is the continuous function defined by

$$f(s) = \Delta(-s)\int_G k(-s+t)d\mu(t) .$$

The applications of Fubini's theorem are valid since it is clear that

$$\int_G |k(s)| \int_G |h(t-s)| d|\mu|(t)dm(s) < \infty .$$

This completes the proof of the theorem.

REMARKS. (a) Clearly the function f is, in general, not unique.

(b) For Euclidean groups R^m, $m > 0$, it is easy to see that we may choose f to be infinitely differentiable.

(c) One cannot conclude that a measure which acts U-almost invariantly is either a U-almost invariant measure or even absolutely continuous with respect to m. For example let G be any infinite compact abelian group, X the space spanned by any nonzero continuous character $(., \gamma)$, $U = G$ and μ the measure with unit mass concentrated at zero. Then μ is neither almost invariant nor absolutely continuous, but it does act almost invariantly on X.

The next theorem shows that for connected groups the two notions of acting almost invariantly are identical.

THEOREM 2. *Let G be a connected LC group, X a translation invariant subspace of $C_0(G)$, $\mu \in V(G)$ and U an open symmetric neighborhood of zero in G. Then the following are equivalent*:

(i) *μ acts almost invariantly on X.*

(ii) *μ acts U-almost invariantly on X.*

PROOF.[1] Clearly (i) implies (ii)

Now suppose μ acts U-almost invariantly on X. Then the space C spanned by $\{F_s \mid s \in U\}$ is a finite dimensional subspace of $L(X)$. Let $E = \{s \mid s \in G,\ F_s \in C\}$. Without loss of generality we may write $F_s = \sum_{i=1}^{n} \alpha_i(s) F_{s_i}$, $s \in E$; where $s_1, s_2, \cdots, s_n$ are fixed elements of U.

Clearly E is not empty as $U \subset E$. We shall show that E is both open and closed, and hence, since G is connected, $E = G$; i.e. μ acts almost invariantly on X.

It is immediate from the finite dimensionality of C and the continuity of the mapping $\varphi: s \rightarrow F_s$, cited in the proof of Theorem 1, that E is a closed subset of G.

On the other hand, let $s_0 \in E$. Since U is an open symmetric neighborhood of zero in G there is an open symmetric neighborhood W of zero such that $W + s_i \subset U$, $i = 1, 2, \cdots, n$. Then $W + s_0$ is a neighborhood of s_0, and for each $s + s_0 \in W + s_0$ we have:

$$\begin{aligned} F_{s+s_0}(h) &= \int_G h(t)\, dT_{s+s_0}\mu(t) \\ &= \int_G h(t-s)\, dT_{s_0}\mu(t) \\ &= \sum_{i=1}^{n} \alpha_i(s_0) \int_G h(t-s)\, dT_{s_i}\mu(t) \\ &= \sum_{i=1}^{n} \alpha_i(s_0) F_{s+s_i}(h) \qquad (h \in X)\ . \end{aligned}$$

[1] The author is indebted to J. Lindenstrauss for suggesting the simple proof given here.

But $s+s_i \in W+s_i \subset U$, $i=1, 2, \cdots, n$; and so $F_{s+s_i} \in C$; $i=1, 2, \cdots, n$.

Thus for each $s+s_0 \in W+s_0$, we see that $F_{s+s_0} \in C$; and consequently E is open.

3. Almost invariant measures. Theorem 1 provides us almost immediately with a necessary and sufficient condition for a measure to be U-almost invariant.

THEOREM 3. *Let G be a LC group, $\mu \in V(G)$ and U an open symmetric neighborhood of zero in G. Then the following are equivalent:*

(i) *μ is a U-almost invariant measure on G.*

(ii) *There is a continuous function f on G such that $d\mu(t) = f(t)dm(t)$ and $\{T_s f \mid s \in U\}$ spans a finite dimensional space of functions.*

Proof. Clearly (ii) implies (i). Suppose μ is a U-almost invariant measure. Then evidently μ acts U-almost invariantly on $X = C_c(G)$; and so by Theorem 1 there exists a continuous function f on G such that

$$\int_G h(t)d\mu(t) = \int_G h(t)f(t)dm(t) \qquad (h \in C_c(G)) .$$

Consequently, from the regularity of μ it is easy to deduce that $d\mu(t) = f(t)dm(t)$ and that $\{T_s f \mid s \in U\}$ spans a finite dimensional space of functions; and this completes the proof.

Given a topological group G, let $FDT(G)$ be the space of all continuous complex-valued functions f on G such that $\{T_s f \mid s \in G\}$ spans a finite dimensional space. As an immediate consequence of Theorem 3 we have the following theorem on almost invariant measures.

THEOREM 4. *Let G be a LC group and $\mu \in V(G)$. Then the following are equivalent:*

(i) *μ is an almost invariant measure on G.*

(ii) *There is an $f \in FDT(G)$ such that $d\mu(t) = f(t)dm(t)$.*

REMARKS. (a) For U-almost invariant measures it is clear that the dimensions of the spaces spanned by $\{T_s\mu \mid s \in U\}$ and $\{T_s f \mid s \in U\}$ must be the same.

(b) If μ is almost invariant and $T_s\mu = \sum_{i=1}^n \alpha_i(s) T_{s_i}\mu$, $s \in G$, it can be shown that $\alpha_1, \alpha_2, \cdots, \alpha_n \subset FDT(G)$; and that we may write $f = \sum_{i=1}^n f(s_i)\alpha_i$.

(c) In general for U-almost invariant measures the function f given by Theorem 3 need not belong to $FDT(G)$. For example, let $G = Z$, the additive group of the integers; $U = \{0\}$, and let μ be the measure with unit mass concentrated at zero. Then $f(0) = 1$, $f(t) = 0$, $t \neq 0$; and $f \notin FDT(Z)$.

(d) For a topological group G, let $D(G)$ be the space of all linear combinations of products of continuous complex-valued functions on G which are either additive or multiplicative; i.e. functions f such that either $f(s+t) = f(s) + f(t)$ or $f(s+t) = f(s)f(t)$. If G is an abelian topological group it is known that $FDT(G) = D(G)$ [2, p. 25]. Thus if G is a LCA group we can conclude that the function f of Theorem 4 belongs to $D(G)$.

(e) If $G = R^m$, $m > 0$, then the preceding remark implies that each almost invariant measure μ must be of the form

$$d\mu(t) = \sum_{j=1}^{l} P_j(t) \exp(b_j, t) dm(t) ,$$

where P_j are arbitrary polynomials with complex coefficients, $j = 1, 2, \cdots, l$; b_j are m-vectors of complex numbers, $j = 1, 2, \cdots, l$ and $t = (x_1, x_2, \cdots, x_m)$.

An immediate corollary to Theorem 4 is the following:

COROLLARY. *Let G be a LC group; $\mu \in V(G)$, $\mu \neq 0$, μ singular with respect to right invariant Haar measure. Then for each Borel set W in G with finite positive Haar measure, $\{T_s\mu \mid s \in W\}$ spans an infinite dimensional subspace of $V(G)$.*

Proof. Suppose the contrary, i.e. there exists a Borel set W of finite positive Haar measure for which $\{T_s\mu \mid s \in W\}$ spans a finite dimensional subspace of $V(G)$. Then from a remark of section one there exists an open symmetric neighborhood U of zero in G such that $\{T_s\mu \mid s \in U\}$ also spans a finite dimensional subspace of $V(G)$.

Thus, by Theorem 3, μ would be absolutely continuous with respect to Haar measure, and hence zero; contrary to the hypotheses of the corollary.

Considering measures $\mu \in V(G)$ as acting on the space $C_c(G)$, Theorem 2 implies that for connected LC groups the notions of almost invariant measures and U-almost invariant measures are equivalent. We state this result as Theorem 5.

THEOREM 5. *Let G be a connected LC group, $\mu \in V(G)$ and U an* *open symmetric neighborhood of zero in G. Then the following are equivalent*:

(i) μ *is an almost invariant measure on G.*

(ii) μ *is a U-almost invariant measure on G.*

4. Uniqueness theorems. As noted previously, a measure μ may act U-almost invarianty on a subspace X of $C_0(G)$ without being a U-almost invariant measure. The next two theorems provide conditions which insure that a measure which acts U-almost invariantly is a U-almost invariant measure. The first theorem is a generalization of Theorem 1 in [1], and its proof is patterned after that in [1].

THEOREM 6. *Let G be a LC group, X a dense translation invariant subalgebra of* $C_0(G)$, $\mu \in V(G)$ *and U an open symmetric neighborhood of zero in G. If* μ *acts U-almost invariantly on X then* μ *is a U-almost invariant measure.*

Proof. Without loss of generality we may assume that

$$\int_G h(t-s)d\mu(t) = \sum_{i=1}^{n} \alpha_i(s) \int_G h(t-s_i)d\mu(t) \qquad (s \in U, h \in X). \tag{2}$$

For each $f \in C_c(G)$, since X is dense in $C_0(G)$, there is a function $g \in X$ such that g vanishes at no point of the support of f. Let $k = f/g$. Clearly $k \in C_c(G)$. Again by the denseness of X there is a sequence $< g_m > \subset X$ which converges uniformly to k.

Then it is easy to verify that

$$\lim_m \int_G g_m(t-s)g(t-s)d\mu(t) = \int_G f(t-s)d\mu(t) \qquad (s \in U), \tag{3}$$

and that

$$\begin{aligned} \lim_m \sum_{i=1}^{n} \alpha_i(s) \int_G g_m(t-s_i)g(t-s_i)d\mu(t) \\ = \sum_{i=1}^{n} \alpha_i(s) \int_G f(t-s_i)d\mu(t) \qquad (s \in U). \end{aligned} \tag{4}$$

But $< g_m g > \subset X$ as X is a subalgebra, and hence from (2) the left hand sides of (3) and (4), and thus the right hand sides, are equal.

Since this holds for each $f \in C_c(G)$ we conclude from the regularity of μ that μ is U-almost invariant.

If the group G is compact then the functionals F_s are bounded, and it is easy to see that in this case the preceding theorem remains true if we only require that X be a dense translation invariant subspace. This leads us to search for conditions on X other than the ones that it be a dense subalgebra which will insure that a measure which acts U-almost invariantly is a U-almost invariant measure. A result in this direction is given by the following theorem.

THEOREM 7. *Let G be a LC group, X a translation invariant subspace of $C_0(G)$, $\mu \in V(G)$ and U an open symmetric neighborhood of zero in G. If X contains a compact approximate identity and μ acts U-almost invariantly on X then μ is a U-almost invariant measure.*

Proof. Since μ acts U-almost invariantly on X, $\{F_s \mid s \in U\}$ spans a finite dimensional subspace B of $L(X)$.

Let C be the linear subspace of $V(G)$ spanned by $\{T_s\mu \mid s \in U\}$ and define the mapping $\Phi: C \to B$ by $\Phi(T_s\mu) = F_s$, $s \in U$. Clearly Φ maps C onto B.

Furthermore, we claim Φ is one-to-one. Indeed, let $\nu = \sum_{j=1}^{l} c_j T_{r_j}\mu$ be an element of C such that $\Phi(\nu) = 0$, i.e. $\int_G h(t)d\nu(t) = 0$, $h \in X$. Let $< g_\beta > \subset X$ be a compact approximate identity. Then, since X is translation invariant, for each $f \in C_c(G)$ we have

$$\begin{aligned} 0 &= \lim_\beta \int_G f(r) \int_G g_\beta(t-r)d\nu(t)dm(r) \\ &= \lim_\beta \int_G g_\beta(-r) \int_G f(r+t)d\nu(t)dm(r) \\ &= \int_G f(t)d\nu(t) \end{aligned}$$

since $\int_G f(.+t)d\nu(t)$ is continuous. The applications of Fubini's theorem are valid as both f and $< g_\beta >$ belong to $C_c(G)$.

Thus, by regularity, $\nu = 0$, and hence Φ is one-to-one.

But then Φ is a one-to-one linear mapping of C onto the finite dimensional space B. Therefore C is finite dimensional, i.e. μ is U-almost invariant.

REMARKS. (a) We have not, of course, circumvented the denseness assumption of Theorem 6; as any translation invariant subspace of $C_0(G)$ which contains a compact approximate identity is necessarily dense in $C_0(G)$.

(b) Let $\mu \in V(G)$ act U-almost invariantly on a translation invariant subspace X of $C_0(G)$, and let f be any function given by Theorem 1 such that

$$\int_G h(t)d\mu(t) = \int_G h(t)f(t)dm(t) \qquad (h \in X) .$$

In general the precise nature of f is not clear. A plausible conjecture, in the light of the structure of U-almost invariant measures, might be that one could always find some f as above for which $\{T_s f \mid s \in U\}$ spans a finite dimensional space of functions. Some support for this conjecture can be found in the fact that for compact groups and measures μ which act almost invariantly one can construct such a function f as a linear combination of the characters common to the space X and the support of the Fourier-Stieltjes transform of μ.

5. **Additional comments.** In a previous version of this paper a seemingly more general problem was considered. A subsemi-group S of a LC group G will be called *admissible* if

(i) S is an open subset of G and

(ii) the zero of G is a point of closure of S.

For such subsemigroups (with the obvious changes in the previous notation) one can consider translation invariant subspaces X of $C_0(S)$, measures $\mu \in V(S)$ and open symmetric neighborhoods U of zero in G such that

$$\int_S h(t)dT_s\mu(t) = \sum_{i=1}^{n} \alpha_i(s) \int_S h(t)dT_{s_i}\mu(t) \qquad (s \in U \cap S,\, h \in X) ,$$

i.e. one can consider measures μ on S which act U-almost invariantly on translation invariant subspaces X of $C_0(S)$. Similarly one can consider $\mu \in V(S)$ which are U-almost invariant measures on S, i.e. $\{T_s\mu \mid s \in U \cap S\}$ spans a finite dimensional subspace of $V(S)$.

It is now seen that the most appropriate way to investigate such measures is to reduce the problem to the context of groups which was discussed in the preceding four sections. Let us indicate how this reduction takes place. We shall restrict ourselves to the case where $\mu \in V(S)$ acts U-almost invariantly; the situation for U-almost invariant measures is similar.

Suppose X is a translation invariant subspace of $C_0(S)$, $\mu \in V(S)$ and U is an open symmetric neighborhood of zero in G; and assume that μ acts U-almost invariantly on X. Define a new measure $\bar{\mu} \in V(G)$ by $\bar{\mu}(E) = \mu(E \cap S)$, $E \in B_c(G)$. This clearly defines a measure in $V(G)$ as S is an open subset of G. Also, since S is open the functions in $C_0(S)$ must vanish on the boundary of S, and hence we may consider $C_0(S)$ as a subspace of $C_0(G)$ by defining for each $f \in C_0(S)$, $f(t) \equiv 0$, $t \notin S$. Let Y be the subset of $C_0(G)$ which consists of X, considered as a subspace in $C_0(G)$, and all its translates by elements of G. Clearly Y is a translation invariant subspace of $C_0(G)$. Moreover, it is easy to check that $\int_G |h(t)|\, d\,|\bar{\mu}|(t) < \infty$, $h \in Y$, and that $\{F_s \mid s \in U \cap S\}$ spans a finite dimensional subspace of $L(Y)$.

But $U \cap S$ has finite positive Haar measure since S is open; and so by a remark in the first section there must exist some open symmetric neighborhood W of zero in G such that $\{F_s \mid s \in W\}$ spans a finite dimensional subspace of $L(Y)$, i.e. $\bar{\mu}$ acts W-almost invariantly on Y.

We now can employ the developement of the preceding sections to investigate $\bar{\mu}$, and then restricting the functions and measures so obtained to the admissible subsemi-group S we get the analogous information about the measure μ. In particular, one can in this fashion establish theorems for $\mu \in V(S)$, S an admissible subsemi-group of G, which are analogs of Theorems 1–7 above.

The reduction just obtained makes it clear that nothing really new is to be gained by a separate consideration of admissible subsemi-groups. Therefore a detailed exposition of this situation has been omitted.

REMARKS. It should be noted that a similar development for arbitrary subsemi-groups of G is not possible. Indeed, let $G = R$, the additive group of the real line; S_1 and S_2 the subsemi-groups of G defined by $S_1 = \{s \mid s \geqq 0\}$ and $S_2 = \{s \mid s > 1\}$; and define the measure $\mu_1 \in V(S_1)$ by $\mu_1(E) = 1$ if $0 \in E$, $\mu_1(E) = 0$ if $0 \notin E$; and the measure $\mu_2 \in V(S_2)$ by $\mu_2(E) = 1$ if $3/2 \in E$, $\mu_2(E) = 0$ if $3/2 \notin E$. Clearly neither S_1 nor S_2 is an admissible subsemi-group, as each violates one of the conditions for admissibility. Furthermore it is easy to check that $\{T_s\mu_i \mid s \in S_i\}$ $i = 1, 2$, span finite dimensional spaces, but that there exist no continuous functions f_i on S_i, $i = 1, 2$, for which $d\mu_i(t) = f_i(t)dm(t)$ and $\{T_s f_i \mid s \in Si\}$ span finite dimensional spaces of functions, $i = 1, 2$; i.e. the analog of Theorem 4 fails.

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