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FUNCTIONS WHICH OPERATE ON CHARACTERISTIC FUNCTIONS

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Let G be a locally compact abelian group and $B^+(G)$ the family of continuous, complex-valued non-negative definite functions on G. Set

$$B_1^+(G) = \{f \in B^+(G): f(0) < 1\}$$
$$\Phi(G) = \{f \in B^+(G): f(0) = 1\}$$

A complex-valued function defined on the open unit disk is said to *operate* on $\{B_1^+(G), B^+(G)\}$ if $f \in B_1^+(G)$ implies $F(f) \in B^+(G)$, similarly for $\{\Phi(G), \Phi(G)\}$. Recently C. S. Herz has given a proof of a conjecture of W. Rudin that F operates on $\{B_1^+(G), B^+(G)\}$ if and only if

$$(*) \qquad F(z) = \sum_{m,n=0}^{\infty} c_{mn} z^m \bar{z}^n,\ c_{mn} \geqq 0,\ |z| < 1\ .$$

for a certain class of G. We shall show by independent methods that F operates on $\Phi(R^1)$ if F is given by (*) for $|z| \leqq 1$ and $F(1) = 1$. This answers a question posed by E. Lukacs and provides in addition an alternate proof of Herz's theorem.

Let $\mathfrak{A}, \mathfrak{B}$ denote two familes of functions $a, b: X \rightarrow Y$. A function $F: Z \subseteqq Y \rightarrow Y$ is said *to operate on* $(\mathfrak{A}, \mathfrak{B})$ provided that for each $a \in \mathfrak{A}$ with range $(a) \subseteqq Z$ we have $F(a) \in \mathfrak{B}$. If $\mathfrak{A} = \mathfrak{B}$ we say simply that F *operates on* $\mathfrak{A}$. Recently there has been considerable interest in determining, for particular families $(\mathfrak{A}, \mathfrak{B})$ the class of functions which operate.

If $\mathfrak{A}$ is the family of complex-valued 2π-periodic functions on R^1 which have absolutely convergent Fourier series

$$\mathfrak{A} = \left\{a : a(\theta) \sim \sum_{k=-\infty}^{\infty} a_k e^{ik\theta} \text{ with } \sum_{k=-\infty}^{\infty} |a_k| < \infty\right\}$$

then a classic result of N. Wiener [10] states that $1/a \in \mathfrak{A}$ provided that $a(\theta) \neq 0\ (0 \leqq \theta < 2\pi)$. P. Lévy [3] generalized Wiener's theorem by proving that analytic functions operate on $\mathfrak{A}$.

If $\mathfrak{A}$ is the family of all non-negative-definite matrices $(a_{i,j})$ with $-1 < a_{i,j} < 1$ then I. J. Schoenberg [8] proved that any continuous function F which operates on $\mathfrak{A}$, $F: (a_{i,j}) \rightarrow (F(a_{i,j}))$ must be of the form

$$F(x) = \sum_{n=0}^{\infty} c_n x^n$$
$$(c_n \geqq 0 \quad -1 < x < 1)$$

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The theorem of Wiener-Lévy can be obtained in a more general setting. Let G be a locally compact abelian group and $\hat{G}$ its dual group, i.e. the set of continuous homomorphisms of G into the multiplicative group of complex numbers of modulus one, endowed with the weak topology. For μ a complex-valued, regular measure on G with finite total variation we define its Fourier-Stieltjes transform by

$$\hat{\mu}(\hat{x}) = \int_G \hat{x}(x)\mu(dx) \quad (\hat{x} \in \hat{G})$$

and denote by $B(\hat{G})$ the family of such transforms. Then

THEOREM. *Real entire functions operate on $B(\hat{G})$ (see [7] for definition).*

In particular by taking $G = Z$ (the group of integers) we obtain the Wiener-Lévy theorem.

A few years ago a converse to this theorem was obtained by H. Helson, J. P. Kahane, Y. Katznelson and W. Rudin [1]. They proved that if F operates on $B(\hat{G})$ then F is a real-entire function.

In probability theory the elements of $B(\hat{G})$ which are of most direct interest are those $\hat{\mu}$ which arise from nonnegative measures μ, i.e. according to Bochner's theorem the $\hat{\mu}$ which are nonnegative-definite on $\hat{G}$. Let $B^+(\hat{G})$ denote this family. Rudin has conjectured [6] that the functions which operate on $(B_1^+(Z), B^+(Z))$[1] must have the form

$$F(z) = \sum_{\substack{n,m=0 \\ (c_{m,n} \geq 0)}}^{\infty} c_{n,m} z^n \bar{z}^m .$$

Recently C. S. Herz [2] published a proof of Rudin's conjecture for $(B_1^+(G), B^+(G))$ under certain restrictions on G. His proof consists of (1) showing that if F, defined on the unit disk, operates on $(B_1^+(G), B^+(G))$ then F operates on $(B_1^+(\Gamma_0), B^+(\Gamma_0))$ where Γ_0 is the discrete multiplicative group of complex numbers of modulus one, and (2) characterizing the functions which operate on $(B_1^+(\Gamma_0), B^+(\Gamma_0))$.

Lukacs posed in [5] the question of determining the class of functions which operate on the set of characteric functions $\Phi(R^1)$, where $\Phi(G) = \{f \in B^+(G): f(0) = 1\}$.

We shall answer here the question posed by Lukacs, directly and by quite independent methods. This will actually yield an alternate proof of Herz's more general result by making use of some of his preliminary propositions. In § 1 we state the main theorem and outline the proof. The details occupy us in § 2–§ 4. In § 5 we show how to obtain the more general result.

[1] $Z=$ the additive group of integers with discrete topology, $B_1^+(G) = \{f \in B^+(G): f(0) < 1\}$

1. Statement of the main theorem and outline of the proof.

THEOREM 1. *If F operates on $\Phi(R^1)$ then F is given by*

$$(*)\qquad F(z) = \sum_{\substack{n,m=0\\(c_{n,m}\geqq 0)}}^{\infty} c_{n,m} z^n \bar{z}^m \qquad (|z| \leqq 1)\,.$$

with $\sum_{n,m=0}^{\infty} c_{m,n} = 1$.

Assuming that F is continuous it is first shown that F operates on $B_1^+(R^1)$. It then follows that

$$F(re^{i\theta}) = \sum_{k=-\infty}^{\infty} a_k(r) \exp(ik\theta)$$

$(0 \leqq r \leqq 1)$ where $a_k(r) \geqq 0$ $(k = 0, \pm 1, \pm 2, \cdots)$. Having obtained this representation we prove that not only is $a_k(r)$ nonnegative, but also absolutely monotonic. Thus

$$(1)\qquad F(re^{i\theta}) = \sum_{k=-\infty}^{\infty} \sum_{n=0}^{\infty} a_{k,n} r^n \exp(ik\theta)$$

with $a_{k,n} \geqq 0$. On the other hand, if the theorem is to be true, then

$$F(re^{i\theta}) = \sum_{k=-\infty}^{\infty} \left\{ \sum_{\substack{n,m\geqq 0\\ n-m=k}} c_{n,m} r^{n+m} \right\} \exp(ik\theta)\,.$$

In order to pass from (1) to (*) $a_k(r)$ must actually be of the form

$$a_k(r) = r^{|k|} \sum_{n=0}^{\infty} b_{k,n} r^{2n}$$

with $b_{k,n} \geqq 0$. To prove that the exponents of r in $a_k(r)$ increase by two can be done directly (Lemma 5). To prove that $a_k(r) = O(r^{|k|})$ (near $r = 0$) we introduce the more general representation of F

$$F(r_1 \exp(i\lambda_1 t) + r_2 \exp(i\lambda_2 t) + \cdots + r_n \exp(i\lambda_n t))$$
$$= \sum_{\substack{k_i=-\infty\\ 1\leqq i\leqq n}}^{\infty} \alpha_{k_1,k_2,\cdots,k_n}(r_1, r_2, \cdots, r_n) \exp\left\{i \sum_{j=1}^{n} k_j \lambda_j t\right\}$$

where $(r_1, r_2, \cdots, r_n)$ varies in a suitable cube of R^n. The vanishing of $a_k(r)$ to the correct order is then deduced from the simple observation that $\alpha_{k_1,k_2,\cdots,k_n}(r_1, r_2, \cdots, r_n) = O(r_1 r_2 \cdots r_n)$ if all $k_j \neq 0$ (Lemma 4).

Finally we turn to the question of continuity. Since $F(\phi)$ is a continuous function for every $\phi \in \Phi(R^1)$, the natural approach would be to prove directly that $z_n \to z_0$ implies $F(z_n) \to F(z_0)$ by constructing a ch.f. ϕ together with a bounded sequence $\{t_n\}$ such that $\phi(t_n) = z_n$.[2] However, as the referee has observed it suffices to prove a slightly weaker interpolation property; namely that some $\phi \in \Phi(R^1)$ exists which interpolates, on a bounded sequence, some subsequence of the $\{z_n\}$. His lemma and proof are given in § 4.

2. Several lemmata. In this section we assume that F is continuous on $\Delta = \{z: |z| \leqq 1\}$ and operates on $\Phi(R^1)$.

LEMMA 1. *If* $p \in B_1^+(R^1)$ *then* $F(p) \in B_1^+(R^1)$.

Proof. It suffices by Cramey's criterion [5, p. 65] to show that

$$\int_0^A \int_0^A F(p(t-u)) \exp(ix(t-u))dtdu \geqq 0$$

for all real x and $A > 0$. If the lemma were false there would exist therefore and $A_0 > 0$ and x_0 such that

$$\int_0^{A_0} \int_0^{A_0} F(p(t-u)) \exp(ix_0(t-u))dtdu = -d < 0^3 . \tag{2}$$

The function

$$p_\varepsilon(t) = \begin{cases} (1-p(0))\left(1 - \dfrac{|t|}{\varepsilon}\right) & \text{if } |t| \leqq \varepsilon \\ 0 & \text{if } |t| > \varepsilon \end{cases}$$

is in $B_1^+(R^1)$ for every $\varepsilon > 0$, [5, p. 70] and thus $\phi_\varepsilon = p_\varepsilon + p \in B^+(R^1)$. It is, in fact, in $\Phi(R^1)$ since $\phi_\varepsilon(0) = 1$. Because F operates on $\Phi(R^1)$.

$$\int_0^{A_0} \int_0^{A_0} F(\phi_\varepsilon(t-u)) \exp(ix_0(t-u))dtdu \geqq 0 . \tag{3}$$

On the other hand

$$\left| \int_0^{A_0} \int_0^{A_0} \{F(p(t-u)) - F(\phi_\varepsilon(t-u))\} \exp(ix_0(t-u))dtdu \right|$$
$$= \left| \int_{G_\varepsilon}\int \{F(p(t-u)) - F(\phi_\varepsilon(t-u))\} \exp(ix_0(t-u))dtdu \right| \leqq 4A_0\varepsilon$$
$$G_\varepsilon = \{(t,u): 0 \leqq t \leqq A_0,\ 0 \leqq u \leqq A_0,\ |t-u| \leqq \varepsilon\}$$

since $|F(z)| \leqq 1$ on Δ. If we take $\varepsilon < d/4A_0$ then (3) contradicts (2).

Let n be a positive integer and $2\pi, \lambda_1, \lambda_2, \cdots \lambda_n$ be rationally independent real numbers. For each vector $\boldsymbol{m} = (m_1, m_2, \cdots, m_n)$ with

[2] We were not able to deduce this strong interpolation property for $\Phi(R^1)$ and this necessitated a somewhat round about argument in the original version of this paper.

[3] That the integral in (2) is real follows from the easily verified identity $F(\bar{z}) = \overline{F(z)}$.

integral components and each vector $r = (r_1, r_2, \cdots, r_n)$ with $0 \leqq r_i < 1/n \, (1 \leqq i \leqq n)$ we formally define $a_m(r)$ by

$$(4) \qquad a_m(r) = \operatorname*{limit}_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} F\Big(\sum_{k=1}^{n} r_k \exp(i\lambda_k t)\Big) \exp\Big\{-it\sum_{k=1}^{n} m_k\lambda_k\Big\}\, dt \,.$$

LEMMA 2. *The limit in* (4) *exists and is independent of* $\lambda_1, \lambda_2, \cdots, \lambda_n$ *(provided that* $2\pi, \lambda_1, \lambda_2, \cdots, \lambda_n$ *are rationally independent real numbers).*

Proof. Combining Lemma 1 with the observation that

$$\sum_{k=1}^{n} r_k \exp(i\lambda_k \cdot) \in B_1^+(R^1)$$

we see

$$F\Big(\sum_{k=1}^{n} r_k \exp(i\lambda_k \cdot)\Big) \in B_1^+(R^1)$$

and hence the limit in (4) exists [5, p. 43].

The Kronecker-Weyl theorem [9] next shows that

$$(5) \qquad \begin{aligned} a_m(r) = \Big(\frac{1}{2\pi}\Big)^n \int_0^{2\pi}\int_0^{2\pi}\cdots\int_0^{2\pi} F\Big(\sum_{k=1}^{n} r_k \exp(i\phi_k)\Big) \\ \times \exp -i\sum_{k=1}^{n} m_k\phi_k \quad d\phi_1 d\phi_2 \cdots d\phi_n \end{aligned}$$

and hence $a_m(r)$ is independent of the particular $\{\lambda_j\}$ chosen.

A function f defined on the cube $0 \leqq x_i < a \, (1 \leqq i \leqq n)$ is called *absolutely monotonic function* if

$$\frac{\partial^{j_1+j_2+\cdots+j_n}}{\partial x_1^{j_1}\partial x_2^{j_2}\cdots\partial x_n^{j_n}} f(x_1, x_2, \cdots, x_n) \geqq 0$$

throughout the cube for $j_1, j_2, \cdots, j_n = 0, 1, 2, \cdots$ Just as in the case of one variable, an absolutely monotonic function admits a power series expansion with nonnegative coefficients.

LEMMA 3. *The pointwise limit of absolutely monotonic functions is absolutely monotonic.*

Proof. For $n = 1$ the lemma is well known. We then proceed by induction to $n + 1$. Suppose

$$\operatorname*{limit}_{k\to\infty} f_k(r_1, r_2, \cdots, r_{n+1}) = f(r_1, r_2, \cdots, r_{n+1}) \,.$$

For fixed $r_1, r_2, \cdots, r_n$ we have

$$f_k(r_1, r_2, \cdots, r_{n+1}) = \sum_{j=0}^{\infty} a_{k,j}(r_1, r_2, \cdots, r_n) r_{n+1}^j \rightarrow f(r_1, r_2, \cdots, r_{n+1})$$

and hence

$$f(r_1, r_2, \cdots, r_{n+1}) = \sum_{j=0}^{\infty} a_j(r_1, r_2, \cdots, r_n) r_{n+1}^j$$

with

$$a_j(r_1, r_2, \cdots, r_n) = \operatorname*{limit}_{k\to\infty} a_{k,j}(r_1, r_2, \cdots, r_n) .$$

Since $a_{k,j}(r_1, r_2, \cdots, r_n)$ is an absolutely monotonic function the induction hypothesis implies $a_j(r_1, r_2, \cdots, r_n)$ is likewise so and lemma is proved.

LEMMA 4. *In the cube* $0 \leqq r_i < 1/n \, (1 \leqq i \leqq n)$

(4i) $a_{\boldsymbol{m}}(r)$ *is an absolutely monotonic function*

$$(6) \qquad a_{\boldsymbol{m}}(r) = \sum_{\substack{0 \leqq i_j < \infty \\ 1 \leqq j \leqq n}} \alpha_{i_1, i_2, \cdots i_n}(\boldsymbol{m}) r_1^{i_1} r_2^{i_2} \cdots r_n^{i_n}$$

and

(4ii) *If* $m_i \neq 0$ *for every* $i \, (1 \leqq i \leqq n)$ *then* $\alpha_{i_1, i_2, \cdots, i_n}(\boldsymbol{m}) = 0$ *if* $i_j = 0$ *for some* $j \, (1 \leqq j \leqq n)$.

Proof. 1. Generalizing a result of Rudin [6, p. 618] we will show that if f is continuous in the cube $0 \leqq x_i < a \, (1 \leqq i \leqq n)$ and satisfies

$$(7) \qquad \int_0^{2\pi}\int_0^{2\pi} \cdots \int_0^{2\pi} f(a_1 + b_1 \cos\theta_1, a_2 + b_2 \cos\theta_2, \cdots, a_n + b_n \cos\theta_n) \times \prod_{k=1}^{n} \cos j_k \theta_k d\theta_k \geqq 0$$

for all integers $j_1, j_2, \cdots, j_n = 0, 1, 2, \cdots$ whenever $0 \leqq b_j \leqq a_j$, $a_j + b_j < a$, then f is absolutely monotonic in the cube $0 \leqq x_i < a \, (1 \leqq i \leqq n)$.

2. To see that $a_{\boldsymbol{m}}(r)$ satisfies (7) (with $a = 1/n$) we observe that

$$\begin{aligned} I &= \left(\frac{1}{2\pi}\right)^n \int_0^{2\pi}\int_0^{2\pi} \cdots \int_0^{2\pi} a_{\boldsymbol{m}}(a_1 + b_1 \cos\theta_1, \cdots, a_n + b_n \cos\theta_n) \\ &\quad \times \prod_{k=1}^{n} \cos j_k \theta_k \, d\theta_k \\ &= \left(\frac{1}{2\pi}\right)^n \int_0^{2\pi}\int_0^{2\pi} \cdots \int_0^{2\pi} a_{\boldsymbol{m}}(a_1 + b_1 \cos\theta_1, \cdots, a_n + b_n \cos\theta_n) \\ &\quad \times \exp -i \sum_{k=1}^{n} j_k \theta_k \, d\theta_1 d\theta_2 \cdots d\theta_n \end{aligned}$$

since the integrand in I is an even function of each of the $\{\theta_k\}$. Next, the integral representation of $a_{\boldsymbol{m}}(r)$ and the Kronecker-Weyl theorem yields

$$I = \left(\frac{1}{2\pi}\right)^n \int_0^{2\pi}\int_0^{2\pi}\cdots\int_0^{2\pi}$$
$$\times F((a_1 + b_1 \cos\theta_1)\exp(i\phi_1) + \cdots + (a_n + b_n\cos\theta_n)\exp(i\phi_n))$$
$$\times \exp -i\sum_{k=1}^{n}(j_k\theta_k + m_k\phi_k)\,d\theta_1\cdots d\theta_n d\phi_1\cdots d\phi_n\,.$$

A final application of the Kronecker-Weyl theorem shows

$$I = \operatorname*{limit}_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}F\Big(\sum_{k=1}^{n}(a_k + b_k\cos\zeta_k t)\exp(i\lambda_k t)\Big)$$
$$\times \exp -i\sum_{k=1}^{n}(j_k\zeta_k + m_k\lambda_k)t\,dt^4$$

and this limit is nonnegative because

$$\sum_{k=1}^{n}(a_k + b_k\cos\zeta_k\cdot)\exp(i\lambda_k\cdot)\in B_1^+(R^1)\ ,$$

Lemma 1 and [5, p. 43].

3. Suppose first that f satisfies (7) and is of class C^∞. To show that

$$\frac{\partial^{j_1+j_2++\cdots+j_n}}{\partial x_1^{j_1}\partial x_2^{j_2}\cdots\partial x_n^{j_n}}f(x_1, x_2, \cdots, x_n)\geqq 0 \tag{8}$$

in the cube $0\leqq x_i < a\,(1\leqq i\leqq n)$ we let $N = j_1 + j_2 + \cdots + j_n$ and write, by Taylor's theorem,

$$f(a_1 + b_1\cos\theta_1, \cdots, a_n + b_n\cos\theta_n)$$
$$= \sum_{k=0}^{N}\frac{1}{k!}\Big(b_1\cos\theta_1\frac{\partial}{\partial x_1} + \cdots + b_n\cos\frac{\partial}{\partial x_n}\Big)^k f\,\Big|_{\substack{x_i=a_i\\ 1\leqq i\leqq n}} \tag{9}$$
$$+ \frac{1}{(N+1)!}\Big(b_1\cos\theta_1\frac{\partial}{\partial x_1} + \cdots + b_n\cos\theta_n\frac{\partial}{\partial x_n}\Big)^{N+1} f\,\Big|_{\substack{x_i=a_i+\eta_i b_i\cos\theta_i\\ 1\leqq i\leqq n}}\,.$$

Multiply (9) by $\prod_{k=1}^{n}\cos j_k\theta_k\,d\theta_k$ and integrate from 0 to 2π. Set $b_i = b < \min_k a_k$ and let $b\downarrow 0$ to obtain (8).

4. If f is *a priori* only continuous, we proceed as follows: let $g\colon R^1\to R^1$ satisfy

(i) $g\in C^\infty$

(ii) $g(t) > 0$ if $0 < t < 1$; $g(t) = 0$ otherwise

(iii) $\int_0^1 g(t)dt = 1$.

If f satisfies (7), then so does

$$f_\delta(x_1, x_2, \cdots, x_n) = \int_0^1\int_0^1\cdots\int_0^1$$
$$\times f(x_1 + \delta y_1, \cdots, x_n + \delta y_n)\prod_{k=1}^{n}g(y_k)dy_k$$

[4] The numbers $2\pi, \lambda_1, \cdots, \lambda_n, \zeta_1, \cdots, \zeta_n$ are taken to be rationally independent real numbers.

on the cube $0 \leqq x_i < a - \delta\,(1 \leqq i \leqq n)$. Now $f_\delta \in C^\infty$ and the argument in 3. applies to show that f_δ is absolutely monotonic. But $f_\delta \to f$ (pointwise) in the cube $0 \leqq x_i < a\,(1 \leqq i \leqq n)$ and Lemma 3 permits us to complete the proof of 4(i).

5. If $m_k \neq 0\,(1 \leqq k \leqq n)$ then from (5) we see

$$a_m(0, r_2, \cdots, r_n) = a_m(r_1, 0, r_3, \cdots, r_n) = \cdots = a_m(r_1, r_2, \cdots, r_{n-1}, 0) = 0$$

and this yields (4)ii.

LEMMA 5. *If*

$$a_k(r) = \frac{1}{2\pi}\int_0^{2\pi} F(r\exp(i\phi))\exp(-ik\phi)\,d\phi \qquad k = 0, \pm 1, \pm 2, \cdots \tag{10}$$

then

5(i) $a_k(-r) = (-1)^k a_k(r)$

and

5(ii) $a_k(r) = \sum_{j=0}^{\infty} a_{k,j} r^j \quad -1 \leqq r \leqq 1$

with

$$a_{k,j} \geqq 0 \quad \sum_{j=0}^{\infty} a_{k,j} < \infty\ .$$

Thus

$$a_k(r) = \begin{cases} \sum_{j=0}^{\infty} a_{k,2j} r^{2j} & \textit{if } k \textit{ is an even integer} \\ \sum_{j=0}^{\infty} a_{k,2j+1} r^{2i+1} & \textit{if } k \textit{ is an odd integer}\,. \end{cases}$$

Proof. For 5(i) note

$$a_k(-r) = \frac{1}{2\pi}\int_0^{2\pi} F(r\exp i(\phi + \pi))\exp(-ik\phi)\,d\phi = (-1)^k a_k(r)\ .$$

Proceeding as in the proof of Lemma 4, we show that

$$\int_0^{2\pi} a_k(\cos\theta)\exp -i\nu\theta\,d\theta \geqq 0 \qquad \nu = 0, \pm 1, \pm 2, \cdots$$

so that $a_k(\cos\cdot) \in B^+(R^1)$. It follows from [4, p. 202] that

$$a_k(\cos\theta) = \sum_{j=0}^{\infty} b_{k,j}\cos j\theta$$

with

$$b_{k,j} \geqq 0 \sum_{j=0}^{\infty} b_{k,j} < \infty \ .$$

If T_j denotes the jth Tchebychev polynomial then

$$a_k(x) = \sum_{j=0}^{\infty} b_{k,j} T_j(x) \qquad -1 \leqq x \leqq 1 \ . \tag{11}$$

But for $0 \leqq x \leqq 1$, Lemma 4 yields the representation

$$a_k(x) = \sum_{j=0}^{\infty} a_{k,j} x^j$$

with

$$a_{k,j} \geqq 0 \sum_{j=0}^{\infty} a_{k,j} < \infty \ .$$

Using elementary properties of the Tchebychev polynomials and the fact that the Fourier series of a C^∞ function may be differentiated term-by-term, 5(i) and (11) imply that the equality

$$\sum_{j=0}^{\infty} a_{k,j} x^j = \sum_{j=0}^{\infty} b_{k,j} T_j(x)$$

extends to $-1 \leqq x \leqq 1$, and this proves 5(ii).

3. Proof of Theorem 1 with hypothesis of continuity. $F(r \exp(i\phi))$ is a continuous, periodic, nonnegative definite function. We can therefore write

$$F(r \exp(i\phi)) = \sum_{k=-\infty}^{\infty} a_k(r) \exp(ik\phi) \tag{12}$$
$$0 \leqq r \leqq 1 \qquad 0 \leqq \phi \leqq 2\pi$$

with

$$a_k(r) \geqq 0 \ (k = 0, \pm 1, \pm 2, \cdots) \sum_{k=-\infty}^{\infty} a_k(r) = F(r) \ .$$

In (12) we set $z = r \exp(i\phi)$ and use Lemma 5 to conclude that

$$F(z) = \sum_{n,m=0}^{\infty} c_{n,m} z^n \bar{z}^m + \sum_{1 \leqq m \leqq n < \infty} (d_{n,m} z^n / \bar{z}^m + e_{n,m} \bar{z}^n / z^m) \tag{13}$$

with

$$c_{n,m} \geqq 0 \ (n, m = 0, 1, 2, \cdots)$$
$$d_{n,m} \geqq 0 \ e_{n,m} \geqq 0 \ (1 \leqq m \leqq n < \infty)$$
$$\sum_{n,m=0}^{\infty} c_{n,m} + \sum_{1 \leqq m \leqq n < \infty} (d_{n,m} + e_{n,m}) = 1 \ .$$

We will now show that $d_{n_0, m_0} = 0$. Let $2\pi, \lambda_1, \cdots, \lambda_{n_0}, \lambda$ be rationally independent real numbers and set

$$z = r \exp(i\lambda t) + \sum_{k=1}^{n_0} r_k \exp(i\lambda_k t) \tag{14}$$

in (13) where

$$0 \leqq r < 2/3 \quad r_k = r/2n_0 \quad (1 \leqq k \leqq n_0) .$$

Let $\boldsymbol{m} = (m_0, \underbrace{1, 1, \cdots, 1}_{n_0})$ and note by Lemma 4

$$\begin{aligned} a_{\boldsymbol{m}}(r, r_1, r_2, \cdots, r_{n_0}) &= C_{\boldsymbol{m}} r r_1 r_2 \cdots r_{n_0} + o(r r_1 r_2 \cdots r_{n_0}) \\ &= C_{\boldsymbol{m}}\left(\frac{1}{2n_0}\right)^{n_0} r^{n_0+1} + o(r^{n_0+1}) . \end{aligned} \tag{15}$$

Examing the term $z^\alpha/\bar{z}^\beta$ with z as in (14) we obtain

$$\begin{aligned} &\frac{\left(r \exp(i\lambda t) + \sum_{k=1}^{n_0} r_k \exp(i\lambda_k t)\right)^\alpha}{\left(r \exp(-i\lambda t) + \sum_{k=1}^{n_0} r_k \exp(-i\lambda_k t)\right)^\beta} \\ &= r^{\alpha-\beta}\left(\exp(i\lambda t) + \frac{1}{2n_0}\sum_{k=1}^{n_0} \exp(i\lambda_k t)\right)^\alpha \exp(i\beta\lambda t) \\ &\quad \times \sum_{p=0}^{\infty} b_p\left\{\frac{1}{2n_0}\sum_{k=1}^{n_0} \exp(-i(\lambda_k - \lambda)t)\right\}^p \qquad (b = 1) \end{aligned} \tag{16}$$

so that only the terms $z^\alpha/\bar{z}^\beta$ with $\beta = m_0 - j$, $\alpha = n_0 + j$ $(0 \leqq j \leqq m_0 - 1)$ yield a contribution to $a_{\boldsymbol{m}}(r, r_1, r_2, \cdots, r_{n_0})$. But with z as in (14)

$$\begin{aligned} &\operatorname*{limit}_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} z^{n_0+j}/\bar{z}^{m_0-j} \exp(-i(m_0\lambda + \lambda_1 + \cdots + \lambda_{n_0})t)dt \\ &= D_j r^{n_0-m_0+2j} \end{aligned}$$

with $D_j \neq 0$ for $j = 0$. Thus (15) implies that $d_{n_0, m_0} = 0$. A similar argument shows $e_{n_0, m_0} = 0$ and the theorem is proved with the hypothesis of continuity.

4. The continuity of F.[5] We begin with an interpolation lemma.

LEMMA 6. *Let $z_n \to z_0$ $(|z_n| < 1, n = 0, 1, 2, \cdots)$. There exists a ch.f. ϕ, a sequence (of real numbers) $t_k \to 1$ and a sequence (of integers) $\{n_k\}$ such that $\phi(t_k) = z_{n_k}$.*

Proof. Let $\tau_n = 1 - (2/3)9^{-n}$; then $(9^n/2)\tau_n \equiv (1/6) \pmod 1$ while $(9^{n+m}/2)\tau_n \equiv (1/2) \pmod 1$ for $m > 0$. Hence

[5] We wish to acknowledge our thanks to the referee for the statement and proof of Lemma 6.

$$\cos \frac{\pi}{2} 9^n \tau_n = \frac{\sqrt{3}}{2}, \ \cos \frac{\pi}{2} 9^{n+m} \tau_n = 0 \ (m > 0)$$

and $\cos (\pi/2) 9^n = 0$. Let $\{\eta_n\}$ be a sequence of positive numbers such that

$$|z_0| + \sum_{n=1}^{\infty} \eta_n < 1 .$$

We define inductively a sequence $\{\phi_n\}$ of positive-definite functions as follows; let

$$\phi_0(t) = |z_0| e^{i(\arg z_0)t} .$$

Assume that $\phi_0, \phi_1, \cdots, \phi_p$ have been defined such that $\phi_j(1) = 0$ for $j > 0$. Choose integers m_{p+1} and n_{p+1} such that

$$r_{p+1} = \left| \sum_{j=0}^{p} \phi_j(\tau_{m_{p+1}}) - z_{n_{p+1}} \right| < \frac{\eta_{p+1}}{2}$$

and define

$$\phi_{p+1}(t) = 2r_{p+1}(\cos \varepsilon_{p+1} t)\left(\cos \frac{\pi}{2} 9^{m_{p+1}} t\right) e^{i\lambda_{p+1} t}$$

where ε_{p+1} and λ_{p+1} are chosen such that

$$\phi_{p+1}(\tau_{m_{p+1}}) = z_{n_{p+1}} - \sum_{j=0}^{p} \phi_j(\tau_{m_{p+1}}) .$$

We shall assume that the sequence $\{m_k\}$ is strictly increasing. If we set $t_k = \tau_{m_k}$ and

$$\phi(t) = \sum_{j=0}^{\infty} \phi_j(t) + \varepsilon \varDelta(t)$$

where $\varDelta(x) = \max(0, 1 - 2|x|)$ and $\varepsilon > 0$ is such that $\phi(0) = 1$ then $\phi(t_k) = z_{n_k}$ $(k = 1, 2, \cdots)$ and $\phi \in \Phi(R^1)$.

LEMMA 7. *F is continuous in the open unit disk* $\{z : |z| < 1\}$.

Proof. Suppose not; then there would exist a $z_0, |z_0| < 1$ and a sequence $\{z_n\}$ $(|z_n| < 1)$ such that $z_n \to z_0$ and $F(z_n) \not\to F(z_0)$. By passing to a subsequence if necessary we can assume that $\{F(z_n)\}$ converges. By Lemma 6 there is a ch.f. ϕ and a sequence (of real numbers) $\{t_k\}$ with limit one such that $\phi(t_k) = z_{n_k}$. But then

$$F(z_0) = F(\phi(1)) = \lim_{k \to \infty} F(\phi(t_k)) = \lim_{k \to \infty} F(z_{n_k})$$

which is a contradiction.

REMARK. For future reference let us note that Lemma 1 now shows that F operates on $B_1^+(R^1) \cup \Phi(R^1)$.

LEMMA 8. *F is continuous on* $-1 \leqq x \leqq 1$.

Proof. By observing that $F(\cos\cdot) \in \Phi(R^1)$, we obtain, just as in Lemma 5

$$F(x) = \sum_{n=0}^{\infty} p_n T_n(x)$$

where $p_n \geqq 0$ and

$$\sum_{n=0}^{\infty} p_n = 1 \ .$$

Since $|T_n(x)| \leqq 1$ on $-1 \leqq x \leqq 1$, F is continuous there.

THEOREM 2. *F is continuous on Δ.*

Proof. As we have already remarked, F operates on $B_1^+(R^1) \cup \Phi(R^1)$. Now Lemmata 2–5 carry over mutatis mutandis to prove that

$$F(z) = \sum_{n,m=0}^{\infty} c_{n,m} z^n \bar{z}^m \qquad |z| < 1 \tag{20}$$

where $c_{n,m} \geqq 0$. Setting $z = x$ in (20) and using Lemma 8 we see that

$$\operatorname*{limit}_{x \uparrow 1} \sum_{k=0}^{\infty} \sum_{\substack{n,m \geqq 0 \\ n+m=k}}^{\infty} c_{n,m} x^k = F(1) = 1 \ .$$

But the $\{c_{n,m}\}$ are nonnegative and hence

$$\sum_{n,m=0}^{\infty} c_{n,m} = 1 \ .$$

Thus our series in (20) extends to a continuous function on Δ. We assert that F is equal to this extension. For let $\phi \in \Phi(R^1)$ $t_k \rightarrow t_0$ with $0 < |\phi(t_k)| < 1$, $|\phi(t_0)| = 1$. Then $F(\phi)$ is a continuous function and thus limit $F(\phi(t_k)) = F(\phi(t_0))$. But

$$\begin{aligned} \operatorname{limit} F(\phi(t_k)) &= \operatorname{limit} \sum_{n,m=0}^{\infty} c_{n,m} (\phi(t_k))^n (\overline{\phi(t_k)})^m \\ &= \sum_{n,m=0}^{\infty} c_{n,m} (\phi(t_0))^n (\overline{\phi(t_0)})^m \end{aligned}$$

and thus

$$F(\phi(t_0)) = \sum_{n,m=0}^{\infty} c_{n,m} (\phi(t_0))^n (\overline{\phi(t_0)})^m \ .$$

5. Concluding remarks. In order to obtain the general theorem we require two propositions due to Herz [2 p. 165, p. 167].

PROPOSITION 1. If a locally compact abelian group H has elements of arbitrarily high order then every F which operates on $(B_1^+(H), B^+(H))$ is continuous.

PROPOSITION 2. If a locally compact abelian group H has elements of arbitrarily high order, then every F which operates on $(B_1^+(H), B^+(H))$ operates on $(B_1^+(Z), B^+(Z))$.

REMARKS. 1. In Propositions 1 and 2 it is assumed that F is defined on $\{z: |z| < 1\}$.

2. Proposition 1 does not include our Lemma 7 since we assume merely that F operates on $\Phi(R^1)$, not on $(B_1^+(R^1), B^+(R^1))$.

THEOREM 2. *If a locally compact abelian group H has elements of arbitrarily high order, then F operates on $(B_1^+(H), B^+(H))$ if and only if*

$$F(z) = \sum_{n,m=0}^{\infty} c_{n,m} z^n \bar{z}^m , \qquad (|z| < 1)$$

where $c_{n,m} \geqq 0$.

Proof. By Propositions 1 and 2 we may assume that $H = Z$ and that F is continuous. It suffices, by the proof of Theorem 1, to show that F operates on $(B_1^+(R^1), B^+(R^1))$. Suppose $\lambda \in B_1^+(R^1)$ and set $\phi = F(\lambda)$. Since ϕ is continuous all that must be verified is that ϕ is a nonnegative-definite function. For any $\delta > 0$, the sequence $\{\lambda_n = \lambda(n\delta)\}$ is nonnegative definite and therefore by the hypothesis $\{\phi(n\delta)\}$ is a nonnegative definite sequence for any $\delta > 0$. Since ϕ is continuous

$$\int_0^A \int_0^A \phi(u - v) \exp(ix(u - v)) du dv$$
$$= \operatorname*{limit}_{\delta \downarrow 0} \sum_{n,m=1}^{A/\delta} \phi((n - m)\delta) \exp ix\delta(n - m)\, \delta^2 .$$

But since $\{\phi(n\delta)\}$ is a nonnegative-definite sequence for each $\delta > 0$

$$\sum_{n,m=1}^{A/\delta} \phi((n - m)\delta) \exp ix\delta(n - m) \quad \delta^2 \geqq 0$$

and hence by Cramer's criterion ϕ is nonnegative definite.

We conclude with a few remarks.

1. There is a formal relation between the result of [1] and our Theorem 1. Every real-entire function F can be written in the form

$$F = (F_1 - F_2) + i(F_3 - F_4)$$

where F_1, F_2, F_3 and F_4 satisfy (*). On the other hand every $\hat{\mu} \in B(\hat{G})$ is of the form

$$\hat{\mu} = (\hat{\mu}_1 - \hat{\mu}_2) + i(\hat{\mu}_3 - \hat{\mu}_4)$$

where $\hat{\mu}_1$, $\hat{\mu}_2$, $\hat{\mu}_3$ and $\hat{\mu}_4$ are in $B^+(\hat{G})$. A direct proof of our theorem starting from this observation would be desirable.

2. The proof given here of Theorem 1 demonstrates in one stroke that F is real-analytic in Δ and if it is expressed as a power series in z and $\bar{z}$ it has nonnegative coefficients. If one could prove directly that F operates on all Fourier transforms assuming values in Δ then proof of the theorem could be completed in two steps:

(A) F is real-analytic [7, Chapter VI] and thus

$$F(z) = \sum_{n,m=0}^{\infty} c_{n,m} z^n \bar{z}^m$$

(B) $c_{n,m} \geqq 0 \; (n, m = 0, 1, 2, \cdots)$ The second step is a consequence of the explicit representation

$$c_{n,m} = \underset{r \downarrow 0}{\text{limit}} \; \underset{T \to \infty}{\text{limit}} \frac{1}{r^{n+m}} \frac{1}{2T} \int_{-T}^{T} F\left(\sum_{k=1}^{n+m} r_k \exp(i\lambda_k t)\right) \times \exp\left(\sum_{k=1}^{n} \lambda_k t - i \sum_{k=1}^{m} \lambda_{n+k} t\right) dt^6$$

where the inner limit exists and is positive by virtue of Lemma 1 and [5, p. 43] and the outer limit exists by (A) above.

3. For nondiscrete G with elements of arbitrarily high order one can show by using the methods used in the proof of Theorem 1, that F operates on $\Phi(G)$ if and only if F satisfies (*). If G is discrete this needn't be the case, and F needn't even be continuous as, $F(z) = 0 \; (|Z| < 1)$, $=1 \; (|z| = 1)$, which operates on $\Phi(Z)$ already shows. For such discrete groups we don't know if it is true that F operates on $\Phi(G)$ implies that F must operate on $B_1^+(G)$. If it were true then at least the structure of F for $|z| < 1$ could be determined.

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