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BOUNDARY MEASURES OF ANALYTIC DIFFERENTIALS AND UNIFORM APPROXIMATION ON A RIEMANN SURFACE

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A classical theorem of F. and M. Riesz establishes a one-to-one correspondence between analytic differentials of class H_1 on the interior of the unit disc and finite complex-valued Borel measures on the boundary of the disc which are orthogonal to polynomials. The main result of this paper gives a similar correspondence when the unit disc is replaced by a compact subset, satisfying a finite connectivity condition, of any noncompact Riemann surface. The analytic differentials on the interior of the set satisfy a boundedness condition analogous to the classical H_1 differentials and the measures on the boundary of the set are those orthogonal to all meromorphic functions with a finite number of poles in the complement of the set. This result is then used to obtain theorems on uniform approximation on the set by such meromorphic functions.

This paper extends results of Bishop in [2] and [5] where he considers compact subsets of the plane staisfying a simple connectivity condition.[1] He obtained such a one-to-one correspondence between boundary measures and analytic differentials and used his result together with an approximation theorem for nowhere dense sets to give a proof of Mergelyan's approximation theorem [6]. We are able to extend Mergelyan's theorem to our more general sets and also show that "local" approximation implies approximation on the whole set.

I. Boundary measures of analytic differentials.

A. DEFINITIONS AND PRELIMINARIES.

In this section S will denote an open Riemann surface. If K is a compact subset of S, we denote by $C(K)$ the algebra of all continuous complex-valued functions on K with norm $||f|| = \sup_{x \in K} |f(x)|$, and by $A(K)$ the closed subalgebra of $C(K)$ consisting of those functions which are limits of meromorphic functions on S with finitely many poles in

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[1] The case with smooth boundary is discussed by Royden in [7].

$S \sim K$. By Runge's Theorem when S is the plane, or by the extension of Runge's Theorem due to Behnke and Stein [1, p. 445 and p. 456] in the general case, $A(K)$ can also be characterized as all functions of $C(K)$ which are uniform limits on K of functions analytic in a neighborhood of K.

The sets for which our results are obtained are defined as follows.

DEFINITION 1. A compact subset K of S will be called *n-balanced* if there exists a finite family $\{U_i\}_{i=1}^n$ of components of $S \sim K$ such that any point of the boundary of K lies on the boundary of one of the U_i. An open subset of S will be called *n-balanced* if it is the interior of its closure and its closure is a compact n-balanced set.

The following properties are clear.

LEMMA 1. *The interior of a compact n-balanced set is an open m-balanced set for some $m \leqq n$. The boundary of a compact n-balanced set is a nowhere dense compact n-balanced set.*

The measures on the boundary of K to be considered are now defined.

DEFINITION 2. If K is a compact subset of S, we denote by $M(K)$ all finite complex-valued Borel measures μ on the boundary of K such that $\int f d\mu = 0$ for all $f \in A(K)$.

Several preliminary definitions will be necessary to describe the boundedness condition on the analytic differentials to be studied.

By an arc we will mean a continuous map $f: [a, b] \rightarrow S$ of a closed interval $a \leqq t \leqq b$ into S. We will identify arcs $f: [a, b] \rightarrow S$ and $g: [c, d] \rightarrow S$ whenever $b - a = d - c$ and $g(t) = f(t + a - c)$. The image of $[a, b]$ under f will be denoted by $|f|$. By a subarc of f we mean the restriction of f to a subinterval $[c, d]$, $a \leqq c < d \leqq b$. If $g: [a_1, b_1] \rightarrow S$ is such that $f(b) = g(a_1)$ then by the product of f and g, written fg, we mean the arc $h: [a, b + b_1 - a_1] \rightarrow S$ defined by

$$h(t) = \begin{cases} f(t) & \text{if} \quad a \leqq t \leqq b \\ g(t + a_1 - b) & \text{if} \quad b \leqq t \leqq b + b_1 - a_1 . \end{cases}$$

An arc $f: [a, b] \rightarrow S$ is an analytic arc if f can be extended to be analytic with nonzero derivative in a neighborhood of $[a, b]$. A piece-wise analytic arc is a product of a finite number of analytic arcs. A simple closed curve is an arc $f: [a, b] \rightarrow S$ such that $f(a) = f(b)$, and if $x \neq a$ and $x \neq b$ then $f(x) \neq f(a)$ and f is one-to-one on the open interval (a, b).

DEFINITION 3. If U is an open subset of S we say that a sequence $\{\gamma_i\}$ delimits U if

(i) each γ_i is a finite family of disjoint piecewise analytic simple closed curves α_{ij} such that $|\alpha_{ij}| \subset U$ and $\bigcup_j |\alpha_{ij}|$ is the boundary of an open set $V_i \subset U$ and each α_{ij} is positively oriented with respect to V_i.

(ii) if T is any compact subset of U, then for all sufficiently large i, $T \subset V_i$.

DEFINITION 4. If U is an open subset of S with compact closure K and γ is a finite family of piecewise analytic curves α_j such that $|\alpha_j| \subset U$ and ω is an analytic differential on U, we denote by $\|\omega\|_\gamma$ the norm of the linear functional F on $C(K)$ defined by $F(h) = \int_\gamma h\omega$.

DEFINITION 5. Let U be an open subset of S with compact closure. The class $H(U)$ consists of all analytic differentials ω on U such that there exists a sequence $\{\gamma_i\}$ which delimits U and an $M > 0$ such that $\|\omega\|_{\gamma_i} < M$ for all i.

Our aim is to establish, in case K is an n-balanced set, a one-to-one correspondence between $M(K)$ and $H(U)$, where U is the interior of K. The correspondence will be between a differential and its boundary measure, in the following sense.

DEFINITION 6. Let U be an open subset of S with compact closure and let B be its boundary. A finite complex-valued Borel measure μ on B is said to be a boundary measure of $\omega \in H(U)$ if the sequence of Definition 5 can be chosen so that

$$\int_{\gamma i} h\omega \longrightarrow \int hd\mu \quad \text{as} \quad i \longrightarrow \infty$$

for all $h \in C(U \cup B)$.

We do not need any restrictions on K other than compactness in order to show the existence of a boundary measure for every differential $\omega \in H(U)$. The following theorem has the same proof as Theorem 1 in [5].

THEOREM 1. *Let U be an open subset of S with compact closure K. Then any $\omega \in H(U)$ has a boundary measure $\mu \in M(K)$.*

In order to "fit together" sequences which delimit two different open sets to obtain a sequence which delimits the union, we will need the following lemma.

LEMMA 2. *Let γ and δ each be a finite family of disjoint piece-*

wise analytic simple closed curves, α_j and β_j respectively, such that $\bigcup_j |\alpha_j|$ is the boundary of an open set Γ with each α_j positively oriented with respect to Γ and similarly $\bigcup_j |\beta_j|$ is the boundary of an open set Δ with each β_j positively oriented with respect to Δ. Then there exists a finite collection of analytic coordinate functions h_i with domain V_i, V_i a neighborhood of a point $p_i \in S$ (the p_i need not be distinct), so that given any neighborhood U_i of $h_i(p_i)$ such that $U_i \subset h_i(V_i)$ and any $\varepsilon_i > 0$, there exists φ, a finite family of disjoint piecewise analytic simple closed curves ψ_j, such that $\bigcup_j |\psi_j|$ is the boundary of an open set Φ and

(i) *each ψ_j is positively oriented with respect to Φ*

(ii) *each ψ_j is the product of a finite number of subarcs, each of which is either a subarc of some α_j or β_j or is an arc f such that for some i, the arc $h_i \circ f$ has length less than ε_i and $|h_i \circ f| \subset U_i$.*

(iii) $\Gamma \cup \Delta \subset \Phi \subset \Gamma \cup \Delta \cup \bigcup_{i=1}^{n} h_i^{-1}(U_i)$

The proof is left as an exercise for the reader.[2]

B. Plane Sets.

In this section we consider the special case where S is the plane.

The proofs of the following lemma and theorem are the same as Lemma 4 and Theorem 1 in [5].

Lemma 3. *If K is a compact n-balanced subset of the plane and if μ and ν are both in $M(K)$ and $\int (t-z)^{-1} d\mu(t) = \int (t-z)^{-1} d\nu(t)$ for all z in the interior of K, then $\mu = \nu$.*

Theorem 2. *Let U be an n-balanced open subset of the plane and K be its closure. Then given $\omega \in H(U)$, its boundary measure, which exists by Theorem 1, is unique and if $\omega = f(z)dz$ then*

$$f(z) = (2\pi i)^{-1} \int (t-z)^{-1} d\mu(t)$$

for all $z \in U$.

The next lemma is a modification of Lemma 6 in [3]. The assumption that ν is orthogonal to all functions analytic in a neighborhood of K rather than just all polynomials enables us to obtain the measure β_{x_0} with support in K. The proof is not given, as the same proof applies with only obvious minor modifications and we prove a general version for any open Riemann surface as Lemma 5 below.

[2] A proof may be found in the author's thesis.

LEMMA 4. *Let K be a compact subset of the complex plane. Let ν be a measure on K orthogonal to $A(K)$. Then for almost all real numbers x_0, there exists a measure β_{x_0} on the set $K \cap \{z : Re\ z = x_0\}$ such that*

$$\int_{R_{x_0}} h d\nu = -\int_{L_{x_0}} h d\nu = \int h d\beta_{x_0}$$

for all $h \in A(K)$, where

$$R_{x_0} = K \cap \{z : Re\, z \geqq x_0\} \quad and \quad L_{x_0} = K \cap \{z : Re\, z \leqq x_0\} .$$

THEOREM 3. *Let K be a compact n-balanced subset of the complex plane with interior U. Then if $\mu \in M(K)$, there exists an analytic differential $\omega \in H(U)$ such that μ is the boundary measure of ω.*

Proof. The proof is by induction on n. If $n = 1$, K is balanced in the sense of [5] and Theorem 3 of [5] is the required result.

Suppose for $n > 1$ the theorem is true for m-balanced sets for all $m < n$. For $z \in U$, define

$$f(z) = (2\pi i)^{-1}\int (t - z)^{-1} d\mu(t) .$$

Now suppose x_0 is as in Lemma 4 and furthermore that $\{z : Re\, z = x_0\}$ intersects the interior of at least one of the bounded components U_i of Definition 1. Then L_{x_0} and R_{x_0} are both m-balanced for some $m < n$. Thus since $\mu \mid L_{x_0} + \beta_{x_0} \in M(L_{x_0})$ and $\mu \mid R_{x_0} - \beta_{x_0} \in M(R_{x_0})$ by Lemma 4 and Runge's theorem, the induction hypothesis applies and they are boundary measures of analytic differentials $f_1(z)dz$ and $f_2(z)dz$ respectively.

For z in the interior of L_{x_0},

$$\begin{aligned} f_1(z) &= (2\pi i)^{-1} \int (t - z)^{-1} d(\mu \mid L_{x_0} + \beta_{x_0})(t) \\ &= (2\pi i)^{-1} \int (t - z)^{-1} d(\mu \mid L_{x_0} + \beta_{x_0})(t) \\ &+ (2\pi i)^{-1} \int (t - z)^{-1} d(\mu \mid R_{x_0} - \beta_{x_0})(t) \\ &= (2\pi i)^{-1} \int (t - z)^{-1} d\mu(t) = f(z) \end{aligned}$$

and for z in the interior of R_{x_0} we have similary

$$f_2(z) = (2\pi i)^{-1} \int (t - z)^{-1} d\mu(t) = f(z) .$$

Now let $x_0 < x_1$ both restricted as above. Then $\mu \mid R_{x_0} - \beta_{x_0}$ is a boundary measure for $f(z)dz$ on the set R_{x_0}. Denote the delimiting sequence by $\{\gamma_j\}$. Also $\mu \mid L_{x_1} + \beta_{x_1}$ is a boundary measure for $f(z)dz$ on the set L_{x_1}. Denote the delimiting sequence by $\{\delta_j\}$. Suppose Γ_j is the open set bounded by γ_j and Δ_j the open set bounded by δ_j, as required in Definition 3. We apply Lemma 2 to $\gamma_j, \delta_j, \Gamma_j, \Delta_j$ where U_i are chosen so that $h_i^{-1}(U_i) \subset U$ and ε_i chosen so that the length of the arc in U_i which is not from δ_j or γ_j is less than η_i and $\Sigma\eta_i \sup_{z\in U_i} |f(z)| < 1$.

The lemma yields φ_j a finite union of disjoint piecewise analytic simple closed curves in U which form the boundary of the open set Φ_j, and $\Gamma_j \cup \Delta_j \subset \Phi_j \subset U$. If S is a compact subset of U, let $x_0 < x_2 < x_1$. Then $S_1 = S \cap \{z : Re\, z \leqq x_2\}$ is a compact subset of the interior of L_{x_1} and $S_2 = S \cap \{z : Re\, z \geqq x_2\}$ is a compact subset of the interior of R_{x_0}. Thus for all j sufficiently large,

$$S_1 \subset \Delta_j \quad \text{and} \quad S_2 \subset \Gamma_j \quad \text{and} \quad S = S_1 \cup S_2 \subset \Delta_j \cup \Gamma_j \subset \Phi_j .$$

Therefore $\{\varphi_j\}$ is a delimiting sequence for U. Furthermore,

$$\begin{aligned}\|\omega\|_{\varphi_j} = \int_{\varphi_j} |f(z)|\,|dz| &\leqq \int_{\gamma_j} |f(z)|\,|dz| + \int_{\delta_j} |f(z)|\,|dz| \\ &+ \Sigma\eta_i \sup_{z\in U_i} |f(z)| \leqq \|\omega\|_{\gamma_j} + \|\omega\|_{\delta_j} + 1 .\end{aligned}$$

Thus $\omega \in H(U)$.

By Theorems 1 and 2 there exists a boundary measure ν on the closure of U such that for $z \in U$,

$$f(z) = (2\pi i)^{-1}\int (t-z)^{-1} d\nu(t) \quad \text{and} \quad \nu \in M(\text{clsr } U) \subset M(K) .$$

Applying Lemma 3 to μ and ν we see that $\mu = \nu$ and thus μ is the boundary measure of ω.

C. Subsets of an Open Riemann Surface.

In this section we consider the general case where S is any open Riemann surface. The function $(t-z)^{-1}$ used in the plane case must be replaced by the elementary differential of Behnke and Stein [1]. The result needed is the following: there exists an elementary differential $\alpha(p)$ which for fixed p is a meromorphic differential on S with exactly one pole, a simple pole at p with residue 1. Furthermore, if h is an analytic coordinate function on an open set $V \subset S$ and $\alpha(p) = A(z, p)dz$ on $h(V)$, then $A(z, p)$ is meromorphic in p on S with exactly one pole, a simple pole at $h^{-1}(z)$. Thus if $h^{-1}(z_0) \notin K$, $A(z_0, p) \in A(K)$.

We prove the following generalization of Lemma 4.

LEMMA 5. *Let K be a compact subset of S. Let ν be a measure on K orthogonal to $A(K)$. Then if f is a nonconstant function analytic on S, for almost all real numbers x_0, there exists a measure β_{x_0} on the set $K \cap \{p : Re f(p) = x_0\}$ such that*

$$\int_{R_{x_0}} h d\nu = -\int_{L_{x_0}} h d\nu = \int h d\beta_{x_0}$$

for all $h \in A(K)$ where

$$L_{x_0} = \{p : Re f(p) \leqq x_0\} \cap K \text{ and } R_{x_0} = \{p : Re f(p) \geqq x_0\} \cap K .$$

Proof. Since f is nonconstant, for all but finitely many real numbers x, the differential of f does not vanish on $K \cap \{p : Re f(p) = x\}$. Let x_1 have this property and let $x_2 > x_1$ be such that the differential of f does not vanish on $K \cap \{p : x_1 \leqq Re f(p) \leqq x_2\}$. Since the differential of f does not vanish, there exists a neighborhood of any point of $K \cap \{p : x_1 \leqq Re f(p) \leqq x_2\}$ on which f is a coordinate function. Cover $K \cap \{p : x_1 \leqq Re f(p) \leqq x_2\}$ by finitely many neighborhoods $\{U_i\}_{i=1}^n$ such that the closure of U_i is compact and contained in V_i and f is a coordinate function on V_i. Denote by f_i^{-1} the inverse of f as a coordinate function on V_i.

There exists a nonnegative measure μ on K such that $|\nu(B)| \leqq \mu(B)$ for all Borel sets B. Let ϕ be the nonnegative, nondecreasing function defined by $\phi(x) = \mu(\{p : Re f(p) \leqq x\})$. Then $\phi'(x)$ will exist for almost all x. Let x_0 be such that $\phi'(x_0)$ exists and $x_1 \leqq x_0 \leqq x_2$. Thus ν vanishes on all subsets of $L_{x_0} \cap R_{x_0}$ and since $h \in A(K)$ implies $\int h d\nu = 0$ we have

$$\int_{R_{x_0}} h d\nu = -\int_{L_{x_0}} h d\nu \quad \text{for all} \quad h \in A(K) .$$

Suppose now that h is a meromorphic function with finitely many poles outside K. Let W be an open neighborhood of K on which h is analytic. Let $W_i = W \cap U_i$. Choose ε, $0 < \varepsilon < 1$ and let

$$T = \bigcup_{i=1}^{n} \{p \in W_i : Re f(p) = x_0 \text{ and } \operatorname{dist}(f(p), f(K \cap W_i)) \leqq \varepsilon\} .$$

Let $||h|| = \sup_{p \in T} |(h(p)|$.

If $Re f(p) > x_0$, define $h_1(p) = (2\pi i)^{-1}\int_T h\alpha(p)$ and if $Re f(p) < x_0$, define $h_2(p) = (2\pi i)^{-1}\int_T h\alpha(p)$ where in each case integration is in a positive direction with respect to $\{p : Re f(p) \leqq x_0\}$. Suppose p_0 is interior to T relative to $\{z : Re z = x_0\}$. Then for some i_0, $f(p_0)$ is interior to $f(W_{i_0} \cap T)$ relative to $\{z: Re\, z = x_0\}$. Let $\tau_{i_0} = f(T \cap W_{i_0})$. Since the W_i cover T, we can choose, for $i \neq i_0$, measurable sets $\tau_i \subset \{z: Re\, z = x_0\} \cap f(W_i)$ so that $f_i^{-1}(\tau_i)$ are pairwise disjoint and each is disjoint from $f_{i_0}^{-1}(\tau_{i_0})$ and so that $T = \bigcup_{i=1}^n f_i^{-1}(\tau_i)$. Then if $p \in U_{i_0}$, $(2\pi i)^{-1}\int_T h\alpha(p)$ becomes

$$(2\pi i)^{-1}\int_{\tau_{i_0}} h(f_{i_0}^{-1}(\zeta))(\zeta - f(p))^{-1}d\zeta + (2\pi i)^{-1}\sum_{i=1}^n \int_{\tau_i} h(f_i^{-1}(\zeta))g_i(f(p), \zeta)d\zeta$$

where g_i is analytic in $f(\operatorname{clsr} U_{i_0})$ in the first variable and in $f(\operatorname{clsr} U_i)$ in the second variable. The first term has continuous boundary values both from the right and the left at p_0 with difference $h(p_0)$ and the integrals in the summation are all continuous in p at p_0. Thus h_1 and h_2 have continuous boundary values $h_1(p_0)$ and $h_2(p_0)$ and

$$h_1(p_0) - h_2(p_0) = h(p_0) .$$

If we define $h_1(p) = h(p) + h_2(p)$ in $Re\, f(p) < x_0$ and $h_2(p) = h(p) + h_1(p)$ in $Re\, f(p) > x_0$, then h_1 and h_2 are analytic in a neighborhood of K and h $= h_1 - h_2$. Thus

$$\int_{R_{x_0}} h d\nu = \int_{R_{x_0}} h_1 d\nu - \int_{R_{x_0}} h_2 d\nu = \int_{R_{x_0}} h_1 d\nu + \int_{L_{x_0}} h_2 d\nu$$

and

$$\begin{aligned} \left|\int_{R_{x_0}} h_1 d\nu\right| &= \left|\int_{R_{x_0}}\left[(2\pi i)^{-1}\int_T h\alpha(p)\right]d\nu(p)\right| \\ &\leqq \int_{\{p:\, Re\, f(p) > x_2\}}\left|\int_T h\alpha(p)\right|d\mu(p) \\ &\quad + \sum_{i=1}^n \int_{R_{x_0}\cap U_i}\left|\int_T h\alpha(p)\right|d\mu(p) . \end{aligned}$$

Cover $K \cap [p: Re\, f(p) \geqq x_2\}$ by a finite number of open analytic neighborhoods, which are the domains of analytic coordinate functions ψ_k, each with range the unit circle D. Continuing the inequalities we have

$$\begin{aligned} &\leqq \|h\| \sum_{i=1}^n \int_{R_{x_0}\cap U_i}\left(\int_{\tau_i} |\zeta - f(p)|^{-1} d\zeta\right)d\mu(p) \\ &+ \|h\| \sum_{i=1}^n \int_{R_{x_0}\cap U_i}\left(\sum_{j=1}^n \int_{\tau_j} |g_{ij}(f(p), \zeta)|\, d\zeta\right)d\mu(p) \\ &+ \|h\| \sum_k \int_{\psi^{-1}(D)}\left(\sum_{i=1}^n \int_{\tau_i} |\gamma_{ki}(\psi_k(p), \zeta)|\, d\zeta\right)d\mu(p) \end{aligned}$$

where g_{ij} is analytic in the first variable in $f(\operatorname{clsr} U_i)$ and in the second variable in $f(\text{clsr } U_j)$ and γ_{ki} is analytic in the first variable in D and the second variable in $f(\text{clsr } U_i)$. The g_{ij} and γ_{ki} are therefore bounded and we have a constant L, independent of ε, so that the above expression is less than or equal to

$$\| h \| \left\{ n \int_{x_0}^{x_2} \int_{-M}^{M} [(x_0 - x)^2 + t^2]^{-1/2} dt \, d\phi(x) + L \right\}$$

where M is chosen, independent of ε, so that if $y = Im\, f(p)$ and $v = Im \zeta$ where $p \in R_{x_0}$ and ζ is in some τ_i, then $| y - v | < M$.

A bound N, independent of ε, is found for

$$\int_{x_0}^{x_2} \int_{-M}^{M} [(x_0 - x)^2 + t^2]^{-1/2} dt \, d\phi(x)$$

as in [3, p. 42]. Thus

$$\left| \int_{R_{x_0}} h_1 d\nu \right| \leqq \| h \| \, (nN + L)$$

and a similar estimate can be made for $\left| \int_{L_{x_0}} h_2 d\nu \right|$. Combining these we have

$$\left| \int_{R_{x_0}} h d\nu \right| \leqq Q \, \| h \| \, .$$

where Q is independent of ε, and thus

$$\left| \int_{R_{x_0}} h d\nu \right| \leqq Q \sup \{ | h(p) | : p \in K \cap \{ p : \, Re\, f(p) = x_0 \} \} \, .$$

Therefore $h \longrightarrow \int_{R_{x_0}} h d\nu$ is a bounded linear functional on a dense subset of $A(K) \mid K \cap \{p\colon Re\, f(p) = x_0\}$ and therefore on $A(K) \mid K \cap \{p\colon Re\, f(p) = x_0\}$. By the Hahn-Banach theorem we can extend this bounded linear functional to $C(K \cap \{p\colon Re\, f(p) = x_0\})$ and then apply the Riesz representation theorem to obtain the desired measure β_{x_0}.

LEMMA 6. *Suppose K is an n-balanced compact subset of S. Suppose f is a nonconstant analytic function on S and $K_1 = K \cap \{p\colon Re\, f(p) \geqq x_0\}$. Then K_1 is a compact m-balanced set for some $m \leqq n$.*

Proof. K_1 is clearly compact.

Let $\{U_i\}_{i=1}^n$ be the finite set of components of $S \sim K$ from Definition 1. A point q on the boundary of K_1 is either on the boundary of K or in the intersection of the interior of K with the boundary of $\{p: Re\, f(p) \geqq x_0\}$. In the former case, q is on the boundary of some U_i and therefore on the boundary of the component of $S \sim K$ which contains U_i, which we call V_i. There are $n\, U_i$ and therefore $m\, V_i$ with $m \leqq n$. In the later case, q is on the boundary of some component Q of $\{p: Re\, f(p) < x_0\}$. Suppose $Q \subset K$, then clsr $U \subset K$ and clsr Q is compact. Q is open, so $Re\, f(p) = x_0$ on the boundary of Q. Since clsr Q is compact, $Re\, f(p)$ must assume its minimum on clsr Q is compact, $Re\, f(p)$ must assume its minimum on clsr Q and by the minimum modulus theorem for real parts of analytic functions, the minimum must be assumed on the boundary, but there $Re\, f(p) = x_0$. Thus $Re\, f(p) \geqq x_0$ on Q which is a contradiction. Since Q is not contained in K, it must interset some U_i. Therefore $Q \subset V_i$ and q is on the boundary of V_i. This shows K_1 is m-balanced.

LEMMA 7. *Under the hypotheses of Lemma 5, the measure $\nu \mid R_{x_0} - \beta_{x_0}$ is orthogonal to $A(R_{x_0})$ and the measure $\nu \mid L_{x_0} + \beta_{x_0}$ is orthogonal to $A(L_{x_0})$.*

Proof. Let h be a rational function on S with poles at $p_1, p_2, \cdots, p_n$ in $S \sim R_{x_0} = S \sim K \cap \{p: Re\, f(p) < x_0\}$. Let $p_1, \cdots, p_k$ be those poles not in $S \sim K$. Each $p_i, i = 1, \cdots, k$ is in some component Q_i of $\{p: Re\, f(\mathrm{p}) < x_0\}$. By the proof of Lemma 6, such a component cannot be contained in K. Thus we may choose $q_i, i = 1, \cdots, k,\ q_i \in Q_i \sim K$ and let J_i be a curve in Q_i joining p_i and q_i. Let

$$B = S \sim \bigcup_{i=1}^{k} J_i \sim \bigcup_{i=1}^{k} \{p_i\} \quad \text{and} \quad \widetilde{B} = S \sim \bigcup_{i=1}^{k} \{q_i\} \sim \bigcup_{i=k+1}^{n} \{p_i\} .$$

Then by Theorem 6 in [4], h, which is analytic on B, can be uniformly approximated on R_{x_0}, a compact subset of B, by functions f_j analytic on $\widetilde{B}$. Now letting $B_1 = \widetilde{B}$ and $\widetilde{B}_1 = S$ we apply Theorem 13 in [1, p. 456] and approximate f_j on R_{x_0} by meromorphic functions g_j with poles on the boundary of B_1, i.e., at the points $q_1, \cdots, q_k, p_{k+1}, \cdots, p_n$. But these are all in $S \sim K$. Thus $g_j \in A(K)$. By Lemma 5, $\int g_j d(\nu \mid R_{x_0} - \beta_{x_0}) = 0$. Thus $\int h d(\nu \mid R_{x_0} - \beta_{x_0}) = 0$ and $\nu \mid R_{x_0} + \beta_{x_0}$ is orthogonal to $A(R_{x_0})$. The same argument shows $\nu \mid L_{x_0} + \beta_{x_0}$ is orthogonal to $A(L_{x_0})$.

Given any finite collection of functions $\{g_k\}_{k=1}^{l}$ on S, and a real number x_0, we define an equivalence relation on the points of S as follows. The points p and q are equivalent, if, for all k, $Re\, f_k(p) \leqq x_0$ if and only if $Re\, f_k(q) \leqq x_0$.

LEMMA 8. *Let K be a compact subset of S and $\{U_i\}$ an open* *covering of K. Then there exists a finite collection of nonconstant functions, each analytic on S, such that given any x_0, $1/4 < x_0 < 3/4$, each equivalence class of the relation defined with these functions lies in a single member of the covering.*

Proof. Fix a metric on S. By the Lebesgue covering lemma, there exists $\rho > 0$ such that any set of diameter less than or equal to ρ, containing a point of K, lies in a single member of the covering $\{U_i\}$. Cover K by a finite number of sets of diameter less than $\rho/3$ which are homeomorphic to a closed disc. Call these sets $\{D_i\}_{i=1}^m$. For i, j such that $D_i \cap D_j$ is empty, let f_{ij} be a function analytic on S such that $Re\, f_{ij} < 1/4$ on D_i and $Re\, f_{ij} > 3/4$ on D_j. This is possible since by the Behnke-Stein extension of Runge's theorem [1, p. 445 and p. 456] we can approximate a function which is identically zero on a neighborhood of D_1 and identically one on a neighborhood of D_j by functions analytic on S.

Now if A is an equivalence class of the equivalence relation defined by these functions, we will show diam $A \leqq \rho$.

Let $p_0 \subset A$. Then for some i_0, $p_0 \in D_{i_0}$. Let $i_0, i_1, \cdots, i_k$ be all i such that $D_{i_0} \cap D_i$ is not empty. Let $p \in K \cap \{p: Re\, f_{i_0 j}(p) \leqq 3/4,$ all $j \neq i_0, i_1, \cdots, i_k\}$. Suppose $p \notin \bigcup_{i=i_0}^{i_k} D_i$. Then since $p \in K \subset \bigcup_{i=1}^n D_i$, $p \in D_{j_0}$, some $j_0 \neq i_0, \cdots, i_k$. Thus $f_{i_0 j_0}(p) > 3/4$ which contradicts the choice of p. We have shown

$$K \cap \left\{p:\ Re\, f_{i_0 j}(p) \leqq \frac{3}{4} \text{ all } j \neq i_0, \cdots, i_k\right\} \subset \bigcup_{i=i_0}^{i_k} D_i\ .$$

Now since $p_0 \in D_{i_0}$, $Re\, f_{i_0 j}(p) < 1/4$, all $j \neq i_0, \cdots, i_k$, but $p_0 \in A$, so for all $p \in A$, we have $Re\, f_{i_0 j}(p) < 1/4$ for $i \neq i_0, \cdots, i_k$. Therefore

$$A \subset K \cap \left\{p:\ Re\, f_{i_0 j}(p) \leqq \frac{3}{4} \text{ all } j \neq i_0, \cdots, i_k\right\} \subset \bigcup_{i=i_0}^{i_k} D_i\ .$$

Each $D_{i_0}, \cdots, D_{i_k}$ intersects D_{i_0} and diam $D_i < \rho/3$. Therefore diam $\bigcup_{i=i_0}^{i_k} D_i < \rho$ and the proof is complete.

THEOREM 4. *If K is a compact subset of S and $\{U_i\}_{i=1}^n$ is an open covering of K, and if μ is a measure on K which is orthogonal to $A(K)$, then there exist measures ν_i with support contained in a compact set $T_i \subset K \cap U_i$ such that ν_i is orthogonal to $A(T_i)$ and $\mu = \nu_1 + \nu_2 + \cdots + \nu_n$.*

Proof. Let f_k be the functions of Lemma 8, $k = 1, \cdots, l$. The proof will be by induction on l. If $l = 0$, let $T = K \subset U_{i_0}$ and $\nu_{i_0} = \mu$ is orthogonal to $A(K) = A(T)$.

Suppose the theorem is true for $l-1$. Let $1/4 < x_0 < 3/4$ and $R_{x_0} = K \cap \{p : Re\, f_l(p) \geqq x_0\}$ and $L_{x_0} = K \cap \{p : Re\, f_l(p) \leqq x_0\}$. R_{x_0} and L_{x_0} are both compact, and $\{U_i\}_{i=1}^n$ is a covering for each. An equivalence class of points of R_{x_0} of the relation defined by $f_1, \cdots, f_{l-1}$ lies in a single member of $\{U_i\}_{i=1}^n$. Similarly for an equivalence class of points of L_{x_0}. Thus we may apply the induction hypothesis to the measures $\mu_1 = \mu \mid R_{x_0} - \beta_{x_0}$ which is orthogonal to $A(R_{x_0})$ by Lemma 7, and $\mu_2 = \mu \mid L_{x_0} + \beta_{x_0}$ which is orthogonal to $A(L_{x_0})$ by Lemma 7. Thus we have measures ν_{ji} with support contained in a compact set $T_{ji} \subset U_i \cap K$ which is orthogonal to $A(T_{ji})$ $j = 1, 2,$ $i = 1, \cdots, n$ and

$$\mu_1 = \nu_{11} + \nu_{12} + \cdots + \nu_{1n}\,, \qquad \mu_2 = \nu_{21} + \nu_{22} + \cdots + \nu_{2n}\,.$$

Thus $\mu = \mu_1 + \mu_2 = (v_{11} + \nu_{21}) + (\nu_{12} + \nu_{22}) + \cdots + (\nu_{1n} + \nu_{2n})$ and $\nu_{1i} + \nu_{2i}$ has support contained in $T_{1i} \cup T_{2i} \subset U_i \cap K$. If $f \in A_{1i} T_{1i} \cup T_{2i})$, then $f \mid T_{1i} \in A(T_{1i})$ and $f \mid T_{2i} \in A(T_{2i})$ and

$$\int f d(\nu_{1i} + \nu_{2i}) = \int f d\nu_{1i} + \int f d\nu_{2i} = 0\,.$$

Thus $\nu_{1i} + \nu_{2i}$ is orthogonal to $A(T_{1i} \cup T_{2i})$ and the theorem is proved.

THEOREM 5. *If K is a compact subset of S and if for every $p \in K$, there is a closed neighborhood W of p such that $f \mid W \in A(K \cap W)$, then $f \in A(K)$.*

Proof. Suppose $f \notin A(K)$. Then there exists a measure μ on K such that μ is orthogonal to $A(K)$ and $\int f d\mu \neq 0$. Let V be the interior of W. Then $\{V\}$ is an open covering of K. Let $\{V_i\}_{i=1}^n$ be a finite subcovering. Apply the last theorem with this covering to get measures ν_i with support contained in a compact set $T_i \subset V_i \cap K \subset W_i \cap K$ and ν_i is orthogonal to $A(T_i)$ and $\mu = \nu_1 + \nu_2 + \cdots + \nu_n$. $f \mid W \in A(K \cap W_i)$ implies $f \mid T_i \in A(T_i)$. Thus $\int f dv_i = 0$ and $\int f d\mu = 0$ which is contradiction.

COROLLARY 1. *If K is a compact subset of S and for every $p \in K$ there exists an analytic coordinate function h with $h(p) = 0$ and the range of h is $\{z : |z| < 1\}$ and an r, $0 < r < 1$, such that $A(h(K) \cap \{z : |z| \leqq r\}) = C(h(K) \cap \{z : |z| \leqq r\})$, then $A(K) = C(K)$.*

Proof. Let $f \in C(K)$. For every p,

$$\begin{aligned} f \circ h^{-1} \mid h\{z : |z| \leqq r\} &\in C(h(K) \cap \{z : |z| \leqq r\}) \\ &= A(h(K) \cap \{z : |z| \leqq r\}). \end{aligned}$$

Thus $f \mid K \cap h^{-1}\{z: |z| \leqq r\} \in A(K \cap h^{-1}\{z: |z| \leqq r\})$. Applying the theorem, $f \in A(K)$.

Thus a local condition on a compact set in the plane which implies that any continuous function can be uniformly approximated by rational functions, such as Theorems 2.4 and 3.4 in [6], can be applied in coordinate neighborhoods of every point of K to show $A(K) = C(K)$. As a special case, using Theorem 2.4 of [6] we have the next corollary which we will need to prove uniqueness of boundary measures.

COROLLARY 2. *If K is a nowhere dense compact n-balanced subset of S, then $A(K) = C(K)$.*

We also obtain a generalization from the plane to Riemann surface of the approximation theorem of Bishop [4].

COROLLARY 3. *If K is a compact nowhere dense subset of an open Riemann surface and M is the minimal boundary of $A(K)$, then $M = K$ implies $A(K) = C(K)$.*

Proof. Let h be an analytic coordinate function at $p \in K$ such that $h(p) = 0$ and the range of h is $\{z: |z| < 1\}$. Let r be $0 < r < 1$. Let M' be the minimal boundary of $A(h(K) \cap \{z: |z| \leqq r\})$. Let $z \in h(K)$ and $|z| \leqq r$, then $h^{-1}(z) \in K = M$. There exists $f \in A(K)$ such that $f(h^{-1}(z)) = 1$ and $|f(q)| < 1$ if $q \in K$ and $q \neq h^{-1}(z)$.

$$f \circ h^{-1} \in A(h(K) \cap \{z: |z| \leqq r\}), \quad f \circ h^{-1}(z) = 1, \quad |f \circ h^{-1}(z)| < 1$$

if $\zeta \in h(K)$ and $|\zeta| \leqq r$, $\zeta \neq z$. Thus $z \in M'$. Since

$$M' = h(K) \cap \{z: |z| \leqq r\},$$

by Theorem 4 in [4], we have

$$A(h(K) \cap \{z: |z| \leqq r\}) = C(h(K) \cap \{z: |z| \leqq r\}).$$

Now the theorem applies and we have $A(K) = C(K)$.

LEMMA 9. *Suppose K is an n-balanced compact subset of S. If μ is a measure on the boundary B of K which is orthogonal to all rational functions on S with poles in the interior of K* or *in $S \sim K$, then $\mu = 0$.*

Proof. The hypothesis implies μ is orthogonal to $A(B)$. By Lemma 1, B is an n-balanced nowhere dense compact subset of S. Thus by Corollary 2, $A(B) = C(B)$ and $\mu = 0$.

THEOREM 6. *If K is a compact n-balanced subset of S and ω is an analytic differential on the interior of K, then the boundary measure μ of ω which exists by Theorem 1 is unique, and if h is an analytic coordinate function on an open set $V \subset S$ and $\omega = f(z)dz$ on $h(V)$, then*

$$f(z) = (2\pi i)^{-1}\int - A(z, q)\, d\mu(q)$$

where $\alpha(p) = A(z, p)dz$ on $h(V)$.

Proof. Suppose μ and ν are both boundary measures of ω. Let g be a rational function on S with poles in the interior or the complement of K. Then

$$\int gd(\mu - \nu) = \int gd\mu - \int gd\nu = \lim_n \int_{\delta_n} g\omega - \lim_n \int_{\gamma_n} g\omega\ .$$

If n is large enough so both δ_n and γ_n surround all the poles of g which lie in the interior of K, then

$$\int_{\delta_n} g\omega = \Sigma_{p \in \operatorname{int} K} Res_p(g\omega) = \int_{\gamma_n} g\omega\ .$$

Therefore $\int gd(\mu - \nu) = 0$ and by Lemma 9, $\mu = \nu$.

$A(z, q)$ is meromorphic in q with a simple pole of residue -1 at $h^{-1}(z)$. Thus

$$(2\pi i)^{-1}\int - A(z, q)\, d\mu(q) = (2\pi i)^{-1} \lim_n \int_{\gamma_n} -A(z, q)\omega = \\ - Res_{h^{-1}(z)}(A(z, q)\omega) = f(z)\ .$$

THEOREM 7. *Let K be a compact n-balanced subset of S with interior U and let $\mu \in M(K)$. Then there exists a differential $\omega \in H(U)$ such that μ is the boundary measure of ω.*

Proof. Let $f_1, \cdots, f_l$ be the finite set of functions analytic on S and satisfying the conditions of Lemma 8 using coordinate neighborhoods for the covering. The proof will be by induction on l. If $l = 0$, K lies in a single coordinate neighborhood and we may consider K as a subset of the plane. In this case we have the result in Theorem 3.

Suppose the theorem is true for $l - 1$. Let $1/4 < x_0 < 3/4$ satisfy the conditions of Lemma 5 for f_l, μ, K. Let L_{x_0}, R_{x_0}, and β_{x_0} be as in Lemma 5. By Lemma 6, R_{x_0} and L_{x_0} are compact m-balanced sets for some m and by Lemma 7, $\mu \mid R_{x_0} - \beta_{x_0} \in M(R_{x_0})$ and $\mu \mid L_{x_0} + \beta_{x_0} \in M(L_{x_0})$. Since $f_1, \cdots, f_{l-1}$ partition R_{x_0} and L_{x_0} in the sense of Lemma 8, the induction hypothesis applies. Thus we have analytic differentials ω_1 and ω_2 on the interiors of R_{x_0} and L_{x_0} for which $\mu \mid R_{x_0} - \beta_{x_0}$ and $\mu \mid L_{x_0} + \beta_{x_0}$ are the boundary measures respectively.

If h_1 is an analytic coordinate function on $V_1 \subset \operatorname{int} R_{x_0}$ and $\omega_1 = f_1(z)dz$ on $\mathrm{h}_1(V_1)$ and $\alpha(q) = A_1(z, q)dz$ on $\mathrm{h}_1(V_1)$ then

$$\begin{aligned} f_1(z) &= (2\pi i)^{-1} \int - A_1(z, q) d(\mu \mid R_{x_0} - \beta_{x_0})(q) \\ &= (2\pi i)^{-1} \int - A_1(z, q) d(\mu \mid R_{x_0} - \beta_{x_0})(q) \\ &+ (2\pi i)^{-1} \int - A_1(z, q) d(\mu \mid L_{x_0} + \beta_{x_0})(q) \\ &= (2\pi i)^{-1} \int - A_1(z, q) d\mu(q) \,. \end{aligned}$$

Similarly, if h_2 is an analytic coordinate function on $V_2 \subset \operatorname{int} L_{x_0}$ and $\omega_2 = f_2(z)dz$ on $h_2(V_2)$ and $\alpha(q) = A_2(z, q)dz$ on $h_2(V_2)$ then

$$f_2(z) = (2\pi i)^{-1} \int - A_2(z, q) d\mu(q) \,.$$

Since we have this for almost all x_0 between 1/4 and 3/4 we can define, for any coordinate function h on $V \subset U$, a differential $\omega = f(z)dz$ on $h(V)$ with

$$f(z) = (2\pi i)^{-1} \int - A(z, q) d\mu(q)$$

where $\alpha(q) = A(z, q)dz$ on $h(V)$, $\omega = \omega_1$ on $\operatorname{int} R_{x_0}$, and $\omega = \omega_2$ on $\operatorname{int} L_{x_0}$.

Let $1/4 < x_1 < x_2 < 3/4$ and both x_1 and x_2 satisfy the conditions of Lemma 5. Let the delimiting sequence of Definition 6 for the boundary measures $\mu \mid L_{x_2} + \beta_{x_2}$ and $\mu \mid R_{x_1} - \beta_{x_1}$ be $\{\delta_i\}$ and $\{\gamma_i\}$ respectively. Let Δ_i and Γ_i be the open sets of which δ_i and γ_i are the boundaries. Let p_j, V_j be the finite collection of points and coordinate neighborhoods obtained in Lemma 2 with h_j the analytic coordinate function on V_j. Let U_j be a closed neighborhood of p_j so that $U_j \subset V_j \cap U$. Let k_j be the maximum of $\mid f(h_j(p)) \mid$ for $p \in U$ where $\omega = f(z)dz$ on V_j. Let $\varepsilon_j = (k_j^{-1} 2^{-j})$. Using these U_j and ε_j we apply Lemma 2 to get φ_i a finite union of disjoint piecewise analytic simple closed curves forming the boundary of Φ_i and $\mid \varphi_i \mid \cup \Phi_i \subset U$. Furthermore, since $\{\delta_i\}$ and $(\gamma_i\}$ delimit the interiors of L_{x_2} and R_{x_1}, respectively, and $\Gamma_i \cup \Delta_i \subset \Phi_i$, $\{\varphi_i\}$ delimits U.

Finally we see that

$$\| \omega \|_{\varphi_i} \leqq \| \omega \|_{\gamma_i} + \| \omega \|_{\delta_i} + \Sigma_j k_j \varepsilon_j \leqq \| \omega \|_{\gamma_i} + \| \omega \|_{\delta_i} + 1 \,.$$

Therefore $\omega \in H(U)$ and by Theorem 1, ω has a boundary measure ν on the boundary of clsr U.

Now let g be a rational function on S with poles in U or $S \sim K$. Choose x, $1/4 < x_0 < 3/4$, as in Lemma 5 and so that no pole of g lies on $\{p: Re\, f_1(p) = x_0\}$. Let $\{\sigma_i\}$ and $\{\tau_i\}$ be the delimiting sequence of Definition 6 for the boundary measures $\mu \mid L_{x_0} + \beta_{x_0}$ and $\mu \mid R_{x_0} - \beta_{x_0}$ respectively. Then

$$\int gd(\mu - \nu) = \int gd(\mu \mid R_{x_0} - \beta_{x_0}) + \int gd(\mu \mid L_{x_0} + \beta_{x_0}) - \int gd\nu$$
$$= \lim_i \int_{\tau_i} g\omega + \lim_i \int_{\sigma_i} g\omega - \lim_i \int_{\varphi_{n_i}} g\omega \,.$$

Letting i be large enough so that all the poles of g in U are surrounded by φ_{n_i} and by either τ_i or σ_i and using the residue theorem we have

$$\int gd(\mu - \nu) = \int_{\tau_i} g\omega + \int_{\sigma_i} g\omega - \int_{\varphi_{n_i}} g\omega = 0 \,.$$

Thus by Lemma 9, $\mu - \nu = 0$ and μ is the boundary measure of ω.

COROLLARY 4. *If K is a compact n-balanced subset of S with interior U, then $A(K)$ consists of all functions in $C(K)$ which are analytic on U.*

Proof. Clearly every function in $A(K)$ is analytic on U. Suppose $A(K)$ does not contain all such functions in $C(K)$. Then there exists a continuous linear functional L orthogonal to $A(K)$ with $L(f) \neq 0$ for some $f \in C(K)$, f analytic on U. The boundary of K is the Silov boundary of the algebra of functions in $C(K)$ analytic on U, so there exists a measure μ on the boundary of K so that $\int gd\mu = L(g)$, all $g \in C(K)$, analytic on U. Thus $\mu \in M(K)$ and there exists $\omega \in H(U)$, so that

$$0 \neq L(f) = \int fd\mu = \lim_i \int_{\gamma_j} f\omega = 0$$

since f is analytic on U.

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