PACIFIC JOURNAL OF MATHEMATICS
Vol. 15, No. 4, 1965

THE 2-LENGTH OF A FINITE SOLVABLE GROUP

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One measure of the structure of a finite solvable group G is its p-length $l_p(G)$. A problem connected with this measure is to obtain an upper bound for $l_p(G)$ in terms of $e_p(G)$, which is a numerical invariant of the Sylow p-subgroups of G. This problem has been solved but the best-possible result is not known for $p = 2$. The main result of this paper is that $l_2(G) \leqq 2e_2(G) - 1$, which is an improvement on earlier results. A secondary objective of this paper is to investigate finite solvable groups in which the Sylow 2-group is of exponent 4. In particular it is proved that if G is a finite group of exponent 12, then the 2-length is at most 2.

Introduction and discussion of results. The object of this paper is to obtain bounds for the 2-length of a finite solvable group. Following Hall and Higman [4], we call a finite group G p-solvable if it possesses a normal series such that each factor group is either a p-group or a p'-group. The p-length, $l_p(G)$, of such a group is the smallest number of p-groups which can occur as factor groups in such a normal series. $e_p(G)$ is defined to be the smallest n such that $x^{p^n} = 1$ for all x belonging to a Sylow p-subgroup of G.

For an odd prime p, it is proved in [4] that $l_p(G) \leqq e_p(G)$ if p is not a Fermat prime and $l_p(G) \leqq 2e_p(G)$ if p is a Fermat prime. Furthermore these results are best-possible. A. H. M. Hoare [6] then proved that in a 2-solvable group G, $l_2(G) \leqq 3e_2(G) - 2$ provided that $l_2(G) \geqq 1$. The primary purpose of this paper is to prove the following improvement:

THEOREM A. *If G is a finite solvable group and $l_2(G) \geqq 1$, then $l_2(G) \leqq 2e_2(G) - 1$.*

Feit and Thompson [1] have proved that solvability and 2-solvability are equivalent notions for finite groups. Thus no loss of generality is involved in requiring G to be solvable in the theorem.

Theorem A will be shown to be an easy consequence of the following theorem about linear groups:

THEOREM B. *Let G be a finite solvable linear group over a field*

Received July 13, 1964. This work, supported in part by a National Science Foundation Cooperative Fellowship, is based upon my doctoral dissertation at the California Institute of Technology. I wish to express my heartfelt appreciation to my advisor Marshall Hall, Jr. for his encouragement and helpful suggestions.

F of characteristic 2 *and assume* G *has no nontrivial normal* 2-*subgroup. Then if* N *is the largest normal* 2′-*subgroup of* G *and if* g *is an exceptional element of order* 2^m *in* G, *it follows that* $g^{2^{m-1}}$ *is in the largest normal* 2-*subgroup of* G/N.

Here, following [4], an element x of order p^n in a linear group over a field of characteristic p is said to be exceptional if $(x-1)^{p^{n-1}} = 0$.

Whether or not Theorem A represents a best-possible result is not known, but it seems likely that further improvements can be made. Indeed, the author knows of no group whose 2-length exceeds its 2-exponent. In the special case of finite solvable groups satisfying $e_2(G) = 2$, i.e., solvable groups whose Sylow 2-subgroups are of exponent 4, I think it likely that $l_2(G) \leqq 2$ instead of the bound $l_2(G) \leqq 3$ furnished by Theorem A.

In § 4 of this paper, groups satisfying $e_2(G) = 2$ are studied in more detail. A sufficient condition for $l_2(G) \leqq 2$ in this special case is established, and, as an application, we prove that $l_2(G) \leqq e_2(G)$ if G is a finite group of exponent 12.

2. Proof of Theorem A from Theorem B.

For the rest of this paper we adopt the convention that all groups referred to are assumed finite, and, if G is such a group, then $|G|$ denotes its order. If H is a normal subgroup of G, we write $H \lhd G$.

We now recall the definition of the upper 2-series of the solvable group G:

$$1 = P_0 \leqq N_0 < P_1 < N_1 < \cdots < P_l \leqq N_l = G .$$

Here N_k/P_k is defined to be the greatest normal 2′-subgroup of G/P_k and P_{k+1}/N_k the greatest normal 2-subgroup of G/N_k. The least integer l such that $N_l = G$ is the 2-length $l_2(G)$. (If there is no danger of confusion we write simply l_2.)

It is proved in [4] that the automorphisms of P_1/F, where F/N_0 is the Frattini subgroup of P_1/N_0, induced by G represent G/P_1 faithfully. Thus G/P_1 is faithfully represented as a linear group operating on P_1/F (P_1/F is an elementary abelian 2-group and so is considered as a vector space over the field with 2 elements).

Now if $l_2(G)=1$, the conclusion of A is trivial. Also the p-length group is at most equal to the class of a Sylow p-subgroup [4, Theorem 1.2.6]. An immediate consequence of this is that if G is solvable and $e_2(G) = 1$, then $l_2(G) = 1$. Thus $l_2 = 2$ implies that $e_2 \geqq 2$ so the result again follows. Now if $l_2 > 2$, then $l_2(G/P_2) = l_2(G) - 2 \geqq 1$ so that Theorem A would follow by induction on l_2 if we could prove that

$e_2(G/P_2) \leqq e_2(G) - 1$.

Now suppose g is an element of maximal order 2^m in a Sylow 2-Sylow subgroup of G/P_1. If g is not exceptional, then [4, Lemma 3.1.2] we have $e_2(G) \geqq m + 1$. If g is exceptional, then, since G/P_1 satisfies the hypothesis of Theorem B, $g^{2^{m-1}}$ is in P_2/P_1 if Theorem B is true. Thus, assuming the validity of B, we obtain in all cases $e_2(G/P_2) \leqq e_2(G) - 1$ and Theorem A follows.

3. Proof of Theorem B. Neither the hypothesis nor the conclusion of the theorem is affected by an extension of the field F. Hence, without loss of generality, we assume that F is algebraically closed. Since an element of order 2 cannot be exceptional, m must be greater than 1. Let $h = g^{2^{m-2}}$ and so $h^2 = g^{2^{m-1}}$.

In proving B we will define subgroups H and H_1 such that $H \triangleleft G$, $H_1 \triangleleft H$, $h^2 \in H_1$, and g normalizes H_1. It then will be shown that if x is any element in the largest normal 2-subgroup of $H_1/H_1 \cap N$ then $(h^2, x) = (h, x)^2$. From this it will follow that h^2 is in the largest normal 2-subgroup of $H_1/H_1 \cap N$, and, finally, from this the theorem will follow.

First we need two lemmas which are of use later and which motivate the definition of H. Here, and elsewhere, we denote the space on which G operates by V.

LEMMA 3.1. *If Q is any $2'$-subgroup of G which is normalized by g, then h^2 fixes every minimal characteristic $F - Q$ submodule of V.*

Proof. A minimal characteristic $F - Q$ submodule is simply the join of all those $F - Q$ submodules operator isomorphic to a given irreducible $F - Q$ submodule. Now if Q is a $2'$-group, V can be written as the direct sum of the minimal characteristic $F - Q$ submodules. g normalizes Q so g must permute the minimal characteristic $F - Q$ submodules. If the lemma were not true, then g, as a permutation of these submodules, would have a cycle of length 2^m which would contradict the assumption that g is exceptional.

LEMMA 3.2. *If Q is any abelian $2'$-subgroup of G and x is any element of G normalizing Q and fixing every minimal characteristic $F - Q$ submodule of V, then x centralizes Q.*

Proof. Let V_i be any minimal characteristic $F - Q$ submodule of V. Since Q is abelian and F is algebraically closed, Q operates on V_i as a scalar multiplication, i.e., if $y \in Q$ and $v \in V_i$ then $yv = \chi_i(y)v$ where $\chi_i(y)$ is a scalar. We now obtain

$$\chi_i(x^{-1}yx)v = x^{-1}y(xv) = x^{-1}\chi_i(y)xv = \chi_i(y)v .$$

Thus (y, x) is the identity on V_i for all $y \in Q$ and the lemma follows.

Now let H be the normal subgroup of G consisting of all elements which fix every minimal characteristic $F - Q$ submodule for every normal $2'$-subgroup Q. Since the largest normal 2-subgroup and the largest normal $2'$-subgroup of H are normal in G, we see that H has no normal 2-subgroup greater than the identity and the largest normal $2'$-subgroup of H is $H \cap N$. By Lemma 3.1 h^2 must belong to H.

Let M be the largest normal nilpotent subgroup of H. Clearly M is a $2'$-group and $M \lhd G$. Furthermore, since H is solvable, M contains its own centralizer in H [2].

LEMMA 3.3. *M is of class 2.*

Proof. Since $h^2 \in H$, h^2 does not centralize M. Thus by Lemmas 3.1 and 3.2, M is not abelian. Now let c be the class of M and suppose $c \geqq 3$. Then if $\Gamma_i(M)$ is the ith term in the lower central series of M ($\Gamma_1(M) = M$ and $\Gamma_{i+1}(M) = (\Gamma_i(M), M)$) and if d is the first integer $\geqq (c+1)/2$, we have [3, Chap. 10]

$$(\Gamma_d(M), M) = \Gamma_{d+1}(M) \neq 1 \text{ (since } d \leqq c - 1) ,$$

and

$$(\Gamma_d(M), \Gamma_d(M)) \leqq \Gamma_{2d}(M) = 1 .$$

Thus $\Gamma_d(M)$ is abelian and, of course, normal in G but is not centralized by M. From Lemma 3.2 and the definition of H we see that this is impossible, and so $c = 2$.

$M = M_1 \times M_2 \times \cdots$ where M_i is the Sylow q_i-subgroup of M and q_i is an odd prime. Each M_i is of class at most 2 and so M_i is a regular q_i-group [3, p. 183]. Then the elements of order at most q_i form a characteristic subgroup K_i of M_i. Let $K = K_1 \times K_2 \times \cdots$ An automorphism of M_i of order prime to q_i centralizes K_i only if it is the identity automorphism [7, Hilfssatz 1.5]. Therefore no 2-element of H, except for the identity, centralizes K. Hence K cannot be abelian (since h^2 is a nonidentity 2-element of H) and so K must be of class 2.

We now are prepared to define the subgroup H_1. For this purpose decompose V for each K_i into the sum

$$V = V_{i1} \oplus V_{i2} \oplus \cdots$$

where the V_{ij} are the minimal characteristic $F - K_i$ submodules. Let

$C_{ij} = \{x \mid x \in H \text{ and } (K_i, x) = 1 \text{ on } V_{ij}\}$. C_{ij} is a normal subgroup of H although not necessarily normal in G.

Take H_1 to be the intersection of all the C_{ij} which contain h^2. If h^2 is not in any C_{ij} then set H_1 equal to H. In any event $H_1 \lhd H$ and H_1 is normalized by g. As was the case with H, H_1 has no normal 2-subgroup greater than the identity and the greatest normal $2'$-subgroup is $H_1 \cap N$.

Now let P be a 2-subgroup of H_1 such that P and g belong to the same Sylow 2-subgroup of G and $P(H_1 \cap N)/(H_1 \cap N)$ is the largest normal 2-subgroup of $H_1/(H_1 \cap N)$. Since, modulo N, P is normalized by g, it follows that g normalizes P.

LEMMA 3.4. *If $x \in P$, then $(h^2, x) = (h, x)^2$.*

Proof. First we show that this lemma finishes the proof of Theorem B: h normalizes P so that $(h, x)^2 \in \Phi(P)$ where $\Phi(P)$ is the Frattini subgroup of P. Thus the lemma implies that h^2 centralizes $P/\Phi(P)$. Therefore from [4] we conclude that $h^2 \in P$. Since h^2 is in the greatest normal 2-subgroup of $H_1/(H_1 \cap N)$, it follows that h^2 is in the greatest normal 2-subgroup of $H/(H \cap N)$ from which the conclusion of Theorem B follows.

To prove the lemma, let $k = (h^2, x)(h, x)^{-2}$ and suppose $k \neq 1$. Since k cannot centralize K, (K_i, k) is not the identity on some V_{ij}. Since $k \in H_1$, we must have (K_i, h^2) also not the identity on V_{ij}. (This last statement is the motivation for our choice of H_1).

In what follows let $V' = V_{ij}$, $q = q_i$, and Q, x_1, k_1 the restrictions of K_i, x, k, respectively, to V'. Let g^{2m-n} be the first power of g fixing V' and let g_1 be the restriction of g^{2m-n} to V'. Now h^2 is not the identity on V' and [4, p. 13] g_1 must be exceptional

$$\text{(i.e.,}\quad (g_1 - 1)^{2^{n-1}} = 0) ,$$

and thus n must be at least 2. Let $h_1 = g_1^{2^{n-2}}$. $k_1 = (h_1^2, x_1)(h_1, x_1)^{-2}$ and both (Q, h_1^2) and (Q, k_1) are not the identity.

Since g_1 is exceptional and $(Q, h_1^2) \neq 1$, Q cannot be abelian. Thus Q must be of class 2. V' is the sum of absolutely irreducible $F - Q$ submodules all of which are operator isomorphic to each other. Hence $Z(Q)$, the center of Q, is cyclic and is generated by a scalar matrix. Since Q is of exponent q and $Q' \neq 1$, we see that

$$Z(Q) = Q' = \Phi(Q)$$

and so Q is an extra-special q-group [4, p. 15]. We note also that if S is the 2-group generated by x_1 and g_1, then $(Z(Q), S) = 1$ since $Z(Q)$ is generated by a scalar matrix.

Now let V'' be an irreducible $F-QS$ submodule of V'. V'' is an irreducible $F-Q$ module [4, Lemma 2.2.3], and V' is the sum of $F-Q$ modules operator isomorphic to V''. Thus $(Q, h_1^2) \neq 1$ on V'' and g_1 is exceptional on V''. From [4, Theorem 2.5.4] we have the following:

(1) $2^n - 1$ is a power of q, and

(2) if g_1 is faithfully and irreducibly represented on Q_1/Q' (such a Q_1 can always be found since h^2 is not the identity on Q/Q'), then Q can be written as the central product of Q_1 and a group Q_2 and g_1 transforms Q_2 trivially. It now follows [6] that $2^n - 1 = q$ and $|Q_1/Q'| = q^2$.

The representation of Q on V'' is isomorphic to the representation of Q on V' so that $(g_1, Q_2) = 1$ on V'' implies that $(g_1, Q_2) = 1$. Thus the centralizer of g_1 in the space Q/Q' has co-dimension 2 over $GF(q)$. The minimal equation of h_1 on Q_1/Q' must be $t^2 + 1 = 0$ so that h_1^2 must have the representation

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

on Q_1/Q'. We now can conclude that for every power of g_1 (except for the identity, of course), the co-dimension of its centralizer in Q/Q' is 2. Also, since $q \equiv 3 \pmod 4$, $GF(q)$ contains no primitive 4th root of unity. Thus if $n \neq 2$ then in the completely reduced representation of g_1^2 on Q/Q' there is only one nontrivial block. If $n = 2$, there are two nontrivial blocks.

Now if c is a generator of Q', define $\rho(a, b)$ for $a, b \in Q$ by the equation

$$(a, b) = c^{\rho(a,b)} .$$

$\rho(a, b)$ is bilinear and skew symmetric and gives Q/Q' the structure of a symplectic space over $GF(q)$ [4].

ρ is of maximum rank since $Q' = Z(Q)$ so Q/Q' must have dimension $2r$. Since $(S, Q') = 1$, S preserves the symplectic structure of Q/Q'. Thus the representation of S on Q/Q' may be considered as a subgroup of a Sylow 2-subgroup of the symplectic group on Q/Q'.

Q/Q' is of dimension $2r$ over $GF(q)$ so that Q/Q' can be provided with the structure of a vector space U of dimension r over $GF(q^2)$. If $u_1, \cdots, u_r$ is a basis for U, the expression [4]

$$\rho(\Sigma\alpha_i u_i, \Sigma\beta_i u_i) = \Sigma(\alpha_i\beta_i' - \alpha_i'\beta_i)/\gamma ,$$

where $\alpha' = \alpha^q$ and γ is a primitive 4th root of unity, is a skew symmetric bilinear form on U of rank $2r$ with values in $GF(q)$.

Let θ be a primitive 2^{n+1}-th root of unity in $GF(q^2)$ and let T be the group of transformations of $GF(q^2)$ generated by the two transformations $\alpha \to \theta^2\alpha$ and $\alpha \to \theta\alpha'$. All transformations $\boldsymbol{y}$ of U of the form

$$\boldsymbol{y}(\Sigma\alpha_i u_i) = \Sigma(T_i\alpha_i)u_{\sigma(i)} ,$$

where the T_i are taken from T and σ is a permutation taken from a Sylow 2-subgroup of the symmetric group on the numbers $1, 2, \cdots, r$, form a Sylow 2-subgroup of the Symplectic group on Q/Q' [4].

Thus we may assume that $\boldsymbol{x}_1, \boldsymbol{g}_1, \boldsymbol{h}_1$, the representations of x_1, g_1, h_1, respectively, on Q/Q', are of this form. Since $(Q, h_1^2) \neq 1$ and $(Q, k_1) \neq 1$, we have $\boldsymbol{h}_1^2 \neq 1$ and $(\boldsymbol{h}_1^2, \boldsymbol{x}_1) \neq (\boldsymbol{h}_1, \boldsymbol{x}_1)^2$. We now need more information on $\boldsymbol{g}_1$.

LEMMA 3.5. *The permutation σ associated with $\boldsymbol{g}_1$ is the identity permutation.*

Proof. σ is of order less than the order of g_1 from [4, p. 23]. First suppose σ is of order > 2. Then $n > 2$ and so the representation of g_1^2 on Q/Q' has only one nontrivial irreducible block. But the permutation associated with $\boldsymbol{g}_1^2$ is σ^2 which has at least 2 disjoint nontrivial cycles. Clearly this is a contradiction. Thus $\sigma^2 = 1$.

Now suppose $\sigma \neq 1$. Assume, say, $\sigma(1) = 2$, $\sigma(2) = 1$. The representation of g_1 on Q/Q' has only one nontrivial irreducble block so $\boldsymbol{g}_1$ must be the identity on

$$\sum_{i \neq 1,2} \alpha_i u_i .$$

Now $\boldsymbol{g}_1^2(\alpha_1 u_1 + \alpha_2 u_2) = T_2T_1\alpha_1 u_1 + T_1T_2\alpha_2 u_2$ and so one of T_2T_1 or T_1T_2 must not be the identity of T. But then neither one can be the identity. Therefore the representation of Q/Q' would have 2 nontrivial irreducible blocks. This can happen only if $n = 2$. This implies that T_2T_1 and T_1T_2 are of order 2 and thus must equal the transformation $\alpha \to -\alpha$. (This is the only element of order 2 in T.) Thus the centralizer of g_1^2 in Q/Q' has co-dimension 4 over $GF(q)$ whereas it should have co-dimension 2. This proves that $\sigma = 1$.

Hence $\boldsymbol{g}_1$ fixes each u_i and must act trivially on α_i for all but one value of i, $i = 1$, say. Therefore

$$\boldsymbol{g}_1(\Sigma\alpha_i u_i) = A\alpha_1 u_1 + \sum_{i \neq 1} \alpha_i u_i$$

where A is an element of order 2^n in T. Then

$$\boldsymbol{h}_1(\Sigma\alpha_i u_i) = A^{2^{n-2}}\alpha_1 u_1 + \sum_{i \neq 1} \alpha_i u_i ,$$

and

$$\boldsymbol{h}_1^2(\Sigma\alpha_i u_i) = -\alpha_1 u_1 + \sum_{i\neq 1}\alpha_i u_i .$$

We may assume that

$$\boldsymbol{x}_1(\Sigma\alpha_i u_i) = \Sigma T_i \alpha_i u_{\pi(i)} .$$

Case 1. $\pi(1) \neq 1$. Assume, say, that $\pi^{-1}(1) = 2$. Straight forward calculation yields

$$(\boldsymbol{h}_1, \boldsymbol{x}_1)(\Sigma\alpha_i u_i) = A^{-2^{n-2}}\alpha_1 u_1 + T_2^{-1}A^{2^{n-2}}T_2\alpha_2 u_2 + \sum_{i\neq 1,2}\alpha_i u_i .$$

But $A^{2^{n-1}}$ is the unique element of order 2 in T. Thus

$$(\boldsymbol{h}_1, \boldsymbol{x}_1)^2(\Sigma\alpha_i u_i) = -\alpha_1 u_1 - \alpha_2 u_2 + \sum_{i\neq 1,2}\alpha_i u_i$$

and it is easily verified that this is the same result as $(\boldsymbol{h}_1^2, \boldsymbol{x}_1)$.

Case 2: $\pi(1) = 1$. In this case we easily find that $(\boldsymbol{h}_1^2, \boldsymbol{x}_1)$ is the identity while

$$(\boldsymbol{h}_1, \boldsymbol{x}_1)^2(\Sigma\alpha_i u_i) = (A^{2^{n-2}}, T_1)^2\alpha_1 u_1 + \sum_{i\neq 1}\alpha_i u_i .$$

Now the group T easily is seen to be a generalized quaternion group of order 2^{n+1} so that the only conjugates of A in T are A and A^{-1}. Thus

$$(A^{2^{n-1}}, T_1)^2 = A^{2^{n-1}}T_1^{-1}(A^{2^{n-1}})T_1 = 1 .$$

Thus $(\boldsymbol{h}_1, \boldsymbol{x}_1)$ is also the identity.

Therefore it has been shown that

$$(\boldsymbol{h}_1, \boldsymbol{x}_1)^2 = (\boldsymbol{h}_1^2, \boldsymbol{x}_1)$$

in all cases. This completes the proof of lemma 3.4, and, by a previous argument, Theorem B now is proved.

4. Groups with $e_2 = 2$.

If G is a solvable group whose Sylow 2-groups are of exponent 4, then we know from Theorem A that $l_2(G) \leqq 3$. We now investigate conditions for $l_2(G) \leqq 2$ to hold. The argument is similar to that used in proving Theorem B, but a more restrictive hypothesis is needed. That no loss of generality is involved in assuming the stronger hypothesis is insured by the following reduction theorem, which is stated in a slightly more general form than needed.

A proposition R will be said to be of type 4.1 if it is of the following form:

If G is a finite p-solvable group satisfying condition C, then

$l_p(G) \leqq f(e_p(G))$, where f is a monotonically increasing function defined for nonnegative integral arguments, $f(0) = 0$, and condition C either is vacuous or states that $e_{p_i}(G) \leqq a_i$ for some set, possibly infinite, of primes p_i and nonnegative integers a_i.

Note that the proposition that $l_2(G) \leqq e_2(G)$ if G is a finite solvable group satisfying $e_2(G) \leqq 2$ is of type 4.1. One of the results of this section is that $l_2(G) \leqq e_2(G)$ if G is a finite group of exponent 12. This statement is also of type 4.1 since the condition that G be of exponent 12 is equivalent to stating that $e_2(G) \leqq 2$, $e_3(G) \leqq 1$, and $e_p(G) \leqq 0$ for all other primes.

THEOREM 4.1. *To prove a proposition R of type* 4.1 *it is sufficient to prove the proposition for the following special case*:

(1) *G is the normal product of V by G_1 where V is a vector space over F, a finite field of characteristic p, and G_1 is a p-solvable linear group on V having no normal p-subgroup other than the identity.*

(2) *Any irreducible representation of any p'-subgroup of G_1 over F is in fact absolutely irreducible.*

(3) *All groups of order at most $|G_1|$ satisfy R.*

(4) *V is an irreducible $F - G_1$ module.*

Proof. In proving this theorem we assume R is valid for the special case and then prove it is valid for the general case.

Now suppose G is the group of smallest order which satisfies the hypothesis but not the conclusion of R, and let

$$1 = P_0 \leqq N_0 < P_1 < \cdots < P_l \leqq N_l = G$$

be the upper p-series of G. Since $f(0) = 0$ we must have $l_p(G) > 0$. If F_1/N_0 is the Frattini subgroup of P_1/N_0, then, as is shown in [4], $l_p(G/F_1) = l_p(G)$ so that if $F_1 \neq 1$ we would have a proper factor group of G satisfying the hypothesis but not the conclusion of R.

Hence assume $F_1 = 1$. Thus P_1 is an elementary abelian p-group which we identify with a vector space V_1 over $GF(p)$. G/P_1 is faithfully represented as a linear group G_1 on V_1 and G_1 has no normal p-group greater than the identity.

From [4, p. 4] we may assume that G has only one minimal normal subgroup. This subgroup must be contained in V_1 and we denote it with M. If $M \neq V_1$ and G_1 is faithfully represented on V_1/M then we have $l_p(G/M) = l_p(G)$ so that we would have a contradiction to the minimality of G.

Now suppose $M \neq V_1$ and G_1 is not faithfully represented on V_1/M.

Then the elements of G_1 centralizing V_1/M form a normal subgroup of G_1 greater than the identity. If Q is a minimal normal subgroup of G_1 centralizing V_1/M, then Q must be a p'-group so that V as a Q-module is completely reducible. Thus there exists a Q-module M_1 such that $V_1 = M \oplus M_1$. Q is the identity on M_1 but not on M since Q is faithfully represented on V_1. Now if M_2 is the centralizer of Q in V_1 then M_2 is normal in G, M_2 is not the identity, and M_2 does not contain M. This contradicts the minimality of M.

Thus we see that $M = V_1$ which implies that G_1 is irreducibly represented on V_1. A consequence of this is that if H is any normal subgroup greater than the identity in G_1 then H can have no nonzero fixed vector in V_1 Otherwise all the vectors fixed by H would form a nontrivial submodule of V_1.

Now pick F to be a large enough finite extension of $GF(q)$ such that any irreducible representation of any p'-subgroup of G_1 over F is absolutely irreducible. Let $1 = \theta_0, \theta_1, \cdots, \theta_r$ be a basis for F over $GF(p)$ and let $v_1, v_2, \cdots, v_s$ be a basis for V_1 over $GF(p)$. Finally let V be the vector space over F with basis $v_1, \cdots, v_s$, i.e., the vectors of V are the formal sums

$$\sum_{j=1}^{s} \sum_{i=0}^{r} c_{ij}\theta_i v_j$$

where $c_{ij} \in GF(p)$. G_1 acts on V in the obvious way.

Consider the group $G^* = G_1 V$, i.e., the normal product of V by G_1. If g^* is of order p^m in G^* then either the image g of g^* in G_1 is of order p^m or g is of order p^{m-1} and g is not exceptional on V. In the latter case $(g-1)^{p^{m-1}}v_i \neq 0$ for some v_i from which it follows that g is not exceptional on V_1. Thus $e_p(G) \geqq (m-1) + 1 = m$.

Therefore in any event $e_p(G) \geqq e_p(G^*)$. Since $e_q(G^*) = e_q(G)$ for $q \neq p$, G^* satisfies condition C. Furthermore $l_p(G) = l_p(G^*)$ so that if G^* satisfies R so does G.

Now suppose H is any normal p'-subgroup other than the identity in G_1 and suppose

$$v = \sum_{j=1}^{s} \sum_{i=0}^{r} c_{ij}\theta_i v_j$$

is a nonzero vector fixed by H. Since $v \neq 0$ the coefficient of v_j is not zero for some j, $j = 1$ say. Then there exists $\alpha \in F$ such that $\alpha(\sum_{i=0}^{r} c_{i1}\theta_1) = 1$. H must fix αv which can be written in the form $\alpha v = v' + v''$ where

$$v' = v_1 + \sum_{j=2}^{s} c'_{0j}v_j,\ v'' = \sum_{j=2}^{s} \sum_{i=1}^{r} c'_{ij}\theta_i v_j \ .$$

For H to fix αv it must also fix v' which contradicts the fact that

H has no nonzero fixed vector in V_1. Thus H has no nonzero fixed vector in V.

If V is an irreducible $F - G_1$ module then we have arrived at the special case of the theorem. Therefore assume U is a proper submodule.

If G_1 is not faithfully represented on V/U, then let Q be a minimal normal subgroup of G_1 centralizing V/U. Q must be a p'-group so that V is completely reducible as an $F - Q$ module. Thus there exists a nontrivial $F - Q$ submodule on which Q is the identity. This is impossible since Q can have no nonzero fixed vector.

Hence G_1 is faithfully represented on V/U. Thus $l_p(G^*) = l_p(G^*/U)$ and, of course, $e_p(G^*) \geqq e_p(G^*/U)$ so that if G^*/U satisfies R so does G^* and then so does G.

We still have that any normal nonidentity p'-subgroup H of G_1 has no nonzero fixed vector in V/U since V is completely reducible as an $F - H$ module. Therefore if G_1 is not irreducibly represented on V/U then the same argument as before yields that G_1 is faithfully represented on a nontrivial factor module of V/U. Continuing in this way we ultimately arrive at the special case where G_1 is faithfully and irreducibly represented on some vector space over the field F. This finishes the proof of Theorem 4.1.

Among the results we now shall prove is that if G is of exponent 12 then $l_2(G) \leqq e_2(G)$. Before doing this it might be well to justify this work. For in a group of order 2^a3^b the 2-length and the 3-length can vary at most by one. Thus if it were true that the 3-length of a group of exponent 12 was one, then it would be trivial to state that the 2-length was at most two. However in [5, p. 5] is found an example of a group of exponent 12 but with 3-length two.

For the rest of this paper we make the following standing assumptions.

(1) $G = G_1V$, the normal product of V by G_1, where V is a vector space over a finite field F of characteristic 2 and G_1 is a finite, solvable linear group having no normal 2-subgroup other than the identity.

(2) V is an irreducible $F - G_1$ module.

(3) Any representation over F of any p'-subgroup of G_1 is absolutely irreducible.

(4) $e_2(G) \leqq 2$.

We are interested in seeing under what conditions can $l_2(G)$ exceed $e_2(G)$. But if $e_2(G_1) = 0$ then both $e_2(G)$ and $l_2(G)$ are 1, and if $e_2(G_1) = 1$ then $l_2(G_1) = 1$ so that $l_2(G) = e_2(G) = 2$. Thus we may as well assume

(5) $e_2(G_1) = 2$.

Later we shall add to these assumptions the further one that G is of exponent 12. Actually, until we restrict ourselves to groups of exponent 12, we will make no use of the fact that G_1 is irreducibly represented on V.

Now let N be the largest normal $2'$-subgroup of G_1. We shall show that a certain 2-subgroup, to be described later, must be contained in the greatest normal 2-subgroup of G_1/N. In particular if $l_2(G) > 2$ (which is the same as $l_2(G_1) > 1$), we shall see that there must exist an element of order 4 of a special type in G_1.

First let H be the following normal subgroup of G_1: $x \in H$ if, and only if, for every normal nilpotent subgroup Q of class at most 2 in G_1, x fixes every minimal characteristic $F - Q$ submodule of V. A normal nilpotent subgroup of G_1 must be a $2'$-group so that V splits into the sum of minimal characteristic $F - Q$ modules.

From (5) there are elements of order 4 in G_1, and from (4) all such elements must be exceptional. Thus if g is of order 4 in G_1 then g^2 must be in H by lemma 3.1. Hence H is greater than the identity. H has no normal 2-subgroup except for the identity and the largest normal $2'$-subgroup is $H \cap N$.

Let D be the greatest normal nilpotent subgroup of H. $D = D_1 \times D_2 \times \cdots$ where D_i is a Sylow q_i-subgroup of D for an odd prime q_i. H centralizes any normal abelian subgroup of G_1 so that, by the proof of Lemma 3.3, we obtain $c(D) = 2$. Now, as before, let K_i be the subgroup of D_i consisting of all elements of order at most q_i and let $K = K_1 \times K_2 \times \cdots$ We again have that no non-identity 2-element of H centralizes K.

Now take H_1 to be the subgroup of G_1 consisting of all elements which fix every minimal characteristic $F - K_i$ module for all i. $H_1 \lhd G_1$, and, since $c(K_i) \leqq 2$, $H \leqq H_1$. H_1 has no normal 2-subgroup except for the identity and its greatest normal $2'$-subgroup is $H_1 \cap N$.

Let P be a Sylow 2-subgroup of H_1. $P \neq 1$ since if g is any element of order 4 in G_1 then $g^2 \in H$. Now the square of any element of P must be in H. Thus $P/(P \cap H)$ is of exponent 2 and thus abelian. Therefore $P' < H$. We now prove two lemmas which enable us to show directly that PN/N is normal in G_1/N.

LEMMA 4.2. *Suppose that g and h are two elements of P and V' is a minimal characteristic $F - K_i$ submodule of V. Let Q, g_1, and h_1 be the restrictions of K_i, g, and h, respectively, to V'. Then if $(Q, h_1^2) = 1$ it follows that $(Q, (g_1, h_1)) = 1$.*

Proof. Assume $(Q, (g_1, h_1)) \neq 1$. Therefore neither g_1 nor h_1 centralizes Q. If $(Q, g_1^2) = 1$, then straight forward calculation yields

$$(Q, (g_1h_1)^2) = (Q, (g_1, h_1)) \neq 1 ,$$
$$(Q, (g_1h_1, h_1)) = (Q, (g_1h_1)^{-1}) \neq 1 .$$

Thus, replacing g_1 by g_1h_1 if $(Q, g_1^2) = 1$, we may assume that $(Q, g_1^2) \neq 1$ along with $(Q, h_1^2) = 1$ and $(Q, (g_1, h_1)) \neq 1$.

Now exactly as in the proof of Lemma 3.4 we obtain that Q is an extra special q-group (actually $q = 3$ since g_1 is of order 4 and thus exceptional so that $4 - 1$ must be a power of q), Q/Q' is a symplectic space, g_1 and h_1 preserve the symplectic structure of Q/Q', and we may assume that g_1 and h_1 operate on Q/Q' as follows:

$$\boldsymbol{g}_1(\Sigma\alpha_i u_i) = A\alpha_1 u_1 + \sum_{i\neq 1} \alpha_i u_i ,$$
$$\boldsymbol{h}_1(\Sigma\alpha_i u_i) = \Sigma T_i\alpha_i u_{\sigma(i)} ,$$

where σ is a permutation of order $\leqq 2$ (since $(Q, h_1^2) = 1$), and A and the T_i are chosen from a group isomorphic to the quaternion group of order 8 (since $q = 3$). In addition A must be of order 4 since $(Q, g_1^2) \neq 1$.

If σ does not fix 1 then $(\boldsymbol{g}_1, \boldsymbol{h}_1)$ would be of order 4 but its centralizer in Q/Q' would have co-dimension 4 over $GF(3)$. Thus $(\boldsymbol{g}_1, \boldsymbol{h}_1)$ would be of order 4 but not exceptional which is impossible.

Hence σ fixes 1 and, since $(Q, h_1^2) = 1$, we must have

$$\boldsymbol{h}_1(\Sigma\alpha_i u_i) = \pm\alpha_1 n_1 + \sum_{i\neq 1} T_i\alpha_i u_{\sigma(i)} .$$

It is now an easy matter to verify that $(\boldsymbol{g}_1, \boldsymbol{h}_1) = 1$ and the lemma is proved.

COROLLARY. *If $g, h \in P$ and $h^2 = 1$, then $(g, h) = 1$.*

Proof. (g, h) is in P' and thus in H. So if $(g, h) \neq 1$ then $(K_i, (g, h)) \neq 1$ for some K_i. Then lemma states that this cannot happen.

LEMMA 4.3. *If $g, h \in P$, then $(g, h)^2 = 1$.*

Proof. Suppose that $(g, h)^2 \neq 1$. Then for some K_i, $(K_i, (g, h)^2) \neq 1$. Choose V' to be a minimal characteristic $F - K_i$ submodule of V such that $(K_i, (g, h)^2)$ is not the identity on V'. If Q, g_1, and h_1 are defined as in the previous lemma, then, if either (Q, g_1^2) or (Q, h_1^2) is the identity, $(g_1, h_1) = 1$. Therefore assume neither g_1^2 nor h_1^2 centralize Q. Thus g_1 and h_1 are both exceptional of order 4. Q is an extra-special

3-group and we may assume g_1 and h_1 operate on Q/Q' as follows:

$$\boldsymbol{g}_1(\Sigma\alpha_i u_i) = A\alpha_1 u_1 + \sum_{i\neq 1}\alpha_i u_i \ ,$$
$$\boldsymbol{h}_1(\Sigma\alpha_i u_i) = B\alpha_j u_j + \sum_{i\neq j}\alpha_i u_i \ .$$

Now if $j \neq 1$ then $(\boldsymbol{g}_1, \boldsymbol{h}_1) = 1$ and if $j = 1$ then

$$(\boldsymbol{g}_1, \boldsymbol{h}_1)^2(\Sigma\alpha_i u_i) = (A, B)^2\alpha_1 u_1 + \sum_{i\neq 1}\alpha_i u_i \ .$$

But A and B are elements of a quaternion group so that $(A, B)^2$ is the identity and the lemma is proved.

THEOREM 4.4. $PN/N \lhd G_1/N$.

Proof. We shall prove that $P(H_1 \cap N)/(H_1 \cap N) \lhd H_1/(H_1 \cap N)$ which is equivalent to the theorem since $H_1 \lhd G_1$.

Let P_1 be the subgroup of P such that $P_1(H_1 \cap N)/(H_1 \cap N)$ is the largest normal 2-subgroup of $H_1/(H_1 \cap N)$. $P_1 \lhd P$ and P_1 contains the center of P [4, Lemma 1.2.3]. Thus by the corollary to Lemma 4.2, P_1 contains all elements of order 2 in P. The elements of order 2 in P form an elementary abelian group P_2 which is normal, modulo $H_1 \cap N$, in H_1. The elements of $H_1/(H_1 \cap N)$ which centralize both P_2 and P_1/P_2 form a normal subgroup of $H_1/(H_1 \cap N)$. But if any $2'$-element centralized both P_2 and P_1/P_2, then, as easily may be seen, this element would centralize P_1 contrary to the fact [4, Lemma 1.2.3] that P_1 contains its centralizer in $H_1/(H_1 \cap N)$. Thus the elements centralizing both P_2 and P_1/P_2 form a normal 2-subgroup of $H_1/(H_1 \cap N)$, and from the corollary to Lemma 4.2 and from Lemma 4.3, P must be contained in this normal 2-subgroup. But P is a Sylow 2-subgroup of H_1 and thus it follows that, modulo $H_1 \cap N$, P is normal in H_1.

COROLLARY. $l_2(H_1) = 1$.

Now let S be a Sylow 2-subgroup of G_1 which contains P. From the theorem it follows that P is normal in S.

LEMMA 4.5. *If P contains all elements of order* 4 *in S, then* $l_2(G_1) = 1$.

Proof. If $S = P$ we are done. Therefore assume $S \neq P$. Then if $x \in S - P$ we must have $x^2 = 1$. Also $x \in S - P$, $y \in P$ imply that $xy \in S - P$ so that $(xy)^2 = 1$ which implies that $x^{-1}yx = y^{-1}$. Thus x induces the automorphism $y \to y^{-1}$ of P. This can be an automorphism only if P is abelian. Now if both x_1 and x_2 are in $S - P$ then x_1x_2

centralizes P. But $e_2(G_1) = 2$ so that P does contain elements of order 4. Hence x_1x_2 cannot be in $S - P$.

Therefore $|S/P| = 2$ and P is abelian. Now if $x \in S - P$, $y \in P$, then $(x, y) = x^{-1}y^{-1}xy = y^2 \in \Phi(P)$ and thus x centralizes $P/\Phi(P)$. Hence [4, Lemma 1.2.5] PN/N connot be the largest normal 2-subgroup of G_1/N. But P is maximal in S so that SN/N must be the largest normal 2-subgroup of G_1/N. This implies that $l_2(G_1) = 1$.

To our assumptions (1)$\sim$(5) we now add

(6) G is of exponent 12.

This implies that K must be a group of exponent 3 and class at most 2. We prove that $l_2(G_1) = 1$ in this case by showing that the hypothesis of Lemma 4.5 are satisfied.

For this purpose assume that g is an element of order 4 in $S - P$. g^2 is in H so $(K, g^2) \neq 1$. Let $V = V_1 \oplus V_2 \oplus \cdots$ be the decomposition of V into minimal characteristic $F - K$ modules. Since $g \in S - P$, g does not fix some V_i. g^2 does fix each V_i and if g^2 is not the identity on a V_i then g must fix that V_i for otherwise g could not be exceptional [4, p. 13]. We now need the following result:

LEMMA 4.6. *There exist x and y in K such that* $((x, g^2), (y, g^2)) \neq 1$.

Proof. Let $C = \{x \mid x \in K, (x, g^2) \in Z(K)\}$. Clearly $C \geqq Z(K)$ but $C \neq K$ since then g^2 would centralize $Z(K)$ and $K/Z(K)$ which would imply that $(K, g^2) = 1$. (g^2 centralizes $Z(K)$ by Lemma 3.1 and 3.2.) $K/Z(K)$ is an elementary abelian 3-group so that there must be a $GF(3) - g$ module of $K/Z(K)$ complementary to $C/Z(K)$. Thus $K/Z(K) = L/Z(K) \oplus C/Z(K)$ and g normalizes L. For all $x \in L - Z(K)$, (x, g^2) is not in $Z(K)$.

Now suppose $x, y \in L - Z(K)$ and $(x, g^2)(y, g^2)^{-1} \in Z(K)$. Since $K/Z(K)$ is abelian, straight forward calculation yields

$$(xy^{-1}, g^2) \equiv (x, g^2)(y^{-1}, g^2) \pmod{Z(K)},$$
$$1 = (yy^{-1}, g^2) \equiv (y, g^2)(y^{-1}, g^2) \pmod{Z(K)}.$$

Thus $(xy^{-1}, g^2) \equiv (x, g^2)(y, g^2)^{-1} \equiv 1 \pmod{Z(K)}$. This implies that $xy^{-1} \in Z(K)$. Therefore we have shown that $(x, g^2) \equiv (y, g^2) \pmod{Z(K)}$ if, and only if, $x \equiv y \pmod{Z(K)}$ for $x, y \in L$.

It immediately follows from this that for any $x \in L$, there exists a y such that $x \equiv (y, g^2) \pmod{Z(K)}$. Now L cannot be abelian since g normalizes L and g^2 does not centralize it. From all this we see that there exist $x, y \in L$ such that $((x, g^2), (y, g^2)) \neq 1$.

Now taking x and y to satisfy the lemma, we may assume without loss of generality that $((x, g^2), (y, g^2))$ is not the identity on V_1. This implies that g^2 is not the identity on V_1 so g must fix V_1.

Since g does not fix every V_i, assume g does not fix V_2. Therefore g^2 is the identity on V_2 which then also must be the case for (x, g^2) and (y, g^2).

V is an irreducible $F - G_1$ module so that there must be an element taking V_1 into V_2. Such an element must be of the form zh where $h \in S$ and z is from a Sylow 3-subgroup of G_1 which necessarily must contain K. We shall derive a contradiction by showing that z and K generate elements of order 9 which is impossible in a group of exponent 12.

If $hV_1 = V_m$ then $zV_m = V_2$. Set $g_1 = hgh^{-1}$. Then

$$((x^{h^{-1}}, g_1^2), (y^{h^{-1}}, g_1^2))$$

is not the identity on V_m. Now suppose $g_1 V_2 = V_2$. Then $gh^{-1}V_2 = h^{-1}V_2$, and, since $gV_2 \neq V_2$, this implies that $h^{-1}V_2 = V_j, j \neq 2$. Then we would have $gV_j = V_j$. But $gh^{-1} \in S$ so that $(gh^{-1})^2 \in H$. Thus $(gh^{-1})^2$ fixes V_2 and, therefore, $gh^{-1}V_j = V_2$. $(h^{-1})^2$ also must fix V_2 so we have $h^{-1}V_j = V_2$. From this we conclude that $V_2 = gh^{-1}V_j = gV_2$ which is a contradiction. Hence $g_1 V_2 \neq V_2$. A consequence of this is that $V_m \neq V_2$ for $V_m = V_2$ would imply that $h^{-1}V_2 = V_1$ which would imply that $g_1 V_2 = hgV_1 = V_2$. Since $V_m \neq V_2$ it follows that z is not the identity and so is of order 3.

If we replace V_1, g, x, and y by $V_m, g_1, x^{h^{-1}}$, and $y^{h^{-1}}$, respectively, we may assume that $zV_1 = V_2, gV_2 \neq V_2$, and $((x, g^2), (y, g^2))$ is not the identity on V_1. Let $x_1 = (x, g^2)$ and $y_1 = (y, g^2)$. x_1 and y_1 must be the identity on V_2 since g_2 is. Since z is of order 3, we have $zV_1 = V_2$, $zV_2 = V_n (n \neq 1, 2)$, and $zV_n = V_1$.

Let $V' = V_1 \oplus V_2 \oplus V_n$. V' is fixed by z and the restrictions of x_1, y_1, and z to V' are

$$z = \begin{pmatrix} 0 & 0 & A \\ B & 0 & 0 \\ 0 & C & 0 \end{pmatrix}, \quad x_1 = \begin{pmatrix} M & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & M_1 \end{pmatrix}, \quad y_1 = \begin{pmatrix} N & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & N_1 \end{pmatrix},$$

where I is the identity and 0 the zero matrix. Now (x_1, y_1) is not the identity on V_1 but $(x_1, y_1) \in Z(K)$ and $Z(K)$ is represented on V_1 as a cyclic group generated by a scalar matrix. Thus $(M, N) = \omega I$ where ω is a primitive third root of unity. From $z^3 = 1$ we obtain $C = A^{-1}B^{-1}$.

Now z, x_1, and y_1 all belong to the same Sylow 3-subgroup of G_1. Thus $(zx_1)^3 = (zy_1)^3 = 1$. From this direct computation yields that $M_1 = A^{-1}M^{-1}A$, $N_1 = A^{-1}N^{-1}A$. Thus $(M_1, N_1) = A^{-1}(M^{-1}, N^{-1})A$. But M and N generate a group of exponent 3 and class 2. It follows easily that $(M^{-1}, N^{-1}) = (M, N) = \omega I$. Thus

$$(x_1, y_1) = \begin{pmatrix} \omega I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \omega I \end{pmatrix}.$$

It is now a simple matter to verify that $(z(x_1, y_1))^3 \neq 1$. Hence $z(x_1, y_1)$ is a 3-element of order greater than 3 which is impossible in a group of exponent 12. This contradiction proves that the hypothesis of Lemma 4.5 is satisfied and thus:

THEOREM 4.7. *If G is a finite group of exponent* 12, *then* $l_2(G) \leqq e_2(G)$.

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