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A SET OF NONNORMAL NUMBERS

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Let P be the set of real polynomials and let $E(P)$ be the the set of real numbers whose nth binary digit from a certain point on is 0 or 1 according as $[\varphi(n)]$ is even or odd for some $\varphi \in P$. We prove that no number in $E(P)$ is normal in the binary system and that $E(P)$ has Hausdorff dimension 0.

Some notations and definitions. It is well known that every real number x of the unit interval which is not a binary fraction can be expanded in the binary system

$$x = \sum_{n=1}^{\infty} \frac{\varepsilon_n(x)}{2^n}$$

where $(\varepsilon_n(x))_{n \in \mathrm{N}}$ is a uniquely determined sequence of functions taking values 0 or 1. The functions $r_n(x) = 1 - 2\,\varepsilon_n(x)$ are known as the Rademacher functions.

We shall say that x is a normal number (in the binary system) if for every positive integer s and every sequence of positive, strictly increasing integers $k_1, k_2, \cdots, k_s$ one has:

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} r_{n+k_1}(x) \cdots r_{n+k_s}(x) = 0 \ . \tag{1}$$

One can prove that this definition is equivalent to the other usual ones [3], [4], [6].

If t is a real number, $[t]$ will denote the greatest integer not greater than t and $\{t\} = t - [t]$ the fractional part of t.

Let P be the set of real polynomials and let $E(P)$ be the set of points x such that for some $\varphi \in P$ and for some $n_0 \geqq 0$, $r_n(x) = \exp i\pi[\varphi(n)]$ for all integers $n > n_0$.

We wish to prove first the following theorem:

THEOREM 1. *$E(P)$ contains only nonnormal numbers.*

This result shows that the measure of $E(P)$ is null, since almost all numbers are normal. Now the question arises if $E(P)$ contains "almost all" (in a sense soon to be made precise) nonnormal numbers or not. We answer this question by stating the known result:

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The Hausdorff dimension of the set of nonnormal numbers is 1, (see for example [1]),

and by proving our second theorem:

THEOREM 2. *The Hausdorff dimension of $E(P)$ is 0.*

2. Proof of Theorem 1. Let x be an element of $E(P)$. We show that for a certain sequence of increasing positive integers $k_1, k_2, \cdots, k_s$ the equation (1) does not hold.

Let φ be a polynomial such that $r_n(x) = \exp i\pi[\varphi(n)]$ for all sufficiently large integers n. Without loss of generality we may suppose that this relation holds for all positive integers, for normality or nonnormality are asymptotic properties. Let the expansion of φ be

$$(2) \qquad \varphi(n) = \alpha_\mu n^\nu + \alpha_{\nu-1}n^{\nu-1} + \cdots + \alpha_1 n + \alpha_0 , \qquad \nu \geqq 1 .$$

If all the numbers $\alpha_j (1 \leqq j \leqq \nu)$ are rational, then x is clearly rational, hence nonnormal. If one of the numbers $\alpha_j (1 \leqq j \leqq \nu)$ is irrational, we can without loss of generality suppose that the leading coefficient α_ν is irrational. Indeed, suppose that $\alpha_\mu (1 \leqq \mu < \nu)$ is irrational and that $\alpha_{\mu+1}, \cdots, \alpha_\nu$ are rational. Let q be the least common denominator of the $\nu - \mu$ fractions $\alpha_{\mu+1} \cdots, \alpha_\nu$. If x is normal, then so is the number y defined by $r_n(y) = \exp i\pi[\varphi(2qn)]$ for all integers n. But clearly $[\varphi(2qn)] \equiv [\psi(n)] \pmod 2$ where $\psi(n) = \alpha_\mu(2q)^\mu n^\mu + \cdots + \alpha_0$. This shows that we can now deal with ψ, the leading coefficient of which is irrational.

From now on in this section, φ is defined by equation (2) where α_ν is an irrational number.

We need the known identity for polynomials of degree ν:

$$(3) \qquad \varphi(n+\nu) \equiv \binom{\nu}{1}\varphi(n+\nu-1) - \binom{\nu}{2}\varphi(n+\nu-2) + \cdots + (-1)^{\nu-1}\binom{\nu}{\nu}\varphi(n) + \nu\,!\,\alpha_\nu$$

and the lemma:

LEMMA 1. *If $F(x_1, x_2, \cdots, x_\nu)$ is a Riemann integrable function which is of period 1 in each variable and if φ is a real polynomial of degree ν, the leading coefficient of which is irrational, then the following equality holds:*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} F(\varphi(n),\ \varphi(n+1), \cdots, \varphi(n+\nu-1)) = \int_{T^\nu} F(x_1, x_2, \cdots, x_\nu) dx_1, \cdots, dx_\nu .$$

This is a very well known corollary of Weyl's theorems on uniform distribution (see for example [2]).

Combining equality (3) and Lemma 1, one can write:

$$\begin{aligned} L &= \lim \frac{1}{N}\sum_{n=1}^{N} \exp i\pi\Big([\varphi(n+\nu)] \\ &\quad - \binom{\nu}{1}[\varphi(n+\nu-1)] + \cdots + (-1)^{\nu}\binom{\nu}{\nu}[\varphi(n)]\Big) \\ &= \lim \frac{1}{N}\sum_{n=1}^{N} \exp i\pi\Big(\Big[\binom{\nu}{1}\varphi(n+\nu-1) \\ &\quad - \binom{\nu}{2}\varphi(n+\nu-2) + \cdots + \nu\,!\,\alpha_\nu\Big] \\ &\quad - \binom{\nu}{1}[\varphi(n+\nu-1)] + \cdots + (-1)^{\nu}[\varphi(n)]\Big) \\ &= \int_{T^\nu} \exp i\pi\Big(\Big[\binom{\nu}{1}2x_\nu - \binom{\nu}{2}2x_{\nu-1} + \cdots + \nu\,!\,\alpha_\nu\Big] \\ &\quad - \binom{\nu}{1}[2x_\nu] + \cdots + (-1)^{\nu}[2x_1]\Big)dx_1 \cdots dx_\nu\,. \end{aligned}$$

By putting $2x_j = y_j$, $j = 1, 2, \cdots, \nu$, the integral becomes

$$\begin{aligned} L = \frac{1}{2^\nu}\int_{(0,2)^\nu} &\exp i\pi\Big(\Big[\binom{\nu}{1}y_\nu - \binom{\nu}{2}y_{\nu-1} + \cdots + \nu\,!\,\alpha_\nu\Big] \\ &- \binom{\nu}{1}[y_\nu] + \cdots + (-1)^{\nu}[y_1]\Big)dy_1 \cdots dy_\nu\,. \end{aligned}$$

Now the identity $[x + \varepsilon y] = [x] + \varepsilon[y] + [\{x\} + \varepsilon\{y\}]$, $\varepsilon = \pm 1$ shows that one has:

$$\begin{aligned} \Big[\binom{\nu}{1}y_\nu - \binom{\nu}{2}y_{\nu-1} + \cdots + \nu\,!\,\alpha_\nu\Big] &= \binom{\nu}{1}[y_\nu] - \binom{\nu}{2}[y_{\nu-1}] + \cdots + [\nu\,!\,\alpha_\nu] \\ &\quad + \Big[\binom{\nu}{1}\{y_\nu\} - \cdots + \{\nu\,!\,\alpha_\nu\}\Big] \end{aligned}$$

so that:

$$\begin{aligned} L &= \frac{\pm 1}{2^\nu}\int_{(0,2)^\nu} \exp i\pi\Big[\binom{\nu}{1}\{y_\nu\} - \binom{\nu}{2}\{y_{\nu-1}\} + \cdots + \{\nu\,!\,\alpha_\nu\}\Big]dy_1 \cdots dy_\nu \\ &= \pm \int_{T^\nu} \exp i\pi\Big[\binom{\nu}{1}y_\nu - \binom{\nu}{2}y_{\nu-1} + \cdots + \{\nu\,!\,\alpha_\nu\}\Big]dy_1 \cdots dy_\nu\,. \end{aligned}$$

Consider the hyperplane $\binom{\nu}{1}y_\nu - \binom{\nu}{2}y_{\nu-1} + \cdots + (-1)^{\nu-1}y_1 = -\{\nu\,!\,\alpha_\nu\}$ in the euclidean space $\mathbf{R}^\nu$. It has rational coefficients except for the constant term, which is irrational. Hence it cannot split the unit cube $(0, 1)^\nu$ into two regions of equal volume. Therefore the integral L

cannot be 0. Finally we notice that L may be written

$$L = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} r_{n+\nu}(x)(r_{n+\nu-1}(x))^{\binom{\nu}{1}} \cdots r_n(x) ;$$

this completes the demonstration.

3. Proof of Theorem 2. Let P_ν denote the set of real polynomials of degree ν, the coefficients of which are all in the interval $[0, 2[$. It is easily seen that to prove Theorem 2, it is sufficient to prove the lemma:

LEMMA 2. *Let E^ν be the set of numbers x such that for some $\varphi \in P_\nu$, $r_n(x) = \exp i\pi[\varphi(n)]$ for all integers n. Then the Hausdorff dimension of E^ν is* 0.

Let $\qquad \varphi(n) = \alpha_\nu n^\nu + \cdots + \alpha_1 n + \alpha_0, \quad \alpha_j \in [0, 2]$

and let $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_\nu)$ be a point in the space $(0, 2)^{\nu+1}$. We are going to estimate the number $N_\nu(p)$ of regions in $(0, 2)^{\nu+1}$ which have the following property: when α ranges over one of these regions, the sequence $[\varphi(1)], [\varphi(2)], \cdots, [\varphi(p)]$ stays invariant. First let us show:

LEMMA 3. *The h-dimensional measure* $(0 \leqq h \leqq 1)$ *of the set* $E_p^\nu = \{x \mid r_n(x) = \exp i\pi[\varphi(n)];\ n = 1, 2, \cdots, p;\ \varphi \in P_\nu\}$ *satisfies the inquality*

$$h\text{-meas } (E_p^\nu) \leqq \frac{N_\nu(p)}{2^{ph}} .$$

Indeed, when φ runs through P_ν, α ranges over $(0, 2)^{\nu+1}$. The set E_p^ν is composed of at most $N_\nu(p)$ intervals, each of which has h-length $\left(\frac{1}{2^p}\right)^h$.

Now, if one notices that $E^\nu = \bigcap_{p=1}^\infty E_p^\nu$, one gets the result that the Hausdorff dimension of E^ν cannot be greater than

$$\delta = \liminf_{p\to\infty} \frac{\log N_\nu(p)}{p \log 2} .$$

We wish to show that $\delta = 0$ and we shall do so by proving our last lemma:

LEMMA 4. *When p goes to infinity, one has*

$$N_\nu(p) = 0(p^{(\nu+1)^2}) .$$

Proof. Let q be an integer such that

$$0 \leqq q \leqq 2\,(n^\nu + n^{\nu-1} + \cdots + n + 1) - 1$$

Consider the set $R_{n,q}$ of the points $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_\nu)$ defined by

$$q \leqq \alpha_\nu n^\nu + \cdots + \alpha_1 n + \alpha_0 < q + 1.$$

Clearly, when α runs through the region $R_{n,q}$, the quantity $[\varphi(n)] = [\alpha_\nu n^\nu + \cdots + \alpha_1 n + \alpha_0]$ stays equal to q. Then let $q_1, q_2, \cdots, q_p$ be any sequence of integers such that $0 \leqq q_j < 2\ (j^\nu + \cdots + j + 1)$, $j = 1, 2, \cdots, p$. When α ranges over the set $\bigcap_{n=1}^{p} R_{n,q_n}$, the sequence $[\varphi(1)], [\varphi(2)], \cdots, [\varphi(p)]$ does not change. But the number of these regions is at most the number of different regions one can obtain by dissecting the space $\mathbf{R}^{\nu+1}$ by hyperplanes $\alpha_\nu n^\nu + \cdots + \alpha_1 n + \alpha_0 = q$. These hyperplanes are at most $M = M_\nu(p) = \sum_{j=1}^{p} 2(j^\nu + \cdots + j + 1) = 0(p^{\nu+1})$. Now, one can show that the space $\mathbf{R}^{\nu+1}$ is dissected into $0(M^{\nu+1})$ regions by M hyperplanes [5] and therefore:

$$N_\nu(p) = 0(p^{(\nu+1)^2}) .$$

REMARK 1. It is easy to generalize Theorem 2 and obtain the following result. Let $(f_n)_{n\in\mathbf{N}}$ be a countable set of real functions such that

$$\lim_{p\to\infty} \frac{\log^+ |f_n(p)|}{p} = 0 , \qquad \forall n \in \mathbf{N} .$$

($\log^+$ denotes the maximum of 0 and log). Let Q be the set of all real finite linear combinations of the family (f_n). Then the Hausdorff dimension of the set $E(Q)$ is 0.

REMARK 2. The proof of Theorem 2 shows that the set E^ν is not dense on the unit interval $(0, 1)$. On the other hand, E^ν is invariant under the mapping $x \to \{2x\}$. From these two remarks, one sees that E^ν is a Rajchman H-set and that $E(p)$ is therefore a set of uniqueness for trigonometric series. This result is to be compared with the following corollary of Pyatetski-Shapiro's theorem:

The set of nonnormal numbers is not a set of uniqueness.

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