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THE BOREL SPACE OF VON NEUMANN ALGEBRAS ON A SEPARABLE HILBERT SPACE

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Let $(S, \mathscr{S})$ be a Borel space (see G.W. Mackey, *Borel structures in groups and their duals*, Trans. Amer. Math. Soc. 85, (1957) 134–165), $\mathscr{H}$ a separable Hilbert space, $\mathfrak{L}$ the bounded linear operators on $\mathscr{H}$ with the Borel structure generated by the weak topology, and $\mathscr{A}$ the collection of von Neumann algebras on $\mathscr{H}$. A *field of $\mathscr{H}$ von Neumann algebras on S* is a map $s \to \mathfrak{A}(s)$ of S into $\mathscr{A}$. We prove that there is a unique standard Borel structures on $\mathscr{A}$ with the property that $s \to \mathfrak{A}(s)$ is Borel if and only if there exist countably many Borel functions $s \to A_i(s)$ of S into $\mathfrak{L}$ such that for each s, the operators $A_i(s)$ generate $\mathfrak{A}(s)$. This is a consequence of the more general result that when it is provided with a suitable Borel structure, the space of weakly* closed subspaces of the dual of a separable Banach space has sufficiently many Borel choice functions.

We show that the commutant, join, and intersection operations on $\mathscr{A}$ are Borel. It follows that the Borel space of factors is standard. The relevance of $\mathscr{A}$ to the theory of group representations is also investigated.

Essentially following von Neumann [9], we say that a field $s \to \mathfrak{A}(s)$ is *Borel* if there exist countably many Borel functions $s \to A_i(s)$ of S into $\mathfrak{L}$ such that for each s the operators $A_i(s)$ generate $\mathfrak{A}(s)$. This definition may be regarded as somewhat artificial. Rather than state which maps of S into $\mathscr{A}$ are Borel, one would conjecture that there is a standard Borel structure on $\mathscr{A}$ for which this characterization of the Borel maps of S into $\mathscr{A}$ is then valid. In § 2 and § 3 we shall show that this is the case. The demonstration depends on two results: a theorem in [4] showing that a certain Borel structure on the closed subsets of a polonais space is standard, and Theorem 2 of this paper. In the latter we prove the existence of Borel choice functions for the weakly* closed subspaces of the dual of a separable Banach space.

The Borel space $\mathscr{A}$ is of importance in representation theory. If G is a second countable locally compact group, and $G^c(\mathscr{H})$ are the weakly continuous unitary represenfions of G on $\mathscr{H}$ with the weak Borel structure (see [8]), the map $L \to L(G)'$ (prime indicates commutant) of $G^c(\mathscr{H})$ into $\mathscr{A}$ is Borel. By proving in § 3 that the factors $\mathscr{F}$ are

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a Borel subset of $\mathscr{A}$, we obtain new proof in § 4 of Dixmier's result that the factor representations $G^f(\mathscr{H})$ form a Borel subset of $G^c(\mathscr{H})$. We are also able to show that the quasi-equivalence relation is a Borel subset of $G^c(\mathscr{H}) \times G^c(\mathscr{H})$.

It is interesting to speculate about the isomorphism relation on $\mathscr{F}$. Conceivably, one might find an argument similar to those in [3] to prove that the quotient space was not smooth, and thus in particular, that there are uncountably many essentially distinct factors on $\mathscr{H}$.

We remark that an analogous problem of a "nonintrinsic" definition of structure, solved for $\mathscr{A}$ below, exists in Spanier's definition of a quasi-topology [12]. As is shown in [12], one must look for structures more general than topologies.

We are indebted to E. Alfsen and E. Størmer, who enabled us to simplify the proofs of Theorem 2 (by the convexity argument for the continuity of L) and Theorem 5, respectively.

2. Separable Banach spaces. Let $\mathfrak{X}$ be a separable real or complex Banach space, $\mathfrak{X}^*$ the dual of $\mathfrak{X}$, $\mathscr{N}(\mathfrak{X})$ the norm closed subspaces of $\mathfrak{X}$, and $\mathscr{W}(\mathfrak{X}^*)$ the weakly* closed subspaces of $\mathfrak{X}^*$. We wish to define a Borel structure on $\mathscr{W}(\mathfrak{X}^*)$. As $\mathfrak{Y} \to \mathfrak{Y}^\perp$ (the annihilator of $\mathfrak{Y}$) is a one-to-one correspondence between $\mathscr{N}(\mathfrak{X})$ and $\mathscr{W}(\mathfrak{X}^*)$, it suffices to find a Borel structure on $\mathscr{N}(\mathfrak{X})$ and then to transfer it to $\mathscr{W}(\mathfrak{X}^*)$.

$\mathscr{N}(\mathfrak{X})$ is a subset of $\mathscr{C}_0(\mathfrak{X})$, the collection of nonempty closed subsets of the polonais space $\mathfrak{X}$. In [4] we showed that convergence of subsets in $\mathscr{C}_0(\mathfrak{X})$ defines a standard Borel structure on $\mathscr{C}_0(\mathfrak{X})$. Recalling the procedure, if F_α is a net in $\mathscr{C}_0(\mathfrak{X})$ let $\underline{\lim}\, F_\alpha$ be those x in $\mathfrak{X}$ for which there is a net $x_\alpha \in F_\alpha$ with $x_\alpha \to x$. Let $\overline{\lim}\, F_\alpha$ be those x in $\mathfrak{X}$ for which there is a subnet F_{α_β} and $x_{\alpha_\beta} \in F_{\alpha_\beta}$ with $x_{\alpha_\beta} \to x$. If $F \in \mathscr{C}_0(\mathfrak{X})$, we say that F_α converges to the limit F, $F_\alpha \to F$, if $F = \underline{\lim}\, F_\alpha = \overline{\lim}\, F_\alpha$. If $\Sigma \subseteqq \mathscr{C}_0(\mathfrak{X})$, we let $\bar{\Sigma}$ be the limits of nets in Σ, and we say that Σ is convergence closed if $\bar{\Sigma} = \Sigma$. The convergence closed sets form a topology, and generate a standard Borel structure on $\mathscr{C}_0(\mathfrak{X})$. We let $\mathscr{N}(\mathfrak{X})$ have the relative Borel structure. It is easily verified that $\mathscr{N}(\mathfrak{X})$ is convergence closed in $\mathscr{C}_0(\mathfrak{X})$, hence $\mathscr{N}(\mathfrak{X})$ and $\mathscr{W}(\mathfrak{X}^*)$ have standard Borel structures.

If d is any metric on $\mathfrak{X}$ compatible with the topology of $\mathfrak{X}$, $x \in \mathfrak{X}$, and $F \in \mathscr{C}_0(\mathfrak{X})$, define $d(x, F) = \operatorname{glb}\{d(x, y) : y \in F\}$. For any positive c,

$$\{F \in \mathscr{C}_0(\mathfrak{X}) : d(x, F) \geqq c\} \tag{1}$$

is convergence closed. It follows that $F \to d(x, F)$ is a Borel function on $\mathscr{C}_0(\mathfrak{X})$. As in the proof of the first theorem in [4], sets of the form (1) separate points in $\mathscr{C}_0(\mathfrak{X})$, and thus as $\mathscr{C}_0(\mathfrak{X})$ is standard, generate the Borel structure. It follows that the Borel structure on

$\mathscr{C}_0(\mathfrak{X})$ is the weakest for which the functions $F \to d(x, F)$ are Borel (actually it would suffice to restrict to the x in a countable dense subset).

Let d be the norm metric on $\mathfrak{X}$. Then for $\mathfrak{Y} \in \mathscr{W}(\mathfrak{X}^*)$, $d(x, \mathfrak{Y}^{\perp}) = \| x + \mathfrak{Y}^{\perp} \|$, the latter being the quotient norm in $\mathfrak{X}/\mathfrak{Y}^{\perp}$. As $\mathfrak{Y}$ is weakly* closed, $\mathfrak{Y}^{\perp}$ $\mathfrak{Y}$, and we have a natural isometry $(\mathfrak{X}/\mathfrak{Y}^{\perp})^* \cong \mathfrak{Y}$. The corresponding isometry of $\mathfrak{X}/\mathfrak{Y}^{\perp}$ into $\mathfrak{Y}^*$ is defined by $x + \mathfrak{Y}^{\perp} \to x \mid \mathfrak{Y}$, where $x \mid \mathfrak{Y}$ in the restriction of x, regarded as an element of $\mathfrak{X}^{**}$, to $\mathfrak{Y}$. We conclude:

THEOREM 1. *Let $\mathfrak{X}$ be a separable Banach space, $\mathscr{W}(\mathfrak{X}^*)$ the weakly* closed subspaces of $\mathfrak{X}^*$. The Borel structure on $\mathscr{W}(\mathfrak{X}^*)$ is standard, and may be described as the smallest structure for which the functions*

$$\mathfrak{Y} \to \| x + \mathfrak{Y}^{\perp} \| = \| x \mid \mathfrak{Y} \| , \qquad x \in \mathfrak{X}$$

are Borel.

If $\mathfrak{X}$ is a real or complex separable Banach space, the *weak* Borel structure* on $\mathfrak{X}^*$ is that generated by the weak* topology. In other words, it is the smallest structure for which the functions $f \to f(x)$, $x \in \mathfrak{X}$ are Borel. Although we shall not use this fact, we remark that this structure is standard (see the proof of [8, Th. 8.1]).

Theorem 2 may be regarded as an elaborate form of the Hahn-Banach Theorem. Recalling the usual argument, suppose that $\mathfrak{X}$ is a real Banach space, and that we wish to construct a function in the closed unit ball $\mathfrak{X}_1^*$ of $\mathfrak{X}^*$. Suppose that f has been defined on a linear subspace $\mathfrak{V}$ of $\mathfrak{X}$, and is in $\mathfrak{V}_1^*$. If we extend f to the space generated by $\mathfrak{V}$ and a vector x, we must insist that

$$| f(x + w) | \leqq \| x + w \| \tag{2}$$

for all $w \in \mathfrak{V}$, i.e.,

$$- \| x + u \| - f(u) \leqq f(x) \leqq \| x + v \| - f(v)$$

for all $u, v \in \mathfrak{V}$. Let

$$\begin{aligned} L(f) &= \text{lub}\,\{- \| x + u \| - f(u) \colon u \in \mathfrak{V}\} , \\ M(f) &= \text{glb}\,\{\| x + v \| - f(v) \colon v \in \mathfrak{V}\} . \end{aligned} \tag{3}$$

These exist as for any $u, v \in \mathfrak{V}$,

$$f(v - u) \leqq \| v - u \| \leqq \| x + v \| + \| x + u \| ,$$

$$\text{i.e.,} \qquad - \| x + u \| - f(u) \leqq \| x + v \| - f(v) . \tag{4}$$

Thus we may rewrite (2):

$$L(f) \leqq f(x) \leqq M(f) \ . \tag{5}$$

We shall assume below that $\mathfrak{B}$ is finite dimensional, and let $\mathfrak{B}^*$ have the norm topology. The functions $f \rightarrow L(f)$ and $f \rightarrow M(f)$ are defined on the closed unit ball $\mathfrak{B}_1^*$. As it is the least upper bound of convex functions, $f \rightarrow L(f)$ is convex, and thus continuous on the interior of of $\mathfrak{B}_1^*$ (see [1, p. 92]). From

$$M(f) = - L(- f) \ , \tag{6}$$

$f \rightarrow M(f)$ is also continuous on the interior of $\mathfrak{B}_1^*$.

THEOREM 2. *Let $\mathfrak{X}$ be a separable Banach space, $\mathscr{W}(\mathfrak{X}^*)$ the weakly* closed subspaces of $\mathfrak{X}^*$. There exist countably many Borel choice functions f_n: $\mathscr{W}(\mathfrak{X}^*) \rightarrow \mathfrak{X}^*$ such that for each $\mathfrak{Y} \in \mathscr{W}(\mathfrak{X}^*)$, the vectors $f_n(\mathfrak{Y})$ are weakly* dense in the closed unit ball $\mathfrak{Y}_1$ of $\mathfrak{Y}$.*

Proof. Suppose that $\mathfrak{X}$ is real. If $\mathfrak{Y} \in \mathscr{W}(\mathfrak{X}^*)$, we may identify $\mathfrak{Y}$ with $(\mathfrak{X}/\mathfrak{Y}^\perp)^*$, the norms and the weak* topologies will coincide.

For each sequence of real numbers $t = (t_1, t_2, \cdots)$ with $0 \leqq t_i \leqq 1$, we shall construct a function $f_t^{\mathfrak{Y}} \in (\mathfrak{X}/\mathfrak{Y}^\perp)_1^*$. Let $x_1, x_2, \cdots$ be norm dense in $\mathfrak{X}$, with $x_1 = 0$. Let $x_n(\mathfrak{Y}) = x_n + \mathfrak{Y}^\perp$, and $\mathfrak{B}_n(\mathfrak{Y})$ be the linear space spanned by $x_1(\mathfrak{Y}), \cdots, x_n(\mathfrak{Y})$ in $\mathfrak{X}/\mathfrak{Y}^\perp$. Define $f_{t_1}^{\mathfrak{Y}}(0) = 0$. Suppose that we have defined $f_{t_1,\cdots,t_n}^{\mathfrak{Y}}$ to be an element of $\mathfrak{B}_n(\mathfrak{Y})_1^*$. Letting $\mathfrak{B}_n(\mathfrak{Y}) = \mathfrak{B}$, $f_{t_1,\cdots,t_n}^{\mathfrak{Y}} = f$, and $x_{n+1}(\mathfrak{Y}) = x$ in our previous discussion, define

$$f_{t_1,\cdots,t_{n+1}}^{\mathfrak{Y}}(x) = t_{n+1}L(f) + (1 - t_{n+1})M(f) \ . \tag{7}$$

If $x \in \mathfrak{B}$, letting $u = v = - x$, we have from (3), (5), and (7)

$$- f(u) \leqq L(f) \leqq f_{t_1,\cdots,t_{n+1}}^{\mathfrak{Y}}(x) \leqq M(f) \leqq - f(v),$$

i.e.,

$$f_{t_1,\cdots,t_{n+1}}^{\mathfrak{Y}}(x) = f(x) \ .$$

Thus defining $f_{t_1,\cdots,t_{n+1}}^{\mathfrak{Y}}$ on $\mathfrak{B}_{n+1}(\mathfrak{Y})$ by

$$f_{t_1,\cdots,t_{n+1}}^{\mathfrak{Y}}(cx + w) = cf_{t_1,\cdots,t_{n+1}}^{\mathfrak{Y}}(x) + f(w) \ ,$$

we obtain an extension of $f_{t_1,\cdots,t_n}^{\mathfrak{Y}}$ to an element of $\mathfrak{B}_{n+1}(\mathfrak{Y})^*$. As $f = f_{t_1,\cdots,t_{n+1}}^{\mathfrak{Y}}$ satisfies (5), it readily follows that $f_{t_1,\cdots,t_{n+1}}^{\mathfrak{Y}}$ is in $\mathfrak{B}_{n+1}(\mathfrak{Y})_1^*$. Define $f_t^{\mathfrak{Y}}$ on the space spanned by the $x_n(\mathfrak{Y})$ to be the union of the functions $f_{t_1,\cdots,t_n}^{\mathfrak{Y}}$. This extends by continuity to an element of $(\mathfrak{X}/\mathfrak{Y}^\perp)_1^*$.

It is clear that any function in $(\mathfrak{X}/\mathfrak{Y}^\perp)_1^*$ must have the form $f_t^{\mathfrak{Y}}$

for some sequence $t = (t_1, t_2, \cdots)$. We claim that the countable family of functions $f^{\mathfrak{Y}}_r$, $r = (r_1, r_2, \cdots)$ with the r_i rational, and all but a finite number equal to 0, are weakly* dense in $(\mathfrak{X}/\mathfrak{Y}^\perp)^*_1$. It suffices to prove that for all n, the functions $f^{\mathfrak{Y}}_{r_1,\cdots,r_n}$ are weakly*, or equivalently, norm dense in the interior of $(\mathfrak{B}_n(\mathfrak{Y}))^*_1$. This is trivial if $n = 1$. Suppose that it is true for n. If $g \in \mathfrak{B}_{n+1}(\mathfrak{Y})^*$ and $\| g \| \leqq 1$, let f be the restriction of g to $\mathfrak{B}_n(\mathfrak{Y})$. From our hypothesis and the earlier discussion, we may select rationals $r_1, \cdots, r_n$ with $f^{\mathfrak{Y}}_{r_1,\cdots,r_n}$ close to f in the norm topology, and $L(f^{\mathfrak{Y}}_{r_1,\cdots,r_n})$ and $M(f^{\mathfrak{Y}}_{r_1,\cdots,r_n})$ close to $L(f)$ and $M(f)$, respectively. Thus by a suitable choice of r_{n+1}, we obtain

$$f^{\mathfrak{Y}}_{r_1,\cdots,r_{n+1}}(x_{n+1}(\mathfrak{Y}))$$

close to $g(x_{n+1}(\mathfrak{Y}))$.

For any sequence $(t_1, t_2, \cdots)$ we have that $\mathfrak{Y} \to f^{\mathfrak{Y}}_t(x_n)$ is Borel (regarding $f^{\mathfrak{Y}}_t$ as an element of $\mathfrak{Y}$). This is trivial if $n = 1$. Suppose that it is true for $k \leqq n$. Then

$$\begin{aligned} f^{\mathfrak{Y}}_t(x_{n+1}) &= f^{\mathfrak{Y}}_{t_1,\cdots,t_{n+1}}(x_{n+1}(\mathfrak{Y})) \\ &= t_{n+1}L(f^{\mathfrak{Y}}_{t_1,\cdots,t}\ ) + (1 - t_{n+1})M(f^{\mathfrak{Y}}_{t},\cdots,_t\ ) \end{aligned} \tag{8}$$

If $\mathfrak{B}_n$ is the linear span of $x_1, \cdots, x_n$,

$$L(f^{\mathfrak{Y}}_{t_1,\cdots,t_n}) = \text{lub}\,\{- \| x_{n+1} + u + \mathfrak{Y}^\perp \| - f^{\mathfrak{Y}}_t(u) : u \in \mathfrak{B}_n\}\ .$$

From Theorem 1 and the induction hypothesis,

$$\mathfrak{Y} \to - \| x_{n+1} + u + \mathfrak{Y}^\perp \| - f^{\mathfrak{Y}}_t(u)$$

is Borel for any $u \in \mathfrak{B}_n$. Restricting to u that are rational linear combinations of the x_k for $k \leqq n$, $\mathfrak{Y} \to L(f^{\mathfrak{Y}}_{t_1,\cdots,t_n})$ is the least upper bound of a countable number of Borel functions, and is thus Borel. From (6) and (8), $\mathfrak{Y} \to f^{\mathfrak{Y}}_t(x_{n+1})$ is Borel. For any $x \in \mathfrak{X}$, $\mathfrak{Y} \to f^{\mathfrak{Y}}_t(x)$ is a limit of functions of the form $\mathfrak{Y} \to f^{\mathfrak{Y}}_t(x_n)$, and hence is Borel. Thus $\mathfrak{Y} \to f^{\mathfrak{Y}}_t$ is Borel.

Finally, suppose that $\mathfrak{X}$ is a complex Banach space. Letting $\mathfrak{X}_R$ be the corresponding real Banach space, $\mathscr{N}(\mathfrak{X})$ is a convergence closed subset of $\mathscr{N}(\mathfrak{X}_R)$. Define a map of $\mathscr{W}(\mathfrak{X}^*)$ into $\mathscr{W}((\mathfrak{X}_R)^*)$ by $\mathfrak{Y} \to \text{Re}\,\mathfrak{Y}$, where the latter consists of all real functions $\text{Re}\,f$ with $f \in \mathfrak{Y}$ (the customary argument shows that $f \to \text{Re}\,f$ is an isometry of $\mathfrak{X}^*$ onto $(\mathfrak{X}_R)^*$). For $\mathfrak{Z} \in \mathscr{N}(\mathfrak{X})$, $\text{Re}\,(\mathfrak{Z}^\perp) = \mathfrak{Z}^\perp$, where annihilators are taken in $\mathfrak{X}^*$ and $(\mathfrak{X}_R)^*$, respectively. It follows that $\mathfrak{Y} \to \text{Re}\,\mathfrak{Y}$ defines a Borel isomorphism of $\mathscr{W}(\mathfrak{X}^*)$ onto a Borel subset of $\mathscr{W}((\mathfrak{X}_R)^*)$. Choose real choice functions f_n: $\mathscr{W}((\mathfrak{X}_R)^*) \to (\mathfrak{X}_R)^*$ with $f_n(\mathfrak{Y})$ weakly* dense in $\mathfrak{Y}_1$ for each $\mathfrak{Y} \in \mathscr{W}((\mathfrak{X}_R)^*)$. Let g_n: $\mathscr{W}(\mathfrak{X}^*) \to \mathfrak{X}^*$ be the corresponding complex functions, i.e., for $\mathfrak{Y} \in \mathscr{W}(\mathfrak{X}^*)$ and $x \in \mathfrak{X}$, let

$$g_n(\mathfrak{Y})(x) = f_n(\mathrm{Re}\,\mathfrak{Y})(x) - if_n(\mathrm{Re}\,\mathfrak{Y})(ix) .$$

Then $\mathrm{Re}\, g_n(\mathfrak{Y}) = f_n(\mathrm{Re}\,\mathfrak{Y}) \in (\mathrm{Re}\,\mathfrak{Y})_1$, implies $g_n(\mathfrak{Y}) \in \mathfrak{Y}_1$. Given an arbitrary $g \in \mathfrak{Y}_1$, $x_1, \cdots, x_k \in \mathfrak{X}$, and $\varepsilon > 0$, choose an f_n with

$$| f_n(\mathrm{Re}\,\mathfrak{Y})(x_j) - \mathrm{Re}\, g(x_j) | < \varepsilon$$
$$| f_n(\mathrm{Re}\,\mathfrak{Y})(ix_j) - \mathrm{Re}\, g(ix_j) | < \varepsilon ,$$

for $j = 1, \cdots, k$. Then as

$$g(x) = \mathrm{Re}\, g(x) - i\, \mathrm{Re}\, g(ix) ,$$

we have

$$| g_n(\mathfrak{Y})(x_j) - g(x_j) | < 2\varepsilon$$

for $j = 1, \cdots, k$. Thus the $g_n(\mathfrak{Y})$ are weakly* dense in $\mathfrak{Y}_1$. Clearly the g_n are Borel.

COROLLARY. *If $(S, \mathscr{S})$ is a Borel space, then a map $s \rightarrow \mathfrak{Y}(s)$ of S into $\mathscr{W}(\mathfrak{X}^*)$ is Borel if and only if there exist countably many Borel functions $s \rightarrow f_n^s$ of S into $\mathfrak{X}^*$, such that for each s, the vectors f_n^s are weakly dense in $\mathfrak{Y}(s)_1$.*

Proof. If $s \rightarrow \mathfrak{Y}(s)$ is Borel, the functions f_n^s are obtained by composing this map with the choice functions of Theorem 2. Conversely, if such functions exist, we have from the isometry

$$\mathfrak{Y}(s) \cong (\mathfrak{X}/\mathfrak{Y}(s)^{\perp})^* ,$$

$$\| x + \mathfrak{Y}(s)^{\perp} \| = \sup \{ | f_i^s(x) | : i = 1, 2, \cdots \}$$

for each $x \in \mathfrak{X}$. Thus $s \rightarrow \| x + \mathfrak{Y}(s)^{\perp} \|$ is Borel for each $x \in \mathfrak{X}$, and by Theorem 1, $s \rightarrow \mathfrak{Y}(s)$ is Borel.

3. Von Neumann algebras. Let $\mathscr{H}$, $\mathfrak{L}$, $\mathscr{A}$, and $\mathscr{F}$ be as in § 1. We have that $\mathfrak{L} = (\mathfrak{L}_*)^*$, where $\mathfrak{L}_*$ is the separable Banach space of ultra-weakly continuous functions on $\mathfrak{L}$ (or by a natural identification, the trace class operators with a suitable norm-see [10]). The ultra-weak and weak* topologies coincide on $\mathfrak{L}$. Thus letting $\mathscr{W}(\mathfrak{L})$ be the ultra-weakly closed subspaces of $\mathfrak{L}$, we may give it the Borel structure described in § 2.

If $\mathfrak{Y} \in \mathscr{W}(\mathfrak{L})$, write $\mathfrak{Y}^*$ and $\mathfrak{Y}'$ for the adjoints of elements in $\mathfrak{Y}$, and the commutant of $\mathfrak{Y}$, respectively. The proof of the following theorem is largely patterned after that of [6, Th. 2.8].

THEOREM 3. *$\mathfrak{Y} \rightarrow \mathfrak{Y}^*$ and $\mathfrak{Y} \rightarrow \mathfrak{Y}'$ define Borel transformations of*

$\mathscr{W}(\mathfrak{L})$.

Proof. For $f \in \mathfrak{L}_*$, define $f^* \in \mathfrak{L}_*$ by $f^*(A) = \overline{f(A^*)}$, the bar indicating complex conjugate. This is an isometry of $\mathfrak{L}_*$, hence the transformation $\mathfrak{B} \to \mathfrak{B}^*$ on $\mathscr{N}(\mathfrak{L})$ is a homeomorphism (in the sense of convergence), and a Borel isomorphism. For $\mathfrak{Y} \in \mathscr{W}(\mathfrak{L})$, $(\mathfrak{Y}^\perp)^* = (\mathfrak{Y}^*)^\perp$, i.e., the adjoint operation on $\mathscr{N}(\mathfrak{L}^*)$ is carried into that on $\mathscr{W}(\mathfrak{L})$, and thus is a Borel isomorphism on the latter.

From Theorem 2, we may let $\mathfrak{Y} \to A_n^{\mathfrak{Y}}$ be Borel choice functions on $\mathscr{W}(\mathfrak{L})$ with $A_n^{\mathfrak{Y}}$ ultra-weakly dense in $\mathfrak{Y}_1$. We have

$$\mathfrak{Y}' = \{B \in \mathfrak{L} : BA_n^{\mathfrak{Y}} - A_n^{\mathfrak{Y}}B = 0 \text{ for } n = 1, 2, \cdots\}.$$

Let $\mathfrak{M}$ and $\mathfrak{M}_*$ be the sequences (A_n) and (f_n) of elements in $\mathfrak{L}$ and $\mathfrak{L}_*$, respectively, with $\sup\{\|A_n\| : n = 1, 2, \cdots\} < \infty$ and $\sum_{n=1}^\infty \|f_n\| < \infty$. With the norms $\|(A_n)\| = \sup\{\|A_n\| : n = 1, 2, \cdots\}$ and $\|(f_n)\| = \sum_{n=1}^\infty \|f_n\|$, $\mathfrak{M}$ and $\mathfrak{M}_*$ are Banach spaces, and defining $(f_n)((A_n)) = \sum_{n=1}^\infty f_n(A_n)$, $\mathfrak{M}$ may be identified with the dual of $\mathfrak{M}_*$. We have

$$\mathfrak{Y}' = \text{kernel } T^{\mathfrak{Y}},$$

where $T^{\mathfrak{Y}} : \mathfrak{L} \to \mathfrak{M}$ is defined by

$$T^{\mathfrak{Y}}(B) = (BA_n^{\mathfrak{Y}} - A_n^{\mathfrak{Y}}B) .$$

we claim that $T^{\mathfrak{Y}}$ is continuous in the weak* topologies. If $(f_n) \in \mathfrak{M}_*$,

$$(f_n)T^{\mathfrak{Y}}(B) = \sum_{n=1}^\infty g_n(B) ,$$

where $g_n(B) = f_n(BA_n^{\mathfrak{Y}} - A_n^{\mathfrak{Y}}B)$. The partial sums $\sum_{n=1}^N g_n$ are weakly* continuous, and converge uniformly on the unit ball $\mathfrak{L}_1$ of $\mathfrak{L}$, as if $B \in \mathfrak{L}_1$,

$$\left| \sum_{n=N+1}^\infty g_n(B) \right| \leqq 2 \sum_{n=N+1}^\infty \|f_n\| .$$

It follows that $B \to (f_n)T^{\mathfrak{Y}}(B)$ is continuous on $\mathfrak{L}_1$, and thus on $\mathfrak{L}$ (see [2, p. 41]). Define $T_*^{\mathfrak{Y}} : \mathfrak{M}_* \to \mathfrak{L}_*$ by

$$T_*^{\mathfrak{Y}}((f_n))(B) = (f_n)(T^{\mathfrak{Y}}(B)) .$$

We have that $(\text{kernel } T^{\mathfrak{Y}})^\perp$ is the closure of the range of $T_*^{\mathfrak{Y}}$. Thus letting B_i be ultra-weakly dense in $\mathfrak{L}_1$ and $g_j = (f_n^j)$ be norm dense in $\mathfrak{M}_*$, we have for any $f \in \mathfrak{L}_*$,

$$\|f + (\mathfrak{Y}')^\perp\| = \text{glb}\{\|f + T_*^{\mathfrak{Y}}(g_j)\| ,\ j = 1, 2, \cdots\}$$

where

$$\begin{aligned}\| f + T_*^{\mathfrak{Y}}(g_j) \| &= \operatorname{lub}\{| f(B_i) + T_*^{\mathfrak{Y}}(g_j)(B_i) |: i = 1, 2, \cdots\} \\ &= \operatorname{lub}\{| f(B_i) + \sum_{n=1}^{\infty} f_n^j(B_i A_n^{\mathfrak{Y}} - A_n^{\mathfrak{Y}} B_i) |: i = 1, 2, \cdots\}.\end{aligned}$$

As $\mathfrak{Y} \to A_n^{\mathfrak{Y}}$ is ultra-weakly Borel, $\mathfrak{Y} \to \| f + (\mathfrak{Y}')^{\perp} \|$ is Borel, and as f is arbitrary, we have from Theorem 1 that $\mathfrak{Y} \to \mathfrak{Y}'$ is Borel.

COROLLARY 1. *$\mathscr{A}$ is a Borel subset of $\mathscr{W}(\mathfrak{Y})$, and thus is standard under the relative Borel structure.*

Proof. $\mathscr{A}$ consists of the $\mathfrak{Y} \in \mathscr{W}(\mathfrak{L})$ invariant under the Borel transformations $\mathfrak{Y} \to \mathfrak{Y}^*$ and $\mathfrak{Y} \to \mathfrak{Y}''$. In general say that θ is a Borel transformation of Borel space $(S, \mathscr{S})$. If Δ is the diagonal of $S \times S$, and $\theta \times \iota: S \to S \times S$ is defined by $\theta \times \iota(s) = (\theta(s), s)$, we have

$$\{s \in S: \theta(s) = s\} = (\theta \times \iota)^{-1}(\Delta) .$$

Thus if $(S, \mathscr{S})$ is standard, Δ is a Borel subset of $S \times S$, and the set of fixed points of θ is Borel.

Given von Neumann algebras $\mathfrak{A}$ and $\mathfrak{B}$, we let $\mathfrak{A} \vee \mathfrak{B}$ denote the von Neumann algebra generated by $\mathfrak{A}$ and $\mathfrak{B}$. Providing $\mathscr{A} \times \mathscr{A}$ with the product structure,

COROLLARY 2. *The maps of $\mathscr{A} \times \mathscr{A}$ into $\mathscr{A}$ defined by $(\mathfrak{A}, \mathfrak{B}) \to \mathfrak{A} \cap \mathfrak{B}$ and $(\mathfrak{A}, \mathfrak{B}) \to \mathfrak{A} \vee \mathfrak{B}$ are Borel.*

Proof. As $\mathfrak{A} \cap \mathfrak{B} = (\mathfrak{A}' \vee \mathfrak{B}')'$, it suffices to prove the second assertion. From Theorem 2, there exist Borel choice functions $A_i: \mathscr{A} \to \mathfrak{L}$ with $A_i(\mathfrak{A})$ ultra-weakly dense in $\mathfrak{A}_1$, for each $\mathfrak{A} \in \mathscr{A}$. For each pair $(\mathfrak{A}, \mathfrak{B}) \in \mathscr{A} \times \mathscr{A}$, let $\mathscr{C}(\mathfrak{A}, \mathfrak{B})$ be the self-adjoint linear algebra generated by the elements $A_i(\mathfrak{A})$ and $A_j(\mathfrak{B})$. Let $B_k(\mathfrak{A}, \mathfrak{B})$ be an enumeration of the finite complex rational combinations of finite products of the elements $A_i(\mathfrak{A})$, $A_j(\mathfrak{B})$ and their adjoints. The $B_k(\mathfrak{A}, \mathfrak{B})$ are norm dense in $\mathscr{C}(\mathfrak{A}, \mathfrak{B})$, hence defining $B_k'(\mathfrak{A}, \mathfrak{B}) = B_k(\mathfrak{A}, \mathfrak{B})$ if $\| B_k(\mathfrak{A}, \mathfrak{B}) \| \leqq 1$, and $B_k'(\mathfrak{A}, \mathfrak{B}) = 0$ otherwise, the $B_k'(\mathfrak{A}, \mathfrak{B})$ are norm dense in $\mathscr{C}(\mathfrak{A}, \mathfrak{B})_1$. From the Kaplansky Density Theorem, the latter is ultra-weakly dense in $(\mathfrak{A} \vee \mathfrak{B})_1$. As $(\mathfrak{A}, \mathfrak{B}) \to B_k'(\mathfrak{A}, \mathfrak{B})$ are Borel, our assertion follows from the corollary to Theorem 2.

COROLLARY 3. *$\mathscr{F}$ is a Borel subset of $\mathscr{A}$, and thus is standard in the relative Borel structure.*

Proof. Let $\mathfrak{J}$ be the von Neumann algebra on $\mathscr{H}$ consisting of complex multiples of the identity operator. Then $\mathscr{F}$ is the inverse image of the element $\mathfrak{J}$ under the Borel map of $\mathscr{A}$ into $\mathscr{A}$ defined by $\mathfrak{A} \to \mathfrak{A} \cap \mathfrak{A}'$.

The argument used in the proof of Corollary 2 shows that a map $s \to \mathfrak{A}(s)$ of a Borel space $(S, \mathscr{S})$ into $\mathscr{A}$ is Borel if and only if there exist Borel functions $s \to A_i(s)$ of S into $\mathfrak{L}$ such that the $A_i(s)$ *generate* $\mathfrak{A}(s)$. Thus we have recaptured the original definition of § 1.

In direct integral theory, it is of some importance to know that various other subsets of $\mathscr{A}$ are measurable (see [9, 11]). We suspect that constructive procedures similar to that used in Theorem 2, would enable one to show that many of these sets are Borel.

4. Representation spaces. Let $\mathscr{H}$, $\mathfrak{L}$, $\mathscr{A}$, and $\mathscr{F}$ be as above, and G be a second countable locally compact group (an analogous theory exists for separable C^*-algebras). Let $G^c(\mathscr{H})$ be the weakly continuous unitary representations of G on $\mathscr{H}$, with the standard Borel structure defined by Mackey (see [8]). Let $G^f(\mathscr{H})$ be the subset of factor representations, i.e. those representations $L \in G^c(\mathscr{H})$ with $L(G)'$ a factor von Neumann algebra.

If $L, M \in G^c(\mathscr{H})$, let $\mathfrak{R}(L, M)$ be the ring of intertwining operators for L and M, i.e., those $B \in \mathfrak{L}$ with $BL(t) = M(t)B$ for all $t \in G$. In particular, $\mathfrak{R}(L, L) = L(G)'$. As was the case for Theorem 3, the following is simply a refinement of [6, Th. 2.8].

THEOREM 4. *The map* $G^c(\mathscr{H}) \times G^c(\mathscr{H}) \to G^c(\mathscr{H})$ *defined by* $(L, M) \to \mathfrak{R}(L, M)$ *is Borel.*

Proof. Let t_n be dense in G, and define $\mathfrak{M}$ and $\mathfrak{M}_*$ as in the proof of Theorem 3. Defining $S^{(L,M)}: \mathfrak{L} \to \mathfrak{M}$ by

$$S^{(L,M)}(B) = (BL(t_n) - M(t_n)B) ,$$

we have that

$$\mathfrak{R}(L, M) = \text{kernel } S^{(L,M)} ,$$

and that $S^{(L,M)}$ is continuous in the weak* topologies. $S^{(L,M)}$ is the adjoint of a map $S_*^{(L,M)}: \mathfrak{M}_* \to \mathfrak{L}_*$, and choosing B_i ultra-weakly dense in $\mathfrak{L}_1$, and $g_j = (f_j^n)$ norm dense in $\mathfrak{M}_*$, we have for any $f \in L_*$,

$$\| f + \mathfrak{R}(L, M)^\perp \| = \text{glb}\,\{\| f + S_*^{(L,M)}(g_j) \| : j = 1, 2, \cdots\} ,$$

where

$$\| f + S_*^{(L,M)}(g_j) \| = \text{lub}\,\{| f(B_i) + \sum_{n=1}^{\infty} f_j^n(B_i L(t_n) - M(t_n)B_i) | : i = 1, 2, \cdots\} .$$

$(L, M) \to f_j^n(B_i L(t_n) - M(t_n)B_i)$ is Borel when $G^c(\mathscr{H}) \times G^c(\mathscr{H})$ is given the product of the Mackey Borel structures, as any ultra-weakly continuous function is a norm limit of weakly continuous functions. It follows that $(L, M) \to \| f + \Re(L, M)^\perp \|$ is Borel, and from Theorem 1, $(L, M) \to \Re(L, M)$ is Borel.

COROLLARY 1. *The map $G^c(\mathscr{H}) \to \mathscr{A}$ defined by $L \to L(G)'$ is Borel*

COROLLARY 2. (*This was first proved by J. Dixmier—see* [5, *Theorem* 1].) *The set $G^f(\mathscr{H})$ of factor representation of G forms a Borel subset of $G^c(\mathscr{H})$, and thus is standard under the relative Borel structure.*

Following Mackey (see [7]), if $L, M \in G^c(\mathscr{H})$, we say that L is *covered* by M, $L \prec M$, if very subrepresentation of L contains a subrepresentation that is unitarily equivalent to a subrepresentation of M. L is *quasi-equivalent* to M, $L \sim M$, if $L \prec M$ and $M \prec L$.

If E is a projection in $L(G)'$, and $E \neq 0$, let L^E denote the corresponding subrepresentation of G on the range of E. If there exists a projection $E \in L(G)'$ with $E \neq 0$ and $L^E \prec M$, let $C(L, M)$ be the least upper bound of all such projections. Otherwise, let $C(L, M) = 0$. $C(L, M)$ is an element of $L(G)' \cap L(G)''$.

THEOREM 5. *The map $G^c(\mathscr{H}) \times G^c(\mathscr{H}) \to \mathfrak{L}$ defined by $(L, M) \to C(L, M)$ is Borel.*

Proof. If $A \in \mathfrak{L}$, let E_A and F_A be the projections on the closure of the range, and the orthogonal complement of the kernel of A. If $A \in \Re(L, M)$, then $F_A \in L(G)'$ and $E_A \in M(G)'$. If $A \neq 0$, and U is the partial isometry in the polar decomposition of A with $U^*U = F_A$, then U determines a unitary equivalence of L^{F_A} and M^{E_A}, and $F_A \leqq C(L, M)$. From Theorems 4 and 2, there exist Borel functions $A_i(L, M)$ that are ultra-weakly dense in the unit ball of $\Re(L, M)$ for each L and M. We claim that

$$C(L, M) = \bigvee_{i=1}^{\infty} F_{A_i(L, M)} , \tag{9}$$

where on the right we have taken the least upper bound in the complete projection lattice of $L(G)'$.

Suppose that there exist L and M with

$$F = C(L, M) - \bigvee_{i=1}^{\infty} F_{A_i(L, M)} \neq 0 .$$

As $L^F \prec M$, there exists a projection $F_0 \leqq F$ with $F_0 \neq 0$ and $F_0 = U^*U$ where $U \in \Re(L, M)$. Choosing i_k for which $A_{i_k}(L, M) \to U$ ultra-weakly,

$$0 = A_{i_k}(L, M)F_0 \to UF_0 = F_0 ,$$

a contradiction.

The map of $\mathfrak{L}$ into itself defined by $A \to F_A$ is Borel. To see this, note that $A \to A^*A$ is weakly Borel, as if $x, y \in \mathscr{H}$, letting x_i be an orthonormal basis we have

$$A^*Ax \cdot y = \sum_{i=1}^{\infty} (Ax \cdot x_i)(Ay \cdot x_i)^- .$$

A similar expansion shows that for positive integers n, $A \to A^n$ is Borel, hence for any polynominal p, $A \to p(A)$ is Borel. Suppose that f is a bounded real Borel function on the reals, and that there is a sequence of real polynomials p_n converging to f point-wise, uniformly bounded on compact sets. If A is a self-adjoint element in $\mathfrak{L}$, we have from spectral theory that $p_n(A) \to f(A)$ weakly. Thus $A \to f(A)$ is Borel. Letting g be the characteristic function of the open set $(0, \infty)$, $A \to F_A = g((A^*A)^{1/2})$ is Borel.

For all i, $(L, M) \to F_{A_i(L,M)}$ is Borel. If $F_1, \cdots, F_n$ are propections, then

$$F_1 \vee \cdots \vee F_n = F_{(F_1 + \cdots + F_n)} ,$$

hence

$$(L, M) \to \bigvee_{i=1}^{n} F_{A_i(L,M)}$$

is Borel. As the projections $\bigvee_{i=1}^{n} F_{A_i(L,M)}$ converge weakly to $\bigvee_{i=1}^{\infty} F_{A_i(L,M)}$, we conclude from (9) that $(L, M) \to C(L, M)$ is Borel.

Ernest remarked in the proof of [5, Prop. 2] that the quasi-equivalence relation on $G^f(\mathscr{H})$ is a Borel subset of $G^f(\mathscr{H}) \times G^f(\mathscr{H})$. The above theorem implies:

COROLLARY 1. *The covering and quasi-equivalence relations are Borel subsets of* $G^c(\mathscr{H}) \times G^c(\mathscr{H})$.

COROLLARY 2. *The quasi-equivalence class* $[L]$ *of a representation* L *in* $G^c(\mathscr{H})$ *is a Borel subset of* $G^c(\mathscr{H})$.

Proof. Let $\pi_i: G^c(\mathscr{H}) \times G^c(\mathscr{H}) \to G^c(\mathscr{H})$, $i = 1, 2$, *be* the projections on the first and second co-ordinates. Then $[L] = \pi_2(\pi_1^{-1}(L) \cap \sim)$, and as π_2 is one-to-one on $\pi_1^{-1}(L) \cap \sim$, and the latter is standard, $[L]$ is Borel.

It would seem likely that the unitary equivalence relation is also a Borel subset of $G^c(\mathscr{H}) \times G^c(\mathscr{H})$. Presumably one must prove the existence of a Borel choice function on spaces of the form $\mathfrak{R}(L, M)$, that selects a unitary operator when such exists. If unitary equivalence were a Borel set, it would follow that the representations $L \in G^c(\mathscr{H})$ with $L(G)'$ finite was also Borel. It should be noted that the unitary analogue of Corollary 2 above is true (see [3, Lemma 2.4]).

If G is the free group on countably many generators, the map described in Corollary 1 of Theorem 4 is onto. As the given structure and the corresponding quotient structure on $\mathscr{A}$ must coincide, a subset of $\mathscr{A}$ will be Borel if and only if the inverse image in $G^c(\mathscr{H})$ is Borel.

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