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GROUP EXTENSION REPRESENTATIONS AND THE STRUCTURE SPACE

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Let K be a locally compact group. K^* will denote the Jacobson structure space of $C^*(K)$, the group C^*-algebra of K. For any unitary representation V of K on a Hilbert space, let E_V denote the projection valued measure on the Borel sets of K^* defined by Glimm (Pacific J. Math. 12 (1962), 885–911; Theorem 1.9). A (not necessarily Borel) subset S of K^* is called E_V-thick if $E_V(S_1) = 0$ for every Borel $S_1 \subseteq K^* \sim S$. For any two representations V_1 and V_2, $\mathscr{R}(V_1, V_2)$ will denote the space of operators intertwining V_1 and V_2.

Suppose K is a closed normal subgroup of the locally compact group G. If V is a representation of K and $x \in G$, V^x is defined by $V^x_k = V_{xkx^{-1}}$, $k \in K$. If $z \in K^*$, $zx = \mathrm{Ker}\,(V^x)$, where V is any irreducible repesentation such that $z = \mathrm{Ker}\,(V)$. (By Ker we mean the kernel in the group C^*-algebra.) This composition turns (K^*, G) into a topological transformation group (Glimm, op. cit., Lemma 1.3). The present paper first shows that the stability subgroups of G at points $z \in K^*$ are closed. Then the following two theorems are proved:

THEOREM 1. **Let $z \in K^*$ and let H be the stability subgroup of G at z. Let L be a representation of H such that $\{z\}$ is $E_{L|K}$-thick. Then $\mathscr{R}(U^L, U^L)$ is isomorphic to $\mathscr{R}(L, L)$ and $\{z\}G$ is $E_{U^L|K}$-thick.**

THEOREM 2. **Let M be a representation of G such that $\{z\}G$ is $E_{M|K}$-thick for some $z \in K^*$. Let H be the stability subgroup of G at z. Suppose G/H is σ-compact. Then there is a representation L of H such that $\{z\}$ is $E_{L|K}$-thick and such that $M \simeq U^L$.**

In the above, U^L denotes the representation of G induced by L.

It is shown further that if $C^*(K)|z$ contains an ideal isomorphic to the algebra of all compact operators on some Hilbert space, then the representation $L|K$ of these theorems is a multiple of the (essentially unique) irreducible representation L^0 of K such that $\mathrm{Ker}\,(L^0) = z$. Finally, it is shown that if M is primary and if K^*/G is almost Hausdorff (i.e., every nonvoid closed subset contains a nonvoid relatively open Hausdorff subset), then M satisfies the hypothesis of Theorem 2.

These results generalize Mackey's Theorem 8.1 [13], in the case of the trivial multiplier. In [13], Mackey attacks the problem of

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reducing the representation theory of a locally compact group G with closed normal subgroup K to the representation theories of K and G/K. His main theorem, Theorem 8.1, supposes the following restrictions on G and K: G satisfies the second axiom of countability and K is type I (in the case of the trivial multiplier). The present paper explores what happens when these restrictions are lifted.

It turns out that a great deal of Mackey's theorem remains true in modified form. The chief modifications are these:

(a) We replace the dual space $\hat{K}$ of K by the structure space K^* of its group C^*-algebra. This is done because K^* is fairly well behaved, being a T_0 topological space, while $\hat{K}$ can be very messy when K is not type I. The first example of § 6 shows how a theory based on $\hat{K}$ cannot get off the ground.

(b) We replace the projection valued measure based on $\hat{K}$ which is canonically associated with the direct integral decomposition of a given representation of K (when K is type I) with the measure based on K^* introduced by Glimm in [10].

These modifications and the lack of separability force us to replace Mackey's highly measure theoretic arguments with arguments more in the spirit of the present author's previous work [1-3] and that of Glimm's paper [10].

After the preliminaries of § 2, we prove our analogue of Mackey's theorem in §§ 3 and 4. Section 5 is concerned with what additional hypotheses are needed to make the analogue exact. The paper closes with some examples in § 6.

The problem dealt with in the last example was the starting point for this investigation. We wish to thank James Glimm for several stimulating conversations on this problem.

2. Preliminaries. Let G be a locally compact group and let $C_0(G)$ be the space of continuous complex valued functions on G with compact support. If $f, g \in C_0(G)$ set $(f \circ g)(x) = \int_G f(y)g(xy^{-1})dy$ and $f^*(x) = \overline{f^*(x^{-1})}\delta_G(x)^{-1}$, where dy denotes right invariant Haar measure and where δ_G is its modular function. $\circ$, $*$, the usual addition of functions, and the usual inductive limit topology on $C_0(G)$ turn it into a topological $*$-algebra. Let L be a unitary representation of G. Then setting $L_f = \int_G f(x)L_x^{-1}\,dx$ (strong operator topology integral) gives us a continuous $*$-representation of $C_0(G)$. Moreover $\{L_f : f \in C_0(G)\}$ has no simultaneous null vectors (we say that L is a *nondegenerate* representation of $C_0(G)$). Conversely, if Φ is a nondegenerate continuous $*$-representation of $C_0(G)$, there is a unique unitary representation L of G such that $L_f = \Phi_f$ for $f \in C_0(G)$.

If L is unitary representation of G, then $\|L_f\| \leqq \|f\|_1$ for $f \in C_0(G)$.

Therefore $||f|| = LUB\{||L_f|| : L$ unitary representation of $G\}$ exists. $||\cdot||$ is a norm on $C_0(G)$ and the completion of $C_0(G)$ with respect to $||\cdot||$ is a C^*-algebra, called the C^*-group algebra of G and denoted by $C^*(G)$. Clearly there is a one-to-one correspondence between nondegenerate *-representations of $C^*(G)$ and nondegenerate continuous *-representations of $C_0(G)$. Thus the representation theory of $C^*(G)$ is "the same" as that of G.

In what follows, G^* will denote the Jacobson structure space of $C^*(G)$; i.e. the space of kernels of irreducible nondegenerate *-representations of $C^*(G)$ equipped with the hull-kernel topology. G^* is a T_0-space.

Let $\mathfrak{A}$ be a C^*-algebra, Z its structure space, and Φ a nondegenerate *-representation of $\mathfrak{A}$ on a Hilbert space $\mathfrak{H}$. Glimm [10] has shown that there is unique projection valued measure E on the Borel field generated by the topology of Z with the following property: if S is a closed subset of Z, then $E(S)$ is the projection on the manifold of $v \in \mathfrak{H}$ such that $\Phi(a)v = 0$ for all $a \in \cap S$. E takes its values in the center of the von Neumann algebra generated by $\Phi(\mathfrak{A})$. In our case, if L is a unitary representation of G, E_L will denote the Glimm measure on G^* associated with the representation of $C^*(G)$ determined by L.

For the formulation of induced representations used in this paper the reader is referred to [1]. If L is a representation of the closed subgroup H of G, we define a regular Borel projection valued measure E^L on G/H as follows: if S is a Borel subset of G/H, then $E^L(S)f = (\chi_S \circ \pi)\cdot f$, where χ_S is the characteristic function of S, π is the canonical projection of G onto G/H, and $f \in \mathfrak{H}(U^L)$. This E^L determines, and is determined by, the *-representation of $C_0(G/H)$ defined by

$$E^L(h) = \int_{G/H} h(p)dE^L(p) ,$$

$h \in C_0(G/H)$ (cf. [3]).

Finally, if E is any projection valued measure on the measurable space $(Z, \mathscr{B})$, any subset $S \subseteq Z$ (not necessarily in $\mathscr{B}$) will be called *E-thick* if $E(T) = 0$ whenever $T \cap S = \varnothing$ (cf. [11], p. 74).

Let G be a locally compact group and let K be a closed normal subgroup. For $f \in C_0(K)$ and $x \in G$, we define $xf \in C_0(K)$ by the formula $(xf)(\xi) = f(x^{-1}\xi x)\Delta(x)$ for $\xi \in K$, where $\Delta(x)$ is the (constant) Radon-Nikodym derivative $[d(x^{-1}\xi x)]/d\xi$. If L is a unitary representation of K and if L^x is defined by $L^x_\xi = L_{x\xi x^{-1}}$, $\xi \in K$, then $L^x_f = L_{xf}$. From this it follows readily that $f \to xf$ is an automorphism of $C_0(K)$ which is isometric in the C^*-norm $||\cdot||$. Therefore this map extends to an automorphism of $C^*(K)$. We define an action of G on K^* by setting $zx = \{f \in C^*(K) : xf \in z\}$ for $z \in K^*$, $x \in G$. Glimm [10] and Fell [7] have shown that the map of $K^* \times G \to K^*$ given by $(z, x) \to zx$ is continuous, giving us a topological transformation group.

LEMMA 1. *Let* (S, G) *be a topological transformation group. Suppose* S *is a* T_0*-space. Then the stability subgroups of* G *are closed.*

Proof. Let H be the stability subgroup of G at $p \in S$. Then $\{p\}H = \{p\}$, so that $\{p\}^- H^- \subseteqq \{p\}^-$. If $x \in H^-$, we have $\{px\}^- = \{p\}^- x \subseteqq \{p\}^-$. Since $x^{-1} \in H^-$, we have $\{p\}^- x^{-1} \subseteqq \{p\}^-$ and hence $\{p\}^- \subseteqq \{p\}^- x = \{px\}^-$; i.e., $\{px\}^- = \{p\}^-$. But S is T_0. Therefore $px = p$ and $x \in H$.

We may now state our main theorems.

THEOREM 1. *Let* G *be a locally compact group and let* K *be a closed normal subgroup of* G. *Let* $z \in K^*$ *and let* H *be the* (*closed*) *stability subgroup of* G *at* z. *Let* L *be a representation of* H *such that* $\{z\}$ *is* $E_{L|K}$*-thick. Then* $\mathscr{R}(U^L, U^L)$ *is isomorphic to* $\mathscr{R}(L, L)$ *and* $\{z\}G$ $E_{U^L|K}$*-thick.*

THEOREM 2. *Let* G *be a locally compact group and let* K *be a closed normal subgroup of* G. *Let* M *be a representation of* G. *Assume that* $\{z\}G$ *is* $E_{M|K}$*-thick for some* $z \in K^*$. *Let* H *be the* (*closed*) *stability subgroup of* G *at* z. *Suppose* G/H *is* σ*-compact. Then there is a representation* L *of* H *such that* $\{z\}$ *is* $E_{L|K}$*-thick and such that* $M \simeq U^L$.

3. **Proof of Theorem 1.** We begin our proofs with the following lemma.

LEMMA 2. *Let* H *and* K *be closed subgroups of the locally compact group* G, K *normal, and* $K \subseteqq H \subseteqq G$. *Let* L *be a representation of* H. *Then for every* $f \in C^*(K)$ *and* $g \in \mathfrak{H}(U^L)$ *we have* $[(U^L \mid K)_f g](x) = (L \mid K)_f^x g(x)$ *for locally almost all* $z \in G$.

Proof. Suppose first that $f \in C_0(K)$ and g is continuous with compact support modulo H. Set

$$u(x) = (L \mid K)_f^x g(x) = \int_K f(\xi) L_{x\xi^{-1}x^{-1}} g(x) d\xi = \int_K f(\xi) g(x\xi^{-1}) d\xi \ .$$

Clearly u is continuous with compact support modulo H and belongs to $\mathfrak{H}(U^L)$. Let $v \in \mathfrak{H}(U^L)$ be continuous with compact support modulo H, and choose $h \in C_0(G)$ such that $\int_H h(\eta x) d\eta = 1$ for x in the support of v. Then

$$(u, v) = \int_G h(x)(u(x), v(x))\,dx = \int_G\int_K h(x)f(\xi)(g(x\xi^{-1}), v(x))d\xi dx$$
$$= \int_K f(\xi)(U^L_{\xi^{-1}}g, v)d\xi = ((U^L \mid K)_f g, v)\ .$$

Since the set of all such v is dense in $\mathfrak{H}(U^L)$, our result holds in this case.

Next suppose $f \in C_0(K)$ and $g \in \mathfrak{H}(U^L)$. Choose a sequence $g_n \in \mathfrak{H}(U^L)$, continuous with compact support modulo H, such that $\| g_n - g \| < 2^{-n}$. Then $\| (U^L \mid K)_f g_n - (U^L \mid K)_f g \| < \| f \| 2^{-n}$. As in the proof of Proposition 1 of [1], $g_n \rightarrow g$ and $(U^L \mid K)_f g_n \rightarrow (U^L \mid K)_f g$ locally almost everywhere.

Finally, if $f \in C^*(K)$ and $g \in \mathfrak{H}(U^L)$, we may choose a sequence $f_n \in C_0(K)$ such that $\| f_n - f \| < 2^{-n}$. Then $\| (L \mid K)^x_f - (L \mid K)^x_f \| < 2^{-n}$ uniformly for all $x \in G$, so that $(L \mid K)^x_{f_n} g(x) \rightarrow (L \mid K)^x_f g(x)$ uniformly on G. Moreover $\| (U^L \mid K)_{f_n} g - (U^L \mid K)_f g \| < 2^{-n} \| g \|$, from which it follows that $(U^L \mid K)_{f_n} g \rightarrow (U^L \mid K)_f g$ locally almost everywhere. Our lemma is thereby proved.

We now assume all the hypotheses of Theorem 1. If π is the natural projection of G into G/H, we define $\alpha : G/H \rightarrow K^*$ by $\alpha(\pi(x)) = zx$ for $x \in G$. α is continuous and one-to-one.

LEMMA 3. *$E_{U^L|K}(S) = E^L(\alpha^{-1}(S))$ for every Borel set $S \subseteqq K^*$.*

Proof. In the first place, we note that $(L \mid K)_f = 0$ for $f \in z$. In fact, $\{z\} \cap \complement\{z\}^- = \emptyset$ implies that $E_{L|K}(\complement\{z\}^-) = 0$ from which we get $E_{L|K}(\{z\}^-) = I$. But this says that $(L \mid K)_f v = 0$ for all $v \in \mathfrak{H}(L)$ and all $f \in \cap \{z\}^- = z$, as desired.

Let S be a closed subset of K^*. Then $\pi(x) \in \alpha^{-1}(S)$ if and only if $zx \in S$, that is, if and only if $zx \supseteqq \cap S$. Therefore $f \in \cap S$ implies $xf \in z$ and hence $(L \mid K)^x_f = 0$. Let $g \in \mathfrak{H}(U^L)$. Let $f \in \cap S$. By Lemma 2, we have $[(U^L \mid K)_f g](x) = 0$ for locally almost all $x \in \pi^{-1}(\alpha^{-1}(S))$. If, moreover, $g \in$ Range $(E^L(\alpha^{-1}(S)))$ then $g(x) = 0$ for locally almost all $x \notin \pi^{-1}(\alpha^{-1}(S))$, so by Lemma 2, $[(U^L \mid K)_f g](x) = 0$ for locally almost all $x \notin \pi^{-1}(\alpha^{-1}(S))$. We conclude that Range $(E^L(\alpha^{-1}(S))) \subseteqq$ Range $(E_{U^L|K}(S))$.

Suppose now that $g \notin$ Range $(E^L(\alpha^{-1}(S)))$. Then g does not vanish for locally almost all $x \notin \pi^{-1}(\alpha^{-1}(S))$. Since g is Bourbaki measurable ([4], p. 180), there exists a compact set $C \subseteqq \complement\pi^{-1}(\alpha^{-1}(S))$ of positive Haar measure upon which g is continuous and does not vanish. Let $x \in C$. Then $z \notin Sx^{-1}$ so that $E_{L|K}(Sx^{-1}) = 0$. Hence there exists $f \in \cap \{Sx^{-1}\}$ such that $(L \mid K)_f g(x) \neq 0$. Setting $f_{(x)} = x^{-1}f$, we have $f_{(x)} \in S$ and $(L \mid K)^x_{f_{(x)}} g(x) \neq 0$. By continuity we have $(L \mid K)^y_{f_{(x)}} g(y) \neq 0$

for y in some neighborhood N_x of x in C. Since C is compact and of positive measure, N_x has positive measure for some $x \in C$. It follows from Lemma 2 that for that x, $(U^L \mid K)_{f(x)} g \neq 0$. Therefore $g \notin \text{Range } (E_{U^L|K}(S))$.

We have proved Lemma 3 for closed S. The general case then follows from the fact that a projection valued Borel measure on a topological space is uniquely determined by its values on closed sets.

LEMMA 4. *Let H be a closed subgroup of the locally compact group G and let E be a regular Borel projection valued measure on G/H. Let $\mathscr{T}$ be a T_0 topology on G/H weaker than the natural topology such that $((G/H)_{\mathscr{T}}, G)$ is a topological transformation group. Then E takes its values in the von Neumann algebra generated by $\{E(S) : S \in \mathscr{T}\}$.*

Proof. Let B be a Borel set in G/H and let T be a self-adjoint bounded operator commuting with all $E(S)$, $S \in \mathscr{T}$. We must show that $E(\complement B) TE(B) = 0$. Since E is regular, it will suffice to show that $E(C_1) TE(C_2) = 0$ for every disjoint compact pair C_1, $C_2 \subseteqq G/H$. A standard compactness argument reduces the problem to the following: if $p_1, p_2 \in G/H$, $p_1 \neq p_2$, find disjoint neighborhoods N_1 of p_1 and N_2 of p_2 such that $E(N_1) TE(N_2) = 0$. To do this we find a $\mathscr{T}$-closed S which separates p_1 and p_2; say, $p_1 \notin S$, $p_2 \in S$. Since S is closed in the natural topology of G/H, we can find a compact neighborhood N of e in G such that $p_1 NN^{-1} \cap S = \varnothing$. Set $N_1 = p_1 N$, $N_2 = p_2 N$. Clearly $N_1 \subseteqq \complement SN$ and $N_2 \subseteqq SN$. Since S is $\mathscr{T}$-closed and N is compact, SN is $\mathscr{T}$-closed. By hypothesis $E(\complement SN) TE(SN) = 0$, and our result follows.

Proof of Theorem 1. Let $\mathscr{T} = \alpha^{-1}$ (topology of K^*). $\mathscr{T}$ and E^L satisfy the hypotheses of Lemma 4. According to Lemma 3 $\{E^L(S) : S \in \mathscr{T}\} \subseteqq \{\text{values of } E_{U^L|K}\}$. This, in turn, is contained in the center $\mathscr{C}$ of the von Neumann algebra generated by $U^L \mid K$. By Lemma 4, $\{\text{values of } E^L\} \subseteqq \mathscr{C}$. Therefore $\mathscr{R}(U^L, U^L) = \mathscr{R}((E^L, U^L), (E^L, U^L) \cong \mathscr{R}(L, L)$ by [2]. Finally, $\{z\}G$ is $E_{U^L|K}$-thick by Lemma 3.

4. Proof of Theorem 2. For the proof of Theorem 2 we need the following lemmas;

LEMMA 5. *Let H be a closed subgroup of the locally compact group G such that G/H is σ-compact. Let $\mathscr{T}$ be a T_0 topology on G/H weaker than the natural topology such that $((G/H)_{\mathscr{T}}, G)$ is a topological transformation group. Let $\mathscr{B}$ be the Borel field generated by $\mathscr{T}$. Let $f \in C_0(G/H)$. Then f is $\mathscr{B}$-measurable.*

Proof. As is well known, it is enough to show that $C \in \mathscr{B}$, where $C = \bigcap_i^\infty O_i$, C is compact and the O_i are open in G/H. (See [11], p. 220. Such a set is called a compact G_δ.) Since $\complement O_i$ is closed, it is σ-compact, whence $\complement C$ is σ-compact. Therefore $\complement C$ has the Lindelöf property. Let $p \in C$, $q \in \complement C$. As in the proof of Lemma 4, there exist open neighborhoods N_{pq} of p and M_{pq} of q and a set $S_{pq} \subseteqq G/H$ such that either S_{pq} or $\complement S_{pq} \in \mathscr{T}$, $N_{pq} \subseteqq S_{pq}$, $M_{pq} \subseteqq \complement S_{pq}$. Since C is compact, we find $p_1, \cdots, p_n$ such that $C \subseteqq \bigcup_i{}_1^n N_{p_iq}$ and set $S_q = \bigcup_i{}_1^n S_{p_iq}$ and $M_q = \bigcap_i{}_1^n M_{p_iq}$. Then $S_q \in \mathscr{B}$, $C \subseteqq S_q$, $M_q \subseteqq \complement S_q$, and M_q is an open neighborhood of q. Since $\complement C$ is Lindelöfian, we find $q_1, q_2, \cdots$ such that $\complement C \subseteqq \bigcup_i{}_1^\infty M_{q_i}$ and set $S = \bigcap_i{}_1^\infty S_{q_i}$. Then $S \in \mathscr{B}$. $C \subseteqq S$, $\complement C \subseteqq \complement S$; that is, $C \in \mathscr{B}$.

LEMMA 6. *Let G be a locally compact group, K a closed normal subgroup, and U a representation of G. Let G act on K^* as above. Then, for any Borel set S in K^* and $x \in G$,* $E_{U|K}(Sx) = U_x^{-1}E_{U|K}(S)U_x$.

Proof. By the uniqueness of Glimm measure, it suffices to prove this for S closed in K^*. We note that $(U\,|\,K)^{x^{-1}} = U_x^{-1}(U\,|\,K)U_x$. Therefore $v \in \text{Range}\, E_{U|K}(Sx)$ if and only if $(U\,|\,K)_f v = 0$ for all $f \in \cap Sx$, if and only if $(U\,|\,K)_f^{x^{-1}} v = 0$ for all $f \in \cap S$, if and only if $U_x^{-1}(U\,|\,K)_f U_x v = 0$ for all $f \in \cap S$. But this is true if and only if $U_x v \in \text{Range}\ E_{U|K}(S)$, if and only if $v \in \text{Range}\ U_x^{-1} E_{U|K}(S)U_x$.

LEMMA 7. *Let U be a representation of the locally compact group K. Let $\mathscr{S}$ be a collection of closed subsets of K^*. Then* Range $E_U(\cap \mathscr{S}) = \cap\{\text{Range}\ E_U(S)\,;\ S \in \mathscr{S}\}$.

Proof. Let $S_0 = \cap \mathscr{S}$. Then $\cap S_0 =$ the closed ideal of $C^*(K)$ generated by $\cup\{\cap S : S \in \mathscr{S}\} =$ the closed linear span in $C^*(K)$ of $\cup\{\cap S : S \in \mathscr{S}\}$. Now let $v \in \cap\{\text{Range}\, E_U(S) : S \in \mathscr{S}\}$. Then, for every $S \in \mathscr{S}$ and every $f \in \cap S$, we have $U_f v = 0$; that is, $U_f v = 0$ for every $f \in \cup\{\cap S : S \in \mathscr{S}\}$. By linearity and continuity, $U_f v = 0$ for every $f \in \cap S$. Therefore $v \in \text{Range}\ E_U(S_0)$. The opposite inclusion is clear, since E_U is monotonic.

Proof of Theorem 2. Let α be as above, let $\mathscr{T} = \alpha^{-1}$ (topology of K^*), and let $\mathscr{B}$ be the Borel field generated by $\mathscr{T}$. Then $\mathscr{B} = \alpha^{-1}$ (Borel field of K^*). As in [11], p. 75, $E_0(\alpha^{-1}(S)) = E_{M|K}(S)$ for all Borel S in K^* defines a projection valued measure E_0 on $\mathscr{B}$. According to Lemma 5, every function in $C_0(G/H)$ is $\mathscr{B}$-measurable. Define $E(f) = \int_{G/H} f dE_0$. Clearly E is a *-representation of $C_0(G/H)$ in the sense of [3]. We assert that (E, M) is a representation of the locally compact transformation group $(G/H, G)$ as defined in [3].

(1) $E(C_0(G/H))\mathfrak{H}(M)$ is dense in $\mathfrak{H}(M)$. In fact, since G/H is σ-compact, there exists a sequence of functions $f_n \in C_0(G/H)$ such that $0 \leqq f_n \uparrow 1$. By the monotone convergence theorem, $E(f_n) \to I$ weakly, and (1) is established.

(2) $M_x E(f) M_x^{-1} = E(R_x f)$ for all $f \in C_0(G/H)$ and all $x \in G$. Here $(R_x f)(p) = f(px)$. For this, it suffices to show that $M_x E_0(B) M_x^{-1} = E_0(Bx^{-1})$ for all $B \in \mathscr{B}$ and $x \in G$. But this follows immediately from Lemma 6 and the definition of E_0.

According to the Corollary of Theorem 2 in [3], (E, M) is unitarily equivalent to an induced representation of $(G/H, G)$; that is, there is a representation L of H such that (E, M) is unitarily equivalent to (E^L, U^L). We shall henceforth assume $E = E^L$ and $M = U^L$. In particular, we have $E_0 \circ \alpha^{-1} = E_{U^{L|K}}$ and also $E_0(C) = E^L(C)$ for every compact G_δ C in G/H.

We must now show that $\{z\}$ is $E_{L|K}$-thick.

Let S be K^* closed. First suppose $z \in S$. We assert that $E_{L|K}(S) = I$. By Lemma 7, it suffices to show that $E_{L|K}(SN) = I$ for every compact neighborhood N of e in G. Let $g \in C_0(G)$, $v \in \mathfrak{H}(L)$. As in [1], we define $\varepsilon(g, v) \in \mathfrak{H}(U^L)$ by

$$\varepsilon(g, v)(x) = \int_H g(\xi x)\delta_H(\xi)^{-1/2}\delta_G(\xi)^{1/2}L_\xi^{-1}v\,d\xi\ .$$

$\varepsilon(g, v)$ is continuous and has compact support modulo H. Let C be a compact G_δ neighborhood of $\pi(e) \subseteqq \pi(N)$. Suppose Support $(g) \subseteqq N \cap \pi^{-1}(C)$. Then

$$\begin{aligned}\varepsilon(g, v) \in \text{Range } E^L(C) &= \text{Range } E_0(C) \subseteqq \text{Range } E_0(\alpha^{-1}(SN)) \\ &= \text{Range } E_{U^{L|K}}(SN)\ .\end{aligned}$$

According to Lemma 2, $f \in \cap SN$ implies $(L \mid K)_f^x \varepsilon(g, v)(x) = 0$ for locally almost all $x \in G$, hence for all $x \in G$ by continuity. In particular, $(L \mid K)_f \varepsilon(g, v)(e) = 0$. Letting $g\delta_H^{-1/2}\delta_G^{1/2} \mid H$ approach the Dirac δ function on H, we get $(L \mid K)_f v = 0$. Since v is arbitrary, $E_{L|K}(SN) = I$.

Now suppose $z \notin S$. We assert that $E_{L|K}(S) = 0$. Let $v \in \text{Range}\, E_{L|K}(S)$. Choose a compact neighborhood N of e in G such that $\pi(NN^{-1}) \cap \alpha^{-1}(S) = \varnothing$. Then $\alpha^{-1}(SN) \cap \pi(N) = \varnothing$. Let $f \in \cap SN$. Let $x \in G$. Then $xf \in \cap SNx^{-1}$. Hence if $\xi \in Nx^{-1} \cap H$, we have $xf \in \cap S\xi$, so that $\xi xf \in \cap S$. Let C be a compact G_δ neighborhood of $\pi(e)$ in $\pi(N)$. Suppose Support $(g) \subseteqq N \cap \pi^{-1}(C)$. Then

$$(L \mid K)_f^x \varepsilon(g, v)(x) = \int_{N_x^{-1} \cap H} g(\xi x)\delta_H(\xi)^{-1/2}\delta_G(\xi)^{1/2}L_\xi^{-1}(L \mid K)_{\xi x f} v\,d\xi = 0$$

(compare the proof of Lemma 6). From Lemma 2,

$$\varepsilon(g, v) \in \text{Range } E_{UL|K}(SN) = \text{Range } E_0(\alpha^{-1}(SN)) .$$

On the other hand $\varepsilon(g, v) \in \text{Range } E^L(C) = \text{Range } E_0(C)$. Since $\alpha^{-1}(SN) \cap C = \varnothing$, $\varepsilon(g, v) = 0$. Therefore $\varepsilon(g, v)(e) = 0$ because $\varepsilon(g, v)$ is continuous. Again letting $g\delta_H^{-1/2}\delta_G^{1/2} \mid H$ approach the Dirac δ function on H, we get $v = 0$. Therefore $E_{L|K}(S) = 0$.

Finally let $\mathscr{C}$ be the class of all Borel sets S in K^* such that either $z \in S$ and $E_{L|K}(S) = I$ or $z \notin S$ and $E_{L|K}(S) = 0$. Clearly $\mathscr{C}$ is a σ-field, and by the foregoing $\mathscr{C}$ contains all the closed sets. Therefore $\mathscr{C}$ consists of all the Borel sets of K^*; that is, $\{z\}$ is $E_{L|K}$-thick.

5. **Connections with Mackey's work.** In the original forms of Theorems 1 and 2 due to Mackey ([13], Theorem 8.1), it is assumed that G satisfies the second axiom of countability and that K is a type I group. K^* is replaced there by $\hat{K}$, the set of all unitary equivalence classes of irreducible representations of K, equipped with the Mackey Borel structure. According to Glimm ([8], Theorem 1) the natural mapping of $\hat{K}$ onto K^*, which sends every irreducible representation into its kernel in $C^*(K)$, is one-to-one if K is of type I and second countable. Moreover, Fell has shown [6] that in this case the Mackey Borel structure is just the σ-field generated by the topology of $\hat{K}(= K^*)$. Our result then specializes to give Mackey's result, except for the following: Mackey shows that $L \mid K$ must be a multiple of the (unique up to unitary equivalence) representation (whose kernel is) z. To get this, in our general setting, seems to require a type restriction on K (or at least on z). The form of our restriction is suggested by Glimm's theorem ([8], Theorem 1) that a separable C^* algebra is of type I if and only if its image under every irreducible representation contains the compact operators. We are led to make the following definition:

DEFINITION. Let $\mathfrak{A}$ be a C^*-algebra and let L be an irreducible representation of $\mathfrak{A}$. L is called *semi-compact* if $L_{\mathfrak{A}}$ contains the compact operators on $\mathfrak{H}(L)$. Ker L will also be called semi-compact.

We know (see Glimm [1], p. 583) that if L is semi-compact and if M is irreducible with $\text{Ker}\, L = \text{Ker}\, M$, then L and M are unitarily equivalent.

LEMMA 8. *Let U be a representation of the C^*-algebra $\mathfrak{A}$ with structure space Z. Let z be semi-compact in Z. Suppose $\{z\}$ is E_U-thick. Then U is a multiple of the* (*essentially unique*) *irreducible representation L^0 of K such that* $\text{Ker}\, L^0 = z$.

Proof. By hypothesis $\mathfrak{A}$ contains an ideal $\mathscr{J} \supseteqq z$ such that $\mathscr{J}/z$

is isomorphic to the algebra of all compact operators on $\mathfrak{H}(L^0)$. As in the proof of Lemma 3, $E_U(\{z\}^-) = I$ implies that $U_a = 0$ for all $a \in z$. Dividing out by z, we may therefore assume $z = \{0\}$. Let $S = \{w \in Z : w \supseteq \mathscr{I}\}$. S is closed. $\mathscr{I} \neq \{0\}$ implies $\{0\} \notin S$. Therfore $E_U(S) = 0$. Since $\mathscr{I} = \cap S$, this says that $U|\mathscr{I}$ is a nondegenerate representation of $\mathscr{I}$. From the known representation theory of the algebra of compact operators on a Hilbert space, we obtain an orthogonal decomposition of $\mathfrak{H}(U)$ into $U|\mathscr{I}$ invariant subspaces $\mathfrak{H}^\gamma$, the restriction of $U|\mathscr{I}$ to each of which is unitarily equivalent to the irreducible representation $L^0|\mathscr{I}$. Let $a \in \mathfrak{A}$, $v \in \mathfrak{H}^\gamma$. Since $U|\mathscr{I}$ is nondegenerate, we can choose a sequence $b_n \in \mathscr{I}$ such that $U_{b_n}U_a v \longrightarrow U_a v$. But $b_n a \in \mathscr{I}$ and hence $U_{b_n a}v \in \mathfrak{H}^\gamma$. Therefore $U_a v \in \mathfrak{H}^\gamma$; that is, the $\mathfrak{H}^\gamma$ are invariant under U. Let $L^\gamma = U$ restricted to act on $\mathfrak{H}^\gamma$. L^γ is irreducible since $L^\gamma|\mathscr{I}$ is. Now Range $E_U(\{\mathrm{Ker}\, L^\gamma\}^-) \supseteq \mathfrak{H}^\gamma \neq \{0\}$. Since $\{0\}$ is E_U-thick, $\{0\} \in \{\mathrm{Ker}\, L^\gamma\}^-$; that is, $\mathrm{Ker}\, L^\gamma = \{0\}$. Since $\mathrm{Ker}\, L^\gamma = \mathrm{Ker}\, L^0$, $L^\gamma \simeq L^0$. Therefore $U \simeq$ a multiple of L^0.

As regards Theorem 2, Mackey shows that if, in addition to the hypotheses on G and K mentioned above, one assumes that K is regularly embedded in G (see [13], p. 302 for the definition), then $E_{M|K}$ is concentrated in an orbit if M is primary. Glimm ([9], Theorem 1) has proved that, in Mackey's case, the assumption of regular embeddedness is equivalent to the topology of $\hat{K}/G$ being almost Hausdorff, in the following sense:

DEFINITION. Let X be a topological space. X is said to be *almost Hausdorff* if every nonvoid closed subset contains a nonvoid relatively open Hausdorff subset.

We propose to turn Glimm's theorem into a definition, even when K is not of type I and second countable.

DEFINITION. Let K be a closed normal subgroup of the locally compact group G. K is *regularly embedded* in G if K^*/G is almost Hausdorff.

REMARK. It follows from [9], p. 133, that K regularly embedded in G implies that every G-orbit in K^* is a Borel set and in fact is relatively open in its closure.

LEMMA 9. *Let K be a regularly embedded closed normal subgroup of the locally compact group G. Let M be a primary representation of G. Then there is a G-orbit of K^* which is $E_{M|K}$-thick.*

Proof. If S is a G-invariant Borel set in K^*, then

$$E_{M|K}(S) \in \mathscr{R}(M, M)$$

by Lemma 6, and hence $E_{M|K}(S)$ belongs to the center of the von Neumann algebra generated by M. Therefore $E_{M|K}(S) = 0$ or I. Let $\mathscr{S}$ be the collection of all closed G-invariant $S \subseteqq K^*$ such that $E_{M|K}(S) = I$. $S_0 = \cap \mathscr{S} \in \mathscr{S}$ by Lemma 7. Let θ be the natural projection of K^* onto K^*/G. If R is any nonvoid relatively open subset of $\theta(S_0)$, then $S_0 - \theta^{-1}(R)$ is a proper closed G-invariant subset of S_0. Hence $E_{M|K}(S_0 - \theta^{-1}(R)) = 0$ so that $E_{M|K}(\theta^{-1}(R)) = I$. Now $\theta(S_0)$ is a nonvoid closed subset of K^*/G. There exists a nonvoid relatively open Hausdorff subset R_0 of $\theta(S_0)$. We assert that R_0 reduces to a point. If not, then R_0 contains two nonvoid disjoint subsets R_1 and R_2 which are open relative to R_0 and hence to $\theta(S_0)$. Then $E_{M|K}(\theta^{-1}(R_i)) = I$ for $i = 1, 2$, an impossibility. So R_0 reduces to a point, $\theta^{-1}(R_0)$ is a G-orbit in K^*, and our lemma is proved.

REMARK. This is not the only reasonable definition of regular embeddedness. Indeed, if G satisfies the second axiom of countability, we could simply require that K^*/G be T_0 or, more generally, be countably separated. The conclusion of Lemma 9 would then follow (cf. [9], p. 126). If K is not type I, the relations between these properties and the almost Hausdorff property is obscure.

6. Three examples. Our first example shows that, despite Lemma 1, the stability subgroup of G at a point in $\hat{K}$ may be very bad. Let G be the group whose underlying topological space is $\boldsymbol{T} \times \boldsymbol{Z} \times \boldsymbol{C}$, where $\boldsymbol{T} = \{\xi \in \boldsymbol{C} : |\xi| = 1\}$ and $\boldsymbol{Z}$ is the discrete integers. The group multiplication is given by

$$(\xi, m, a)(\zeta, n, b) = (\xi\zeta, m + n, a\zeta e^{in} + b) .$$

Let $K = \{1\} \times \boldsymbol{Z} \times \boldsymbol{C}$ and $N = \{1\} \times \{0\} \times \boldsymbol{C}$. N and K are normal subgroups of G, and N is abelian. We identify $\hat{N} = N^*$ with $\boldsymbol{C}$ as follows: each $\lambda \in \boldsymbol{C}$ corresponds to the character $\chi_{(\lambda)}: (1, 0, a) \rightarrow e^{i \operatorname{Re}(a\lambda)}$. In terms of this identification, the action of K on N^* is given by $\lambda^{(1,m,a)} = \lambda e^{-im}$. By Theorems 1 and 2, ${}^{\lambda}L = {}_{K}U^{\chi_{(\lambda)}}$ is irreducible if $\lambda \neq 0$; moreover ${}^{\lambda}L \simeq {}^{\mu}L$ if and only if λ and μ belong to the same K-orbit in N^*.

We next calculate ${}^{\lambda}L^{(\xi,m,a)}$. To this end, we realize ${}^{\lambda}L$ in the Hilbert space of all square summable functions f on $\boldsymbol{Z}$ according to the rule: $({}^{\lambda}L_{(l,n,b)}f)(k) = \exp(i \operatorname{Re}(\lambda b e^{-i(n+k)}))f(n + k)$. Then $({}^{\lambda}L^{(\xi,m,a)}_{(l,n,b)}f)(k) = ({}^{\lambda}L_{(l,n,b\xi^{-1}e^{-im})}f)(k) = \exp(i \operatorname{Re}(\lambda b \xi^{-1} e^{-im} e^{-i(n+k)}))f(n + k)$. It follows that ${}^{\lambda}L^{(\xi,m,a)} = {}^{\lambda\xi^{-1}e^{-im}}L$. Therefore ${}^{\lambda}L^{(\xi,m,a)} \simeq {}^{\lambda}L$ if and only if λ and $\lambda\xi^{-1}$ are in the same K-orbit. Supposing $\lambda \neq 0$, we see that the stability subgroup of G at $({}^{\lambda}L)^{\wedge}$ in $\hat{K}$ is $\{(\xi, m, a) : \xi = e^{in}$ for some

$n \in \boldsymbol{Z}\}$, a proper dense subgroup of G.

As a consequence of Lemma 1, we see that ${}^{\lambda}L$ and ${}^{\mu}L$ have the same kernel in $C^*(K)$ if $|\lambda| = |\mu|$. (That this sufficient condition is also necessary may be seen by applying the structure theory of Glimm [10].) We also see that Theorems 1 and 2 are useless in this case in analyzing the irreducible representations M of G for which $\{\mathrm{Ker}\,{}^{\lambda}L\}G$ is $E_{M|K}$-thick. This is precisely because the stability subgroup of G at $\mathrm{Ker}\,{}^{\lambda}L$ in K^* is G itself.

Our second example shows that in Theorem 2 some restriction on G/H, such as σ-compactness, is necessary. Let G_1 be the "$ax + b$ group"; that is, the group whose underlying topological space is $\boldsymbol{R} \times \boldsymbol{R}$ and whose group multiplication is given by $(a, b)(c, d) = (a + c, be^c + d)$. Let G be the same group, except that the topology is modified by making the first factor discrete. Let $K_1 = \{0\} \times \boldsymbol{R} \subseteqq G_1$ and $K = \{0\} \times \boldsymbol{R} \subseteqq G$. Let φ be the (continuous) identity map of G onto G_1. Let χ be a nontrivial character of K_1. By Theorem 1, $M_1 = {}_{G_1}U^{\chi}$ is an irreducible representation of G_1 and $\{\chi\}G_1$ is $E_{M_1|K_1}$-thick in $\hat{K}_1 = K_1^*$. Let $M = M_1 \circ \varphi$, an irreducible representation of G. Because $\varphi | K$ is an isomorphism of K onto K_1 which is equivariant with respect to G, when G and G_1 are identified as abstract groups under φ, $\{\chi \circ (\varphi | K)\}G$ is $E_{M|K}$-thick in $\hat{K} = K^*$. The stability subgroup of G at $\{\chi \circ (\varphi | K)\}$ is K. But M is not induced from any representation of K, because $\dim \mathfrak{H}(M) = \aleph_0$ while $\dim \mathfrak{H}({}_GU^L) \geqq 2^{\aleph_0}$ for any representation L of K. One may see that, in this case, the proof of Theorem 2 breaks down right at the beginning: the representation E of $C_0(G/K)$ defined there is identically zero.

Finally, we show that the first part of Theorem 6.2 of [5] is an easy consequence of our Theorem 1. In fact, we have the following generalization: Let L be an irreducible representation of the closed normal subgroup K of the locally compact group G and let H be the stability subgroup of G at $\hat{L}$ in $\hat{K}$. If ${}_GU^L$ is irreducible, then $H = K$; if L is semi-compact, the converse holds. In fact, suppose $H \neq K$ and choose $x \in H$, $x \notin K$. Define T on $\mathfrak{H}(U^L)$ by setting $(Tf)(y) = Vf(xy)$, where V implements the equivalence of L^x and L; i.e., $L^x_\xi = V^{-1}L_\xi V$, $\xi \in K$. Then $(Tf)(\xi y) = Vf(x\xi y) = VL_{x\xi x^{-1}}V^{-1}Vf(xy) = L_\xi(Tf)(y)$ for $f \in \mathfrak{H}(U^L)$, $\xi \in K$, $y \in G$, and it follows easily that $Tf \in \mathfrak{H}(U^L)$ and that T is bounded. T clearly intertwines U^L but is not a scalar multiple of I, so the first assertion is established. The converse assertion follows from Theorem 1 together with the observation that H is the stability subgroup of G at $\mathrm{Ker}\,L$ in K^* by virtue of the comment following the definition of semi-compactness.

There remains the question of whether the converse is true without the semi-compactness condition. If L is not semi-compact, but if G/K

is discrete, the converse is true ([12], Theorem 3′). However. the general case is open.

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