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SOME AVERAGES OF CHARACTER SUMS

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Let χ and ψ be nonprincipal characters mod p. Let f be a polynomial mod p and let $a_1, \cdots, a_p$ be complex constants. We will assume $a_j = a_k$ for $j \equiv k(p)$, and thus have a_n defined for all n. Define

$$S = \sum_r a_r \chi(f(r)) \tag{1}$$

and

$$J_n(c) = \sum_r \psi(r)\chi(r^n - c) , \tag{2}$$

where the variables of summation run through a complete system of residues mod p.

The averages in question are

$$A_1 = \sum_{a=1}^{p-1} | J_n(a) |^2 \tag{3}$$

and

$$A_2 = \Sigma | S |^2 , \tag{4}$$

where the sum in (4) is over the coefficients mod p of certain fixed powers of the variables in f. Exact formulae for A_1 will be obtained in all cases, and for A_2 in an extensive class of cases.

Specifically, the following theorems are true.

THEOREM I. *Let $f(r) = yr^{m_1} + xr^{m_2} + g(r)$ and assume $(m_2 - m_1, p - 1) = 1$. Let the sum in* (4) *be over all x and y* mod p. *If g has a nonzero constant term and neither m_1 nor m_2 is zero, then*

$$A_2 = p(p-1) \sum_{r=1}^{p-1} | a_r |^2 + p^2 | a_0 |^2 . \tag{5}$$

Otherwise,

$$A_2 = p(p-1) \sum_{r=1}^{p-1} | a_r |^2 . \tag{6}$$

THEOREM II. *Let $d = (n, p-1), \psi(t) = e^{2\pi i (r \operatorname{ind}(t)/s)}$, where, naturally, $s \mid (p-1), (r, s) = 1$ and $g^{\operatorname{ind}(t)} \equiv t(p)$ for g a primitive root* mod p. *If $ds \nmid (p-1)$, then $A_1 = 0$. If $ds \mid (p-1)$ and $\psi\chi^n$ is nonprincipal, then $A_1 = p(p-1)d$. If $ds \mid (p-1)$ and $\psi\chi^n$ is principal, then $A_1 = p(p-1)(d-1) - (p-1)$.*

The following is an immediate consequence of the first theorem.

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THEOREM III. *Let f be as in Theorem I, and assume $|a_r| = 1$, $r = 1, \cdots, p$. Then there exist x_0, y_0, x_1 and y_1 depending on χ, such that the S, as in* (1), *for x_0 and y_0 satisfies $|S| < \sqrt{p}$ and the S, for x_1 and y_1, satisfies $\sqrt{(p-2)} < |S|$.*

Proof of Theorem II. Our principal device is the fact that a function which is periodic mod p has a unique expansion by means of the characters mod p [2]. That is if $h(r) = h(s)$ for $r \equiv s(p)$, then for $n \not\equiv 0(p)$

$$h(n) = \sum_{\theta} b_\theta \theta(n) , \tag{7}$$

where θ runs through the characters mod p. b_θ is given by

$$(p-1)b_\theta = \sum_r h(r)\bar{\theta}(r) . \tag{8}$$

Regarding $J_n(c)$ as a periodic function mod p of c, and expanding $J_n(c)$ in the form (7), we obtain, by standard methods,

$$J_n(c) = \sum_{\rho^n = \psi\chi^n} \pi(\bar{\rho}, \chi)\rho(c) \tag{9}$$

where $\pi(\alpha, \beta)$ is a Jacobi sum [1]

$$\pi(\alpha, \beta) = \sum_r \alpha(r)\beta(1-r) . \tag{10}$$

The sum in (9) is over all characters ρ which satisfy the indicated condition.

The expansion (7) has a Parseval identity

$$\sum_{t=1}^{p-1} |h(t)|^2 = (p-1)\sum_\theta |a_\theta|^2 . \tag{11}$$

Thus we can evaluate A_1 by means of (11) and (9) when we know the value of $|\pi(\alpha, \beta)|^2$. Now [1] $|\pi(\alpha, \beta)|^2 = p$ when $\alpha \neq \varepsilon$, $\beta \neq \varepsilon$ and $\alpha\beta \neq \varepsilon$, where ε is the principal character. If $\alpha = \varepsilon$ or $\beta = \varepsilon$, then $|\pi(\alpha, \beta)|^2 = 1$. If $\alpha\beta = \varepsilon$ with $\alpha \neq \varepsilon$ or $\beta \neq \varepsilon$, then $|\pi(\alpha, \beta)|^2 = p$. By hypothesis, χ is nonprincipal. Thus $|\pi(\bar{\rho}, \chi)|^2$ is p unless $\bar{\rho} = \varepsilon$ or $\bar{\rho}\chi = \varepsilon$. If $\bar{\rho} = \varepsilon$, then $\bar{\rho} = \varepsilon$ and $\psi\chi^n$ is principal. If $\bar{\rho}\chi = \varepsilon$, then $\rho = \chi$ and $\rho^n = \psi\chi^n$ implies $\psi = \varepsilon$ which is excluded by hypothesis. Let N be the number of solutions of $\rho^n = \psi\chi^n$. If $\psi\chi^n$ is nonprincipal then $|\pi(\bar{\rho}, \chi)|^2 = p$ for all N of the ρ and $A_1 = p(p-1)N$. If $\psi\chi^n$ is principal, then $|\pi(\bar{\rho}, \chi)|^2 = p$ for $N-1$ of the ρ and $|\pi(\bar{\rho}, \chi)|^2 = 1$ for $\rho = \varepsilon$. Thus, in this case, $A_1 = (p-1)(p(N-1)+1) = Np(p-1)^2$.

N, the number of solutions of $\rho^n = \psi\chi^n$, is the number of solutions of $\sigma^n = \psi$. It is a standard lemma from the theory of cyclic groups of order k that $a^n = b$ has (n, k) or 0 solutions according to whether or not order $b \mid k/(n, k)$. Also, N is the number of solutions of $x^n = \psi(g)$, for x, in $(p-1)-st$ roots of unity. From either description of N, it follows that $N = d$ or $N = 0$ according as $ds \mid (p-1)$ or $ds \nmid (p-1)$, and the theorem follows.

Proof of Theorem I. Referring to the hypotheses of Theorem I,

$$|S|^2 = \sum_{r,s} a_r \bar{a}_s \chi(yr^{m_1} + xr^{m_2} + g(r))\bar{\chi}(ys^{m_1} + xs^{m_2} + g(s))$$

and thus,

$$A_2 = \Sigma a_r \bar{a}_s \Sigma \chi(yr^{m_1} + xr^{m_2} + g(r))\chi(ys^{m_1} + xs^{m_2} + g(s)) = T_1 + T_2 . \tag{12}$$

T_1 is the sum of the terms in (12) such that $r \not\equiv 0$ and $s \not\equiv 0$. T_2 is the sum of the terms in (12) such that $r \equiv 0$ or $s \equiv 0$. T_1 can be witten

$$T_1 = \sum_{r \not\equiv 0, s} a_r \bar{a}_s \chi^{m_1}(r/s) A(r^{m_2-m_1}, r^{-m_1}g(r); s^{m_2-m_1}, s^{-m_1}g(s)) \tag{13}$$

where

$$A(a, b; c, d) = \sum_{y+cx+d \not\equiv 0} \chi\left(\frac{y + ax + b}{y + cx + d}\right) .$$

Now,

$$A(a, b; c, d) = \sum_x \sum_{y \not\equiv 0} \chi\left(\frac{y + x(a-c) + (b-d)}{y}\right) .$$

Except when $(a-c)x + (b-d) \equiv 0(p)$,

$$\sum_{y \not\equiv 0} \chi\left(\frac{y + (a-c)x + (b-d)}{y}\right) = -1 .$$

Also, $(a-c)x + (b-d) \equiv 0(p)$ when $x \equiv ((b-d)/(a-c))(p)$ or when $a \equiv c$ and $b \equiv d$. Thus, if $a \not\equiv c$ or $b \not\equiv d$, then

$$A(a, b; c, d) = -(p-1) + p - 1 = 0 .$$

If $a \equiv c$ and $b \equiv d$, then

$$A(a, b; c, d) = p(p-1) .$$

In view of this (13) becomes the sum over all r and s such that $r \not\equiv 0 \not\equiv s$ and $r^{m_2-m_1} = s^{m_2-m_1}$, $r^{-m_1}g(r) = s^{-m_1}g(s)$. Since $(m_2 - m_1, p-1) = 1$, we have $r \equiv s$. Thus the sum in (13) is over those r and s such that $r \not\equiv 0 \not\equiv s$ and $r \equiv s$. Thus

$$T_1 = p(p-1) \sum_{r=1}^{p-1} |a_r|^2 .$$

Now

$$T_2 = \sum_{r \not\equiv 0} a_r \bar{a}_0 \sum_{x,y} \chi(yr^{m_1} + xr^{m_2} + g(r))\bar{\chi}(g(0)) + \sum_{s \not\equiv 0} a_0 \bar{a}_s \sum_{x,y} \chi(g(0))\bar{\chi}(ys^{m_1} + xs^{m_2} + g(s)) + |a_0|^2 \sum_{x\ y} \chi(g(0))\bar{\chi}(g(0)) = p^2 |a_0|^2 |\chi(g(0))|^2 , \tag{14}$$

except when $m_1 = 0$ or $m_2 = 0$.

Thus, if $g(0) \equiv 0$,

$$A_2 = p(p-1) \sum_{r \not\equiv 0} |a_r|^2$$

and if $g(0) \not\equiv 0$, then

$$A_2 = p(p-1) \sum_{r \not\equiv 0} |a_r|^2 + p^2 |a_0|^2 ,$$

when $m_1 = 0$ or $m_2 = 0$, then $\chi(g(0))$ in (14) must be changed to $\chi(y + g(0))$ or $\chi(x + g(0))$, and A_2 is given by (6).

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