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A THEOREM ON PARTITIONS OF MASS-DISTRIBUTION

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A 'bisector' of a continuous mass-distribution M in a bounded region on the plane is defined as a straight line such that the two half-planes determined by this line contain half the mass of M each. It is known that there exists at least one point (in the plane) through which pass three bisectors of M.

THEOREM. **Let, for a continuous mass distribution M, the point P through which three bisectors pass be unique. Then all bisectors of M pass through p.**

The following corollary also is established: For a convex figure K (i.e., compact convex set with nonempty interior) to be centrally symmetric, it is necessary and sufficient that the point through which three bisectors of area pass be unique.

In what follows, M stands for any continuous mass-distribution in a compact domain in the plane. A line l is called a *bisector* of M if the two half-planes determined by l contain equal masses of M.

The following results are well-known regarding bisectors of M. (see, for example, [4], Problem 3–1, 3–2, and [1]).

(1) Let l be any line in the plane. There is a bisector of M parallel to l.

(2) There exists a point P in the plane and two perpendicular lines through P such that the portions of M contained in each of the four 'wedges' determined by these lines have the same mass, namely, a quarter of that of M.

(3) There exists a point in the plane through which three distinct bisectors of M pass.

Further, let l_0 be a bisector of M and 0 a fixed point on l_0. Let $l(\alpha)$ be a bisector of M, inclined to l_0 at an angle α and intersecting l_0 in P_α. It is easy to verify that we can choose the bisector $l(\alpha)$ such that the distance $0P_\alpha$ is a continuous function of α. We shall make use of this observation in the following.

In this paper we shall investigate the nature of the points through which three distinct bisectors of M pass. Specifically, let P be a point

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in the plane such that three distinct bisectors l_1, l_2, l_3 of M pass through P; and let $l(\alpha)$ be a bisector of M not passing through P. We shall prove the existence of a point $P_1 \neq P$ such that three distinct bisectors pass through P_1 too.

First, let $l(\alpha)$ be parallel to one of l_1, l_2, l_3; say, to l_1. Since l_1 and $l(\alpha)$ are both bisectors, it follows that the portion of M contained between these lines l_1 and $l(\alpha)$ has zero mass (see Fig. 1).

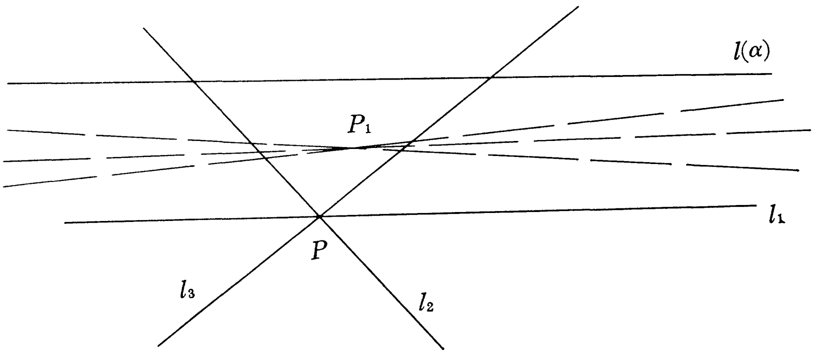

Fig. 1

Since M is enclosed in a bounded domain D, we can choose a point P_1 midway between l_1 and $l_{(\alpha)}$, and three distinct lines through P_1 such that each of these three lines intersects (if at all) l_1 and $l_{(\alpha)}$ outside D. In other words, these three lines are bisectors of M.

Secondly, let $l_{(\alpha)}$ intersect the lines l_1, l_2, l_3 (see Fig. 2), and let X be the point of intersection of $l_{(\alpha)}$ with l_2. (We number the lines l_1, l_2, l_3, such that X lies between the points of intersection of $l_{(\alpha)}$ with l_1 and l_3).

With reference to a fixed direction, let θ_1 and θ_2 be the directions of l_1 and l_3 respectively, and let α be that of $l_{(\alpha)}$.

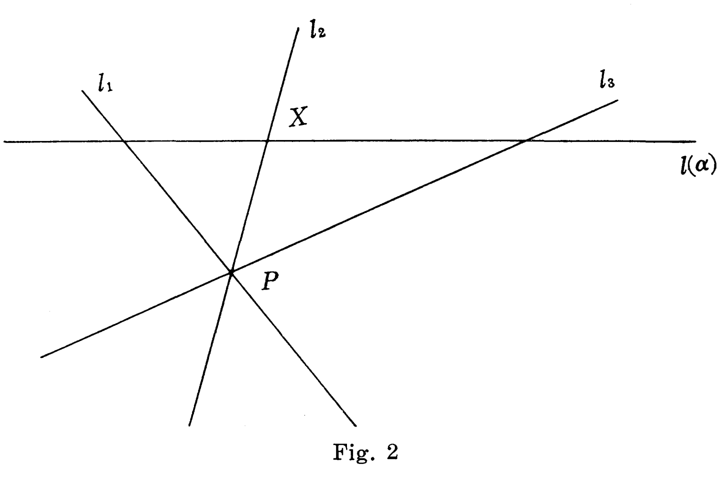

Fig. 2

When θ varies from θ_1 to θ_2 we can choose the bisectors $l(\theta)$ such that $l(\theta_1) = l_1$, $l(\theta_2) = l_3$, and $PX = x(\theta)$ is a continuous function of θ. (The equality $l(\theta_1) = l_1$ means that the lines $l(\theta_1)$ and l_1 coincide).

Since $x(\theta_1) = 0 = x(\theta_2)$, and for the given bisector $\lambda(\alpha)$, $x(\alpha) \neq 0$, if follows that there are two distinct values α_1 and α_2 for which

$$x(\alpha_1) = x(\alpha_2) \neq 0 \ .$$

Let P_1 be the position of X corresponding to $x(\alpha_1)$. Thus three distinct bisectors l_2, $l(\alpha_1)$ and $l(\alpha_2)$ pass through P_1, and $P_1 \neq P$.

This proves the required assertion, that is, if a bisector r of M does not pass through P and three distinct bisectors pass through P then there is a point P_1 distinct from P through which also pass three distinct bisectors.

Hence we have the

THEOREM. *Let, for a continuous mass distribution M, the point P through which three distinct bisectors of M pass be unique. Then all bisectors of M pass through P. (In particular, every line through P bisects M).*

Something more can be asserted about the mass distribution M in the following special case. Consider a compact convex figure K (i.e., a compact convex set with nonempty interior) and interpret mass as the area. Since the bisector in any direction is unique, it follows from the above theorem that every line through P is bisector of K where P is the unique point through which three bisectors pass. Consider two such bisectors inclined at a small angle θ, as in Figure 3.

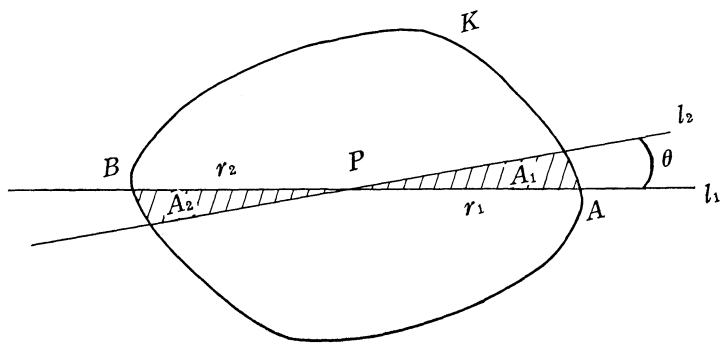

Fig. 3

Let l_1 intersect the boundary of K in A and B, and let $PA = r_1$, $PB = r_2$ Denote by A_1 and A_2, respectively, the areas of the portions of K in the two wedges (shaded in the figure) between l_1, l_2. We have, for small θ, the approximate equalities

$$A_1 \doteq \frac{1}{2} r_1^2 \theta$$

$$A_2 \doteq \frac{1}{2} r_2^2 \theta$$

$A_1 = A_2$ since l_1, l_2 are both bisectors, and hence $r_1 = r_2$ by making θ approach zero. As this is true for any position of l_1, it follows that the figure K is centrally symmetric and P is its centre.

Of course, the converse also is true because any line through the centre of any centrally symmetric figure (convex or not) is a bisector of it.

Thus we have the following corollary.

COROLLARY. *Let K be a compact convex figure. The following four statement are equivalent*:

(a) *the point P through which three bisectors of K pass is unique,*
(b) *all bisectors of K are concurrent in P;*
(c) *there exists a point P such that any line through it is a bisector of K;*
(d) *K is a centrally symmetric figure with P as its centre.*

REMARKS. 1. K. Zarankiewicz appears to have proved a similar theorem for convex figures (see [3], page 264, note 10). Our result is in a more general setting, and is, surprisingly, quite strong. The author believes that his proof is different from that given by Zarankiewicz.

2. A stronger statement of the theorem is not possible, in the sense that out of the four statements (a), (b), (c), (d) mentioned in the corollary, it is not true in general that (c) implies (b), (since a bisector in a direction need not be unique). Also mass-distributions can be constructed easily for which (a) is true but (d) is not. (I am grateful to the referee for bringing to my notice an example where (a), (b), (c) are true but (d) is not).

3. Consider the set of points through which three bisectors pass. Very little is known about this set (see, however, [2]).

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