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CONVEXITY WITH RESPECT TO EULER-LAGRANGE DIFFERENTIAL OPERATORS

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This paper is concerned with the problem of characterizing sub-(L) functions, where L is the Euler-Lagrange operator for the functional $I_{cd}[y] = \int_c^d \left[\sum_{j=0}^{n} p_j(D^j y)^2 \right]$, with n a positive integer, $[c, d]$ a subinterval of a fixed interval $[a, b]$, and $p_0, p_1, \cdots, p_n$ continuous real-valued functions on $[a, b]$ with $p_n(x) > 0$ on this interval. Under certain hypotheses on the operator L, it is shown that if f is a function in the domain of L on a subinterval $[c, d]$ of $[a, b]$, then the statement that f is sub-(L) on $[c, d]$ is equivalent to each of the following conditions: (i) $(-1)^n Lf(x) \leqq 0$ on $[c, d]$ (ii) $I_{cd}[y] \geqq I_{cd}[f]$ whenever y is a function having continuous derivatives of the first $n-1$ orders with $D^{n-1}y$ having a piecewise continuous derivative on $[c, d]$ such that $D^{j-1}y$ and $D^{j-1}f$ have the same value at c and at d for j in $\{1, \cdots, n\}$, and $y(x) - f(x) \leqq 0$ on $[c, d]$.

Section 2 is devoted to the definition and equivalent formulizations of Euler-Lagrange operators and to a discussion of adjoint operators. In § 3 it is shown that, under a hypothesis which is equivalent to the operator L being nonoscillatory on $[a, b]$, L admits a factorization of the form $(-1)^n L_0^* L_0$, where $L_0 y = \sum_{j=0}^{n} r_j D^j y$ for suitable $r_0, r_1, \cdots, r_n$. Under the further hypothesis that the operator L_0 possesses the "property W" of Polya [3], it is established that L can be written as a composition of first order real linear operators.

In § 4, the analogue of Polya's mean-value theorem in [3] is obtained for L under the above hypotheses on L and L_0. This theorem is used in §§ 5 and 6 to give characterizations, which are generalizations of results of Bonsall [1] and Reid [5] on convexity with respect to second order operators, of sub-(L) functions in terms of the operator L and the functional I_{cd}, as well as a theorem on the constancy of sign of the Green's function for a certain incompatible boundary-value problem.

Finally, in § 7, it is proved under the above assumptions on L and L_0 that the null-space of L is a $2n$-parameter family in the sense of Tornheim [7], although the relationship between sub-(L) functions and

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functions which are convex with respect to this family remains undecided.

Matrix notation will be used throughout; in particular, a vector is a matrix having one column. If M is a matrix, then M^* denotes its transpose. If M is a symmetric matrix, (i.e., $M = M^*$), then M is *nonnegative* ($M \geqq 0$) if and only if $\eta^* M \eta$ is a nonnegative real number for all admissible vectors η. The symbol E_k is used to denote the $k \times k$ identity matrix, 0 is used to denote the zero matrix of arbitrary dimensions, and, for j in $\{1, \cdots, n\}$, e^j denotes the vector $[\delta_{ij}]_{i=1}^n$. If M is a function matrix, (i.e., a matrix of real functions), then M is said to be continuous, differentiable, etc., whenever each of its elements has the corresponding property. If M is a differentiable function matrix, then DM denotes the matrix of derivatives of the elements of M.

All functions appearing are real-valued. In particular, if k is a nonnegative integer and $[c, d]$ is a subinterval of $[a, b]$, then $C^k[c, d]$ denotes the class of all real-valued functions which have continuous derivatives of the first k orders on $[c, d]$. The symbol $\varDelta^n[c, d]$ will stand for the class of all functions w in $C^{n-1}[c, d]$ for which $D^{n-1}w$ has a piecewise continuous derivative on $[c, d]$, and $\varDelta_0^n[c, d]$ is the class of all those functions w in $\varDelta^n[c, d]$ such that $D^{j-1}w(c) = 0 = D^{j-1}w(d)$ for j in $\{1, \cdots, n\}$. Also, R^k denotes the class of all k-tuples of real numbers. Finally, if f is an integrable function and c is a point in its domain, then $\int_c f$ denotes the function whose value at x is $\int_c^x f$.

2. **Properties of differential operators.** Let $[a, b]$ be a nondegenerate compact interval and, for each α and each β in $\{0, 1, \cdots, n\}$, let $f_{\alpha\beta}$ be a continuous real-valued function on $[a, b]$. The first problem of this section is to examine the form of the Euler-Lagrange operator L which corresponds to the functional I_{cd}, where $[c, d]$ is a subinterval of $[a, b]$, defined on $\varDelta^n[c, d]$ by

$$I_{cd}[y] = \int_c^d \left[\sum_{\alpha,\beta=0}^n f_{\alpha\beta} D^\alpha y D^\beta y \right]. \tag{2.1}$$

By definition, a function y belongs to the domain of L on a subinterval $[c, d]$ if and only if $y \in C^n[c, d]$ and there exists a function $\varphi[y]$ in $C^0[c, d]$ such that for every w in $\varDelta_0^n[c, d]$, the relation

$$\int_c^d \left[\sum_{\alpha,\beta=0}^n f_{\alpha\beta} D^\alpha y D^\beta w \right] = \int_c^d \varphi[y] w \tag{2.2}$$

holds. In this case, $\varphi[y]$ is uniquely determined, and Ly is defined to be $(-1)^n \varphi[y]$. The following result gives an explicit form for the operator L.

THEOREM 2.1. *If L is the Euler-Lagrange operator for the functional defined by* (2.1), *then y belongs to the domain of L on a subinterval $[c, d]$ of $[a, b]$ if and only if $y \in C^n[c, d]$ and there exist functions $\mu_1[y], \cdots, \mu_n[y]$ in $C^1[c, d]$ such that*

$$(2.3)\qquad \begin{aligned} \mu_n[y] &= \sum_{\alpha=0}^{n} f_{\alpha n} D^\alpha y \,, \\ \mu_{i-1}[y] &= \sum_{\alpha=0}^{n} f_{\alpha i-1} D^\alpha y - D\mu_i[y] \,, \qquad i \text{ in } \{2, \cdots, n\} \,. \end{aligned}$$

In this case,

$$Ly = (-1)^{n+1}\Big(D\mu_1[y] - \sum_{\alpha=0}^{n} f_{\alpha 0} D^\alpha y\Big) \,,$$

that is,

$$\begin{aligned} Ly = D\Big(D\Big(\cdots D\Big(D\Big(\sum_{\alpha=0}^{n} f_{\alpha n} D^\alpha y\Big) - \sum_{\alpha=0}^{n} f_{\alpha n-1} D^\alpha y\Big) \cdots\Big) \\ + (-1)^{n+1} \sum_{\alpha=0}^{n} f_{\alpha 1} D^\alpha y\Big) + (-1)^n \sum_{\alpha=0}^{n} f_{\alpha 0} D^\alpha y \,. \end{aligned}$$

First, if y is in the domain of L on $[c, d]$, then y satisfies (2.2) with $\varphi[y] = (-1)^n Ly$. Let $\rho_0[y], \rho_1[y], \cdots, \rho_n[y]$ be the functions defined recursively by

$$(2.4)\qquad \begin{aligned} \rho_0[y] &= \sum_{\alpha=0}^{n} f_{\alpha 0} D^\alpha y - \varphi[y] \\ \rho_i[y] &= \sum_{\alpha=0}^{n} f_{\alpha i} D^\alpha y - \int_c \rho_{i-1}[y] \,, \qquad i \text{ in } \{1, \cdots, n\} \,. \end{aligned}$$

Then, for every w in $\varDelta_0^n[c, d]$ and each k in $\{1, \cdots, n\}$,

$$\int_c^d \Big[\Big(\sum_{\beta=k}^{n} \sum_{\alpha=0}^{n} f_{\alpha\beta} D^\alpha y D^\beta w\Big) + \rho_{k-1}[y] D^{k-1} w\Big] = 0 \,.$$

In particular, if $k = n$ then

$$\int_c^d \Big[\Big(\sum_{\alpha=0}^{n} f_{\alpha n} D^\alpha y\Big) D^n w + \rho_{n-1}[y] D^{n-1} w\Big] = 0 \,,$$

and one final integration by parts gives $\int_c^d \rho_n[y] D^n w = 0$. Since w is an arbitrary member of $\varDelta_0^n[c, d]$, the fundamental lemma of the calculus of variations implies there is a polynomial function Q_{n-1} of degree at most $n - 1$ such that $\rho_n[y] = Q_{n-1}$. If Q_{n-1-j} denotes the jth derivative of Q_{n-1} for j in $\{1, \cdots, n-1\}$, then, for i in $\{1, \cdots, n\}$ let $\mu_i[y]$ be

$$(-1)^{n-i} Q_{i-1} + \int_c \rho_{i-1}[y] \,.$$

Then

$$\mu_n[y] = \rho_n[y] + \int_c \rho_{n-1}[y] = \sum_{\alpha=0}^{n} f_{\alpha n} D^\alpha y \ ,$$

and, for i in $\{2, \cdots, n\}$, $D\mu_i[y]$ exists, is continuous, and

$$\begin{aligned} D\mu_i[y] &= (-1)^{n-i} Q_{i-2} + \rho_{i-1}[y] \\ &= (-1)^{n-i} Q_{i-2} + \sum_{\alpha=0}^{n} f_{\alpha i-1} D^\alpha y - \int_c \rho_{i-2}[y] \\ &= \sum_{\alpha=0}^{n} f_{\alpha i-1} D^\alpha y - \mu_{i-1}[y] \ . \end{aligned}$$

Thus, the relations (2.3) hold, and, since Q_0 is a constant function,

$$D\mu_1[y] = \rho_0[y] = \sum_{\alpha=0}^{n} f_{\alpha 0} D^\alpha y - \varphi[y] \ ,$$

so

$$Ly = (-1)^n \varphi[y] = (-1)^{n+1}\Big(D\mu_1[y] - \sum_{\alpha=0}^{n} f_{\alpha 0} D^\alpha y\Big) \ .$$

Conversely, suppose $y \in C^n[c, d]$ and there exist functions $\mu_1[y], \cdots, \mu_n[y]$ in $C^1[c, d]$ satisfying (2.3). If

$$\varphi[y] = \sum_{\alpha=0}^{n} f_{\alpha 0} D^\alpha y - D\mu_1[y] \ ,$$

then, for any w in $\Delta_0^n[c, d]$,

$$\begin{aligned} &\int_c^d \Big[\sum_{\alpha,\beta=0}^{n} f_{\alpha\beta} D^\alpha y D^\beta w\Big] \\ &= \int_c^d \Big[\mu_n[y] D^n w + \sum_{\beta=1}^{n-1} (D\mu_{\beta+1}[y] + \mu_\beta[y]) D^\beta w + (D\mu_1[y] + \varphi[y]) w\Big] \\ &= \int_c^d \Big[D\Big(\sum_{i=0}^{n-1} \mu_{i+1}[y] D^i w\Big) + \varphi[y] w\Big] \\ &= \int_c^d \varphi[y] w \ . \end{aligned}$$

Hence, y is in the domain of L with

$$Ly = (-1)^n \varphi[y] = (-1)^{n+1}\Big(D\mu_1[y] - \sum_{\alpha=0}^{n} f_{\alpha 0} D^\alpha y\Big) \ .$$

Since the coefficients $f_{\alpha\beta}$ are only assumed to be continuous, L is in general not a $2n$th order differential operator in the classical sense but is an example of what has come to be known as a "quasidifferential operator". However, if the "leading coefficient" f_{nn} vanishes at no point of $[a, b]$, then the equation $Ly = \varphi$ is equivalent to a first order $2n$-dimensional vector system.

THEOREM 2.2. *Suppose* $f_{nn}(x) \neq 0$ *on* $[a, b]$, A *and* B *are the* $n \times n$ *matrices*

$$\begin{bmatrix} 0 & & & \\ \vdots & & E_{n-1} & \\ 0 & & & \\ -f_{0n}/f_{nn} & -f_{1n}/f_{nn} & \cdots & -f_{n-1n}/f_{nn} \end{bmatrix}, \quad \begin{bmatrix} & & 0 \\ 0 & & \vdots \\ & & 0 \\ 0 & \cdots\ 0 & 1/f_{nn} \end{bmatrix},$$

respectively, F *is the* $n \times n$ *matrix*

$$\begin{bmatrix} 0 \cdots 0 & f_{n0}/f_{nn} \\ & f_{n1}/f_{nn} \\ -E_{n-1} & \vdots \\ & f_{nn-1}/f_{nn} \end{bmatrix},$$

and C *is the* $n \times n$ *matrix whose element in the ith row and jth column is* $f_{j-1\,i-1} - (f_{ni-1}f_{j-1\,n})/f_{nn}$. *Then* $Ly = \varphi$ *if and only if* $u = [D^{i-1}y]_{i=1}^n$, $v = [\mu_i[y]]_{i=1}^n$ *is a solution of*

$$(2.5) \qquad Du = Au + Bv\ , \qquad Dv = Cu + Fv + (-1)^{n+1}\varphi e^1\ .$$

Moreover, if $f_{nn}(x) > 0$ *on* $[a, b]$, *then the matrix* $B(x) \geqq 0$ *on* $[a, b]$, *and if the matrix* $[f_{\alpha\beta}]_{\alpha=0}^{n}\,{}_{\beta=0}^{n}$ *is symmetric then so is the matrix* C, *and* (2.5) *becomes*

$$(2.6) \qquad Du = Au + Bv\ , \qquad Dv = Cu - A^*v + (-1)^{n+1}\varphi e^1\ .$$

The first part of the theorem follows immediately from Theorem 2.1, particularly the fact that the functions $\mu_1[y], \cdots, \mu_n[y]$ are determined uniquely by (2.3) for a given y in the domain of L. The last statement of the theorem is obvious from the definitions of the matrices involved.

We shall be concerned in particular with the homogeneous vector systems

$$(2.5') \qquad Du = Au + Bv\ , \qquad Dv = Cu + Fv\ ,$$

$$(2.6') \qquad Du = Au + Bv\ , \qquad Dv = Cu - A^*v\ .$$

and the related matrix systems

$$(2.5'') \qquad DU = AU + BV\ , \qquad DV = CU + FV\ ,$$

$$(2.6'') \qquad DU = AU + BV\ , \qquad DV = CU - A^*V\ .$$

For convenience, if each of U and V is an $n \times r$ function matrix, then $(U; V)$ will stand for the $2n \times r$ function matrix whose jth column consists of the functions $U_{1j}, \cdots, U_{nj}, V_{1j}, \cdots, V_{nj}$.

The following property of the system (2.5′) will be especially important for discussing the oscillation properties of L in the case that $f_{nn}(x) > 0$ and the matrix $[f_{\alpha\beta}]$ is symmetric.

THEOREM 2.3. *The system* (2.5′) *is identically normal on* $[a, b]$, *that is, if* $(u; v)$ *is a function vector which satisfies* (2.5′) *and there is a nondegenerate subinterval* I *of* $[a, b]$ *on which* u *vanishes identically, then both* u *and* v *vanish identically on* $[a, b]$.

If $(u; v)$ satisfies (2.5′) with $u(x) \equiv 0$ on a nondegenerate subinterval I of $[a, b]$, then the relations

$$v_n = f_{nn}Du_n + \sum_{\alpha=0}^{n-1} f_{\alpha n}u_{\alpha+1} ,$$

$$v_{i-1} = \sum_{\alpha=0}^{n-1} f_{\alpha i-1}u_{\alpha+1} + f_{ni-1}Du_n - Dv_i , \qquad i \text{ in } \{2, \cdots, n\}$$

imply that $v(x) \equiv 0$ on I and, therefore, both u and v must vanish identically on all of $[a, b]$.

Indeed, if $(u; v)$ is a solution of (2.5′) with $u_1(x) \equiv 0$ on a nondegenerate subinterval I of $[a, b]$, then the first $n - 1$ component equations of (2.5′) imply that $u(x) \equiv 0$ on I, so u and v vanish identically on I. Thus, in view of the results of Theorems 2.2 and 2.3, together with the elementary existence and uniqueness theorems for first-order vector differential equations, it follows that if $f_{nn}(x) \neq 0$ on $[a, b]$ then the null-space of L has a basis of $2n$ linearly independent functions, so that L deserves to be called a "$2n$th order operator".

We conclude this section with the well-known formulization of the adjoint L_0^* of an operator L_0 which is defined by

$$L_0 y = \sum_{j=0}^{n} r_j D^j y , \tag{2.7}$$

where the coefficients $r_0, r_1, \cdots, r_n$ are continuous real-valued functions on $[a, b]$. By definition, a function z belongs to the domain of L_0^* on a subinterval $[c, d]$ of $[a, b]$ if and only if $z \in C^0[c, d]$ and there exists a function $\varphi[z]$ in $C^0[c, d]$ such that, for every w in $\Delta_0^n[c, d]$,

$$\int_c^d zL_0 w = \int_c^d \varphi[z]w .$$

In this case, $\varphi[z]$ is unique, and $L_0^* z$ is defined to be $\varphi[z]$. Using much the same integration-by-parts technique, and subsequent application of the fundamental lemma of the calculus of variations as in the proof of Theorem 2.1, we find that z belongs to the domain of L_0^* on $[c, d]$ if and only if $z \in C^0[c, d]$ and there exist functions $\nu_1[z], \cdots, \nu_n[z]$ in $C^1[c, d]$ such that

$$\begin{aligned} &\nu_n[z] = r_n z \;, \\ (2.8) \qquad &\nu_{i-1}[z] = r_{i-1}z - D\nu_i[z] \;, \qquad i \text{ in } \{2, \cdots, n\} \;, \end{aligned}$$

in which case $L_0^* z = r_0 z - D\nu_1[z]$.

It is easily verified that if $r_n(x) \neq 0$ on $[a, b]$ and G is the $n \times n$ function matrix

$$G = \begin{bmatrix} 0 & & & \\ \vdots & & E_{n-1} & \\ 0 & & & \\ -r_0/r_n & -r_1/r_n & \cdots & -r_{n-1}/r_n \end{bmatrix},$$

then $L_0 y = f$ if and only if there exists a function vector $u = [u_i]_{i=1}^n$ such that $Du = Gu + (f/r_n)e^n$ and $y = u_1$, and $L_0^* z = g$ if and only if there exists a function vector $v = [v_i]_{i=1}^n$ such that $Dv = -G^* v - ge^1$ with $z = v_n/r_n$.

3. Factorization of Euler-Lagrange operators. In this section we shall consider a particular functional of the form (2.1) which is given by

$$(3.1) \qquad I_{cd}[y] = \int_c^d \Big[\sum_{j=0}^n p_j (D^j y)^2 \Big] \;,$$

where $p_0, p_1, \cdots, p_n$ are continuous real-valued functions on $[a, b]$ with $p_n(x) > 0$ on this interval, and $[c, d]$ is a subinterval of $[a, b]$. We then have the following special case of results of § 2.

THEOREM 3.1. *If L is the Euler-Lagrange operator for the functional I_{cd} given by* (3.1), *then a function y belongs to the domain of L on a subinterval $[c, d]$ of $[a, b]$ if and only if $y \in C^n[c, d]$ and there exist functions $\mu_1[y], \cdots, \mu_n[y]$ in $C^1[c, d]$ such that*

$$\begin{aligned} &\mu_n[y] = p_n D^n y \;, \\ (3.2) \qquad &\mu_{i-1}[y] = p_{i-1} D^{i-1} y - D\mu_i[y] \;, \qquad i \text{ in } \{2, \cdots, n\} \;. \end{aligned}$$

In this case $Ly = (-1)^{n+1}(D\mu_1[y] - p_0 y)$, that is,

$$\begin{aligned} Ly = D(D(\cdots D(D(p_n D^n y) - p_{n-1} D^{n-1} y) \cdots) + (-1)^{n+1} p_1 Dy) \\ + (-1)^n p_0 y \;. \end{aligned}$$

Moreover, the equation $Ly = \varphi$ is equivalent under the transformation

$$\begin{aligned} u_i &= D^{i-1} y \;, \qquad i \text{ in } \{1, \cdots, n\} \;, \\ v_i &= \mu_i[y] \;, \qquad i \text{ in } \{1, \cdots, n\} \end{aligned}$$

to the vector system

$$Du = Au + Bv\,, \qquad Dv = Cu - A^*v + (-1)^{n+1}\varphi e^1\,, \tag{3.3}$$

where

$$A = \begin{bmatrix} 0 & & \\ \vdots & E_{n-1} & \\ 0 & \cdots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} & & 0 \\ & 0 & \vdots \\ & & 0 \\ 0 & \cdots\; 0 & 1/p_n \end{bmatrix}, \quad C = \begin{bmatrix} p_0 & & & 0 \\ & p_1 & & \\ & & \ddots & \\ 0 & & & p_{n-1} \end{bmatrix}.$$

In particular, the equation $Ly = 0$ *is equivalent under the above transformation to the identically normal system*

$$Du = Au + Bv\,, \qquad Dv = Cu - A^*v\,. \tag{3.3$'$}$$

As was indicated in § 2, we shall also make use of the related matrix equation

$$DU = AU + BV\,, \qquad DV = CU - A^*V\,. \tag{3.3$''$}$$

In particular, consider the following condition:

(H_1). *There exist solutions* $y_1, \cdots, y_n$ *of* $Ly = 0$ *such that if* $U = [D^{i-1}y_j]_{i=1}^{n}{}_{j=1}^{n}$ *and* $V = [\mu_i[y_j]]_{i=1}^{n}{}_{j=1}^{n}$ *then* $U^*(x)V(x) - V^*(x)U(x) \equiv 0$ *on* $[a, b]$ *and* $U(x)$ *is nonsingular on* $[a, b]$.

Since the matrix $(U; V)$ based on the matrices U and V of (H_1) is a solution of (3.3″), $U^*V - V^*U$ is a constant function matrix, and the particular condition that this constant matrix be the zero matrix is what has been termed the condition that $(U; V)$ be a "select solution" of (3.3″), or that the columns of $(U; V)$ be "mutually conjugate" or "conjoined" solutions of (3.3′), (see, e.g., Reid [4]).

Hypothesis (H_1) has an important bearing on conditions of oscillation involving L and on the variational behavior of the functional I_{cd}. At the present, however, we are concerned with the following property of L.

THEOREM 3.2. *If* (H_1) *holds, then there exist continuous real-valued functions* $r_0, r_1, \cdots, r_n$ *on* $[a, b]$ *with* $r_n(x) > 0$ *on this interval such that if* L_0 *is the nth order differential operator defined by*

$$L_0 y = \sum_{j=0}^{n} r_j D^j y\,, \tag{3.4}$$

and L *is the Euler-Lagrange operator for the functional* (3.1), *then*

$$L = (-1)^n L_0^* L_0 .$$

Moreover, the functions $y_1, \cdots, y_n$ *specified in* (H_1) *form a basis for the null-space of* L_0.

It is useful for the proof of this theorem to introduce the following notation. Let R be the $n \times n$ matrix

$$\begin{bmatrix} & & 0 \\ & 0 & \vdots \\ & & 0 \\ 0 & \cdots\ 0 & p_n \end{bmatrix},$$

let $P = C$, and let ω be the function defined on $[a, b] \times R^n \times R^n$ by the formula $2\omega(x, \sigma, \tau) = \tau^* R(x)\tau + \sigma^* P(x)\sigma$. Then L is also the Euler-Lagrange operator for the functional

$$\int_c^d 2\omega(x, \eta(x), D\eta(x))dx$$

subject to the restraints

$$D\eta_i = \eta_{i+1}, \qquad i \text{ in } \{1, \cdots, n-1\} .$$

Now, if U and V are as in (H_1), and, for a subinterval $[c, d]$ of $[a, b]$, $y \in C^n[c, d]$ and $w \in \Delta^n[c, d]$, then with

$$\eta^1 = [D^{i-1}y]_{i=1}^n , \qquad \eta^2 = [D^{i-1}w]_{i=1}^n ,$$

we have

$$\begin{aligned} &(D\eta^1)^* R(D\eta^2) + \eta^{1*} P\eta^2 \\ &\quad = (UD[U^{-1}\eta^1])^* R(UD[U^{-1}\eta^2]) + D[\eta^{1*}(VU^{-1})\eta^2] . \end{aligned} \tag{3.5}$$

This identity is essentially formula (5.3) in Reid [6]. Since

$$UD[U^{-1}\eta^\alpha] = U(DU^{-1})\eta^\alpha + D\eta^\alpha$$

and the matrix U is independent of both y and w, as is also the matrix R, it follows that there exist continuous functions $r_0, r_1, \cdots, r_n$ independent of y and w such that if L_0 is defined by (3.4), then

$$(UD[U^{-1}\eta^1])^* R(UD[U^{-1}\eta^2]) = (L_0 y)(L_0 w) . \tag{3.6}$$

In particular, $r_n = p_n^{1/2}$, so $r_n(x) > 0$ on $[a, b]$. If w also belongs to $\Delta_0^n[c, d]$, then (3.5) and (3.6) imply that

$$\int_c^d [(D\eta^1)^* R(D\eta^2) + \eta^{1*} P\eta^2] = \int_c^d [(L_0 y)(L_0 w)] . \tag{3.7}$$

Theorem 2.1, with $f_{\alpha\beta} = r_\alpha r_\beta$, and the remarks at the end of § 2 concerning the adjoint L_0^* of an operator L_0 of the form (3.4), show that $(-1)^n L_0^* L_0$ is the Euler-Lagrange operator for the functional given by $\int_c^d (L_0 y)^2$ on $\Delta^n[c, d]$ and that y belongs to the domain of $(-1)^n L_0^* L_0$ if and only if $y \in C^n[c, d]$ and

$$\int_c^d [(L_0 y)(L_0 w)] = \int_c^d [L_0^*(L_0 y) w]$$

whenever $w \in \Delta_0^n[c, d]$. On the other hand, the left-hand member of (3.7) is just $\int_c^d \sum_{j=0}^n p_j D^j y D^j w$. These remarks together with the definition of L show that a function y in $C^n[c, d]$ belongs to the domain of L if and only if it belongs to the domain of $(-1)^n L_0^* L_0$ and, in this case, $Ly = (-1)^n L_0^* L_0 y$.

Finally, if y is one of the functions $y_1, \cdots, y_n$ specified in (H_1) and $\eta^1 = [D^{i-1} y]_{i=1}^n$, then $U^{-1}\eta^1$ is constant and (3.6) implies that $L_0 y = 0$. The linear independence of $\{y_1, \cdots, y_n\}$ follows from the assumption that $U(x)$ is nonsingular on $[a, b]$.

In [3], Polya showed that, under a certain hypothesis which he called "property W", the operator L_0 can be written as a composition of first order operators. We shall show that, under this same hypothesis, the operator L_0^* can also be written in this form, and, therefore, so can L if the additional hypothesis (H_1) holds. The "property W" of Polya shall be referred to in this paper as:

(H_2). *There exist solutions $y_1, \cdots, y_n$ of $L_0 y = 0$ such that if W_k denotes the Wronskian*

$$W(y_1, \cdots, y_k) = \det [D^{i-1} y_j]_{i=1}^{k} {}_{j=1}^{k}, \tag{3.8}$$

then $W_k(x) > 0$ on $[a, b]$ for each k in $\{1, \cdots, n\}$.

It should be noticed that if hypotheses (H_1) and (H_2) were always to be applied simultaneously, then one could assume without loss of generality that the functions $y_1, \cdots, y_n$ specified in (H_1) also satisfied the condition on the corresponding Wronskians which is stated in (H_2). This follows directly from the last statement of Theorem 3.2 and the identical normality of (3.3′). However, we shall be interested in certain statements which are true under (H_2) alone.

The following known property of Wronskians is stated here for convenience.

LEMMA. *If each of $f_1, \cdots, f_k, f$ belongs to $C^k[a, b]$, and $W(f_1, \cdots, f_k)$ vanishes at no point of $[a, b]$, then*

$$\begin{aligned} &D[W(f_1, \cdots, f_{k-1}, f)/W(f_1, \cdots, f_{k-1}, f_k)] \\ &\quad = W(f_1, \cdots, f_{k-1})W(f_1, \cdots, f_{k-1}, f_k, f)/[W(f_1, \cdots, f_{k-1}, f_k)]^2 . \end{aligned}$$

This equality is most easily seen by noting that each side is the value at f of a kth order linear differential operator whose null-space has $\{f_1, \cdots, f_k\}$ as a basis. Hence, the two expressions must be proportional, and examination of the leading coefficients shows that the expressions are, in fact, identical.

THEOREM 3.3. *If* (H_2) *holds, then there exist positive functions* $\pi_0, \pi_1, \cdots, \pi_n$ *with* π_j *in* $C^{n-j}[a, b]$ *for* j *in* $\{0, 1, \cdots, n\}$ *such that if* Γ_j *and* Λ_j *are the operators defined recursively by:*

$$(3.9)\quad \begin{aligned} &\Gamma_0 z = \pi_n z , && \Lambda_0 y = (1/\pi_0)y , \\ &\Gamma_j z = \pi_{n-j} D\Gamma_{j-1} z , && j \text{ in } \{1, \cdots, n-1\} ,\ \Lambda_j y = \pi_j D\Lambda_{j-1} y , \\ &\Gamma_n z = (-1)^n (1/\pi_0) D\Gamma_{n-1} z , && j \text{ in } \{1, \cdots, n\} , \end{aligned}$$

then $L_0 = \Lambda_n$ *and* $L_0^* = \Gamma_n$.

It is to be emphasized that a real-valued function f belongs to the domain of Γ_j (respectively, Λ_j) on a subinterval $[c, d]$ of $[a, b]$ if and only if f is continuous on $[c, d]$ and if $j \in \{1, \cdots, n\}$, then $\Gamma_{j-1} f$ (respectively, $\Lambda_{j-1} f$) is in $C^1[c, d]$.

By a theorem of Polya [3], if $W_0 = 1$, W_k is as specified in (H_2) for k in $\{1, \cdots, n\}$, $\pi_0 = W_1$, $\pi_j = W_j^2/(W_{j-1}W_{j+1})$ for j in $\{1, \cdots, n-1\}$, and $\pi_n = r_n W_n/W_{n-1}$, then $L_0 = \Lambda_n$. Furthermore, since each y_k appearing in (H_2) is necessarily in $C^n[a, b]$, it follows that each π_j is in $C^{n-j}[a, b]$, and there exist continuous functions p_{ij}, i in $\{0, 1, \cdots, n\}$, j in $\{0, 1, \cdots, n\}$, such that

$$(3.10)\qquad \Lambda_j y = \sum_{i=0}^{j} p_{ij} D^i y , \qquad \text{for } j \text{ in } \{0, 1, \cdots, n\} .$$

Moreover, $p_{jj} = W_j/W_{j+1}$ for j in $\{0, 1, \cdots, n-1\}$ and $p_{nn} = r_n$, so that $p_{ii}(x) > 0$ on $[a, b]$ for i in $\{0, 1, \cdots, n\}$. This implies that a function w is in $\Delta_0^n[c, d]$ for a subinterval $[c, d]$ of $[a, b]$ if and only if $w \in \Delta^n[c, d]$ and $\Lambda_j w$ vanishes at c and at d for j in $\{0, 1, \cdots, n-1\}$.

As to the factorization (3.9) of L_0^*, notice that if z is in the domain of Γ_n on a subinterval $[c, d]$ of $[a, b]$ and $w \in \Delta_0^n[c, d]$, then repeated use of integration by parts and the fact that Λ_j is of the form (3.10) gives $\int_c^d z L_0 w = \int_c^d z \Lambda_n w = \int_c^d w \Gamma_n z$. Hence, by definition of L_0^*, z belongs to the domain of L_0^* on $[c, d]$, and $\Gamma_n z = L_0^* z$. In particular, since Γ_n is clearly a linear operator, the null-space of Γ_n has dimension at most n.

On the other hand, suppose that, for k in $\{1, \cdots, n\}$,

$$z_k = W(y_1, \cdots, y_{k-1}, y_{k+1}, \cdots, y_n)/(r_n W_n) . \tag{3.11}$$

Then $\{z_1, \cdots, z_n\}$ is a basis for the null-space of L_0^*. For the discussion of adjoints in § 2 shows that $L_0 y = 0$ and $L_0^* z = 0$ are, respectively, equivalent to vector systems

$$Du = Gu , \tag{3.12}$$

$$Dv = -G^* v . \tag{3.13}$$

But if Y is the function matrix $[D^{i-1}y_j]_{i=1}^{n}{}_{j=1}^{n}$, then Y is a fundamental matrix for (3.12), by choice of $y_1, \cdots, y_n$. Hence, the matrix Y^{*-1} is a fundamental matrix for (3.13). It follows that the elements in the last row of Y^{*-1}, each multiplied by $1/r_n$, form a basis for the null-space of L_0^*. But these elements, after a proper choice of sign, are just the functions z_k.

Now, $z_n = W_{n-1}/(r_n W_n) = 1/\pi_n$, so $D(\pi_n z_n) = 0$ and $\Gamma_n z_n = 0$. For k in $\{1, \cdots, n-1\}$ and j in $\{0, 1, \cdots, n-(k+1)\}$, it will be shown by induction on j that $\Gamma_j z_k$ is defined, and

$$\Gamma_j z_k = W(y_1, \cdots, y_{k-1}, y_{k+1}, \cdots, y_{n-j})/W_{n-j-1} .$$

For the case $j = 0$, $\Gamma_0 z_k = \pi_n z_k = W(y_1, \cdots, y_{k-1}, y_{k+1}, \cdots, y_n)/W_{n-1}$, since $\pi_n = r_n W_n/W_{n-1}$. Assume the result holds for some index j in $\{0, 1, \cdots, n-k-2\}$. Then

$$\begin{aligned}&(-1)^{n-j-1-k}\Gamma_j z_k \\ &\quad = W(y_1, \cdots, y_{k-1}, y_{k+1}, \cdots, y_{n-j})/W(y_1, \cdots, y_{k-1}, y_{k+1}, \cdots, y_{n-j-1}, y_k) .\end{aligned}$$

Since both Wronskians appearing have at least one derivative, so does $\Gamma_j z_k$, and by the above lemma,

$$\begin{aligned}&(-1)^{n-j-1-k}[D\Gamma_j z_k][W(y_1, \cdots, y_{k-1}, y_{k+1}, \cdots, y_{n-j-1}, y_k)]^2 \\ &\quad = W(y_1, \cdots, y_{k-1}, y_{k+1}, \cdots, y_{n-j-1})/W(y_1, \cdots, y_{k-1}, y_{k+1}, \cdots, y_{n-j-1}, \\ &\qquad y_k, y_{n-j}) .\end{aligned}$$

Therefore,

$$D\Gamma_j z_k = W(y_1, \cdots, y_{k-1}\, y_{k+1}, \cdots, y_{n-j-1})\, W_{n-j}/[W_{n-j-1}]^2 ,$$

and then

$$\Gamma_{j+1} z_k = \pi_{n-j-1} D\Gamma_j z_k = W(y_1, \cdots, y_{k-1}, y_{k+1}, \cdots, y_{n-j-1})/W_{n-j-2} ,$$

which completes the induction. In particular,

$$\Gamma_{n-k-1} z_k = W(y_1, \cdots, y_{k-1}, y_{k+1})/W_k ,$$

so

$$D\Gamma_{n-k-1} z_k = W_{k-1} W_{k+1}/W_k^2 = 1/\pi_k .$$

Thus $\Gamma_{n-k}z_k = \pi_k D\Gamma_{n-k-1}z_k = 1$, so $D\Gamma_{n-k}z_k = 0$ and $\Gamma_n z_k = 0$. This shows that $\{z_1, \cdots, z_n\}$ is not only a subset of the domain of Γ_n, but, by the previous remarks, it is also a basis for the null-space of Γ_n.

Now, suppose z is in the domain of L_0^* on $[c, d]$. The form of the operator Γ_n clearly implies the existence of a function z_0 such that $\Gamma_n z_0 = L_0^* z$. But then z_0 is in the domain of L_0^* on $[c, d]$ as well, and $\Gamma_n z_0 = L_0^* z_0$, so $L_0^*(z - z_0) = 0$. Therefore there exist constants $c_1, \cdots, c_n$ such that $z - z_0 = \sum_{k=1}^n c_k z_k$, so z is a linear combination of elements in the domain of Γ_n and must therefore be in the domain of Γ_n. Moreover, $\Gamma_n z = \Gamma_n z_0 + \sum_{k=1}^n c_k \Gamma_n z_k = L_0^* z$. Thus, the operators L_0^* and Γ_n are identical.

Throughout the remainder of this discussion, I_{cd} will denote the functional given by (3.1) *for which L will be the Euler-Lagrange operator with the corresponding operators L_0 and L_0^* as in Theorem* 3.2 *and Γ_j and Λ_j defined by* (3.9).

4. A mean-value theorem. In this section, theorems analogous to Polya's Theorem I, II, III of [3] are obtained under hypotheses (H_1) and (H_2) for the operator L. For these theorems and certain preliminary results, we shall adopt the following terminology: if X is a finite set of real numbers, then a number x is said to be *intermediate with respect to X* if and only if x lies in the interior of the smallest compact interval containing X, unless X is a one-point set $\{x\}$, in which case the only intermediate point is defined to be the point x. The first result is an analogue of Polya's Theorem I for the operator L_0^*.

THEOREM 4.1. *Under hypothesis* (H_2), *if z is in the domain of L_0^* on a subinterval I of $[a, b]$ and one of the following conditions holds*:

(i) *z vanishes at $n + 1$ points $t_1 < t_2 < \cdots < t_{n+1}$ of I,*

(ii) *z vanishes at n points $t_1 < t_2 < \cdots < t_n$ of I and there is a j in $\{1, \cdots, n\}$ for which $D(r_n z)(t_j) = 0$,*

then there is a point t intermediate with respect to the set $\{t_i\}$ such that $L_0^* z(t) = 0$.

Notice that no additional condition of differentiability of the function z has been asserted in (ii), since $r_n z$ has a continuous derivative whenever z is in the domain of L_0^*, (see 2.8).

In case (i), it will be shown by induction that for every j in $\{0, 1, \cdots, n\}$ there exist $n - j + 1$ points $t_1^j < t_2^j < \cdots < t_{n-j+1}^j$ in $[t_1, t_{n+1}]$ at which $\Gamma_j z(t_i^j) = 0$. The assertion for $j = 0$ is just the condition (i). If the statement is true for some j in $\{0, 1, \cdots, n - 1\}$, then, by Rolle's theorem, for each i in $\{1, \cdots, n - j\}$ there is a point t_i^{j+1} in (t_i^j, t_{i+1}^j) such that $D\Gamma_j z(t_i^{j+1}) = 0$. Hence $\Gamma_{j+1} z(t_i^{j+1}) = 0$ for i

in $\{1, \cdots, n-j\}$, and $t_1 < t_1^{j+1} < t_2^{j+1} < \cdots < t_{n-j}^{j+1} < t_n$. Thus the induction is complete, and, in particular, there is a point t which lies in (t_1, t_{n+1}) at which $L_0^* z(t) = \Gamma_n z(t) = 0$.

In case (ii), Rolle's theorem implies that for each i in $\{1, \cdots, j-1\}$ there exists a point t_i^1 in (t_i, t_{i+1}), and for each i in $\{j, \cdots, n-1\}$ there exists a point t_{i+1}^1 in (t_i, t_{i+1}), such that $\Gamma_1 z(t_i^1) = 0$, i in $\{1, \cdots, j-1\}$, and $\Gamma_1 z(t_{i+1}^1) = 0$, i in $\{j, \cdots, n-1\}$. But

$$D(r_n z)(t_j) = 0 , \qquad \pi_n z = r_n z W_n / W_{n-1} ,$$

and W_n / W_{n-1} has a derivative, so

$$D(\pi_n z)(t_j) = [D(W_n / W_{n-1}) r_n z + (W_n / W_{n-1}) D(r_n z)](t_j) = 0 ,$$

since z also vanishes at t_j. Hence, if $t_j^1 = t_j$, then the n points $t_1^1 < t_2^1 < \cdots < t_n^1$ of (t_1, t_n) satisfy $\Gamma_1 z(t_i^1) = 0$ for i in $\{1, \cdots, n\}$. The same inductive process used in the proof of (i) then gives the existence of a number t intermediate with respect to $\{t_1, \cdots, t_n\}$ such that $L_0^* z(t) = 0$.

Theorem 4.1, together with results of § 3, result in the following analogue of Polya's Theorem 1 of [3] for the operator L.

THEOREM 4.2. *If* (H_1) *and* (H_2) *hold,* y *is in the domain of* L *on a subinterval* I *of* $[a, b]$, x_1 *and* x_2 *are points of* I *with* $x_1 < x_2$, *and there is a point* x_0 *of* I *different from* x_1 *and* x_2 *such that* $y(x_0) = 0$, *while* y *satisfies the conditions*

$$D^{j-1} y(x_1) = 0 = D^{j-1} y(x_2) , \qquad j \text{ in } \{1, \cdots, n\} , \tag{4.1}$$

then there is a point t *intermediate with respect to* $\{x_0, x_1, x_2\}$ *at which* $Ly(t) = 0$.

An induction argument will show that for each k in $\{1, \cdots, n-1\}$ there exist points $s_1^k < s_2^k < \cdots < s_{k+1}^k$ which are all different from x_1 and x_2 and lie in (x_1, x_2), (x_1, x_0), or (x_0, x_2), depending as $x_1 < x_0 < x_2$, $x_2 < x_0$, or $x_0 < x_1$, such that $\Lambda_k y(s_i^k) = 0$ for i in $\{1, \cdots, k+1\}$, and $\Lambda_k y(x_1) = 0 = \Lambda_k y(x_2)$.

First, the statement that $\Lambda_k y(x_1) = 0 = \Lambda_k y(x_2)$ for k in $\{0, 1, \cdots, n-1\}$ follows from the fact that $\Lambda_k y$ is of the form (3.10) and the hypothesis that y satisfies (4.1). Since $\Lambda_0 y(x_0) = (1/\pi_0) y(x_0) = 0$, and x_0 is different from x_1 and x_2, an application of Rolle's theorem gives the assertion when $k = 1$. If the statement is true for some k in $\{1, \cdots, n-2\}$, then points $s_1^{k+1} < s_2^{k+1} < \cdots < s_{k+2}^{k+1}$ are chosen as follows. If $x_1 < x_0 < x_2$, then the points $s_1^k, s_2^k, \cdots, s_{k+1}^k$ are in (x_1, x_2) and, by Rolle's theorem, choose s_1^{k+1} in (x_1, s_1^k), s_{k+2}^{k+1} in (s_{k+1}^k, x_2), and s_i^{k+1} in (s_{i-1}^k, x_i^k), for i in $\{2, \cdots, k+1\}$, such that $D\Lambda_k y(s_i^{k+1}) = 0$ for i in $\{1, \cdots, k+2\}$. If, on the other hand, $x_2 < x_0$, then there is an index q such that $s_q^k < x_2 < s_{q+1}^k$ while $x_1 < s_i^k$ for i in $\{1, \cdots, k+1\}$. Therefore, choose s_1^{k+1} in (x_1, s_1^k), s_i^{k+1} in (s_{i-1}^k, s_i^k) for i in $\{2, \cdots, q\}$, s_{q+1}^{k+1} in (s_q^k, x_2), s_{q+2}^{k+1} in (x_2, s_{q+1}^k), and s_i^{k+1} in (s_{i-2}^k, s_{i-1}^k) for i in $\{q+3, \cdots, k+2\}$ such that $D\Lambda_k y(s_i^{k+1}) = 0$, i in $\{1, \cdots, k+2\}$. A similar method of choice gives the values s_i^{k+1} in case x_1 lies between x_0 and x_2. But then $\Lambda_{k+1}y(s_i^{k+1}) = (\pi_{k+1}D\Lambda_k y)(s_i^{k+1}) = 0$ for i in $\{1, \cdots, k+2\}$, and the induction is complete.

In particular, there are points $s_1^{n-1} < s_2^{n-1} < \cdots < s_n^{n-1}$ different from x_1 and x_2 at which $\Lambda_{n-1}y$ vanishes, and, as the above construction shows, these points are also intermediate with respect to $\{x_0, x_1, x_2\}$. But $\Lambda_{n-1}y$ also vanishes at x_1 and x_2 so, applying Rolle's theorem once more, there exist points $t_1 < t_2 < \cdots < t_{n+1}$ different from x_1 and x_2 at which $L_0 y(t_i) = \Lambda_n y(t_i) = (\pi_n D\Lambda_{n-1}y)(t_i) = 0$. By Theorem 4.1 there is a point t intermediate with respect to $\{t_1, \cdots, t_{n+1}\}$, (hence, with respect to $\{x_0, x_1, x_2\}$), at which $Ly(t) = (-1)^n L_0^*(L_0 y)(t) = 0$.

Before continuing with the development of this section, we introduce an important property of the operator L. Since the equation $Ly = 0$ is equivalent to the identically normal system (3.3′) in which the matrix $B(x)$ is nonnegative on $[a, b]$, it follows, (see Theorem 5.2 of Reid [6]), that a necessary and sufficient condition for hypothesis (H_1) to hold is that L be *nonoscillatory on* $[a, b]$, that is, if $a \leqq x_1 < x_2 \leqq b$, then the boundary-value problem

$$(4.2)\qquad \begin{aligned} &Ly = 0\ , \\ &D^{j-1}y(x_1) = 0 = D^{j-1}y(x_2)\ , \qquad j \textit{ in } \{1, \cdots, n\}\ , \end{aligned}$$

is incompatible, i.e., has the function which vanishes identically on $[x_1, x_2]$ as its only solution. The equivalence of $Ly = 0$ to (3.3′) then implies that (H_1) is also equivalent to the statement that if $(x_1, y_1^1, y_1^2, \cdots, y_1^n)$ and $(x_2, y_2^1, y_2^2, \cdots, y_2^n)$ are points of R^{n+1} with $a \leqq x_1 < x_2 \leqq b$ and $\varphi \in C^0[x_1, x_2]$, then there exists a unique solution of the nonhomogeneous boundary-value problem

$$(4.2')\qquad \begin{aligned} &Ly = \varphi\ , \\ &D^{j-1}y(x_i) = y_i^j\ , \qquad i \textit{ in } \{1, 2\},\ j \textit{ in } \{1, \cdots, n\}\ . \end{aligned}$$

This enables us to formulate the following extension of Polya's mean-value theorem, the proof of which is identical to that of Polya.

THEOREM 4.3. *Suppose* (H_1) *and* (H_2) *hold, f is a function in the domain of L on a subinterval I of* $[a, b]$, *x_1 and x_2 are points of I with $x_1 < x_2$, and y_{12} denotes the solution of*

$$(4.3)\qquad \begin{aligned} &Ly = 0\ , \\ &D^{j-1}y(x_i) = D^{j-1}f(x_i)\ , \qquad i \textit{ in } \{1, 2\},\ j \textit{ in } \{1, \cdots, n\}\ . \end{aligned}$$

If h_{12} *denotes the solution of*

$$(4.4)\qquad \begin{aligned} Ly &= 1 , \\ D^{j-1}y(x_i) &= 0 , \qquad i \text{ in } \{1, 2\},\ j \text{ in } \{1, \cdots, n\} , \end{aligned}$$

then for each point x *in* I, *there is a point* t_x *in* I *such that*

$$(\mathrm{M})\qquad f(x) = y_{12}(x) + h_{12}(x)Lf(t_x) .$$

If $x = x_1$ or $x = x_2$, then (M) holds for any choice of t_x. If $x \in I$ and x is different from x_1 and x_2, then $h_{12}(x) \neq 0$ by Theorem 4.2, so there is a (unique) number c_x such that $f(x) = y_{12}(x) + h_{12}(x)c_x$. Let θ_x denote the function $f - y_{12} - c_x h_{12}$. Then θ_x is in the domain of $L, D^{j-1}\theta_x(x_i) = 0$ for i in $\{1, 2\}$, j in $\{1, \cdots, n\}$, and $\theta_x(x) = 0$. By Theorem 4.2, there is a point t_x intermediate with respect to $\{x, x_1, x_2\}$ at which $L\theta_x(t_x) = 0$. But

$$L\theta_x = Lf - Ly_{12} - c_x Lh_{12} = Lf - c_x \cdot 1 ,$$

so $c_x = Lf(t_x)$ and (M) follows.

It was noted that the solution h_{12} of (4.4) does not vanish in $[a, b]$ except at x_1 and at x_2. We now determine exactly what the sign of h_{12} is on (x_1, x_2) and on the union of $[a, x_1)$ and $(x_2, b]$.

THEOREM 4.4. *Under hypotheses* (H_1) *and* (H_2), *if* h_{12} *is the solution of* (4.4), *then*

$$\begin{aligned} (-1)^n h_{12}(x) &> 0 , \qquad \text{if } x_1 < x < x_2 , \\ h_{12}(x) &> 0 , \qquad \text{if } a \leqq x < x_1 \text{ or } x_2 < x \leqq b . \end{aligned}$$

Fix x_1 and x_2 in $[a, b]$ with $x_1 < x_2$, and suppose $z = L_0 h_{12}$. As in the proof of Theorem 4.2, one obtains by use of Rolle's theorem a set of n points $t_1^1 < t_1^2 < \cdots < t_1^n$ in (x_1, x_2) such that $z(t_1^k) = 0$ for k in $\{1, \cdots, n\}$. Applying Rolle's theorem as in the proof of Theorem 4.1, for each j in $\{1, \cdots, n\}$ there exist $n - j + 1$ points $t_j^1 < t_j^2 < \cdots < t_j^{n-j+1}$ such that $t_j^k < t_{j+1}^k < t_j^{k+1}$ for j in $\{1, \cdots, n-1\}$, k in $\{1, \cdots, n-j\}$, and $\Gamma_{j-1}z(t_j^k) = 0$ for j in $\{1, \cdots, n\}$, k in $\{1, \cdots, n-j+1\}$. If, for example, $s_j = t_j^{n-j+1}$, then $x_1 < s_n < s_{n-1} < \cdots < s_1 < x_2$ and $\Gamma_{j-1}z(s_j) = 0$ for j in $\{1, \cdots, n\}$. But $Lh_{12} = (-1)^n L_0^* z$, so $L_0^* z = (-1)^n$ and $(1/\pi_0)D\Gamma_{n-1}z = 1$. Therefore,

$$z = (1/\pi_n)\int_{s_1}\Big[(1/\pi_{n-1})\int_{s_2}\Big[(1/\pi_{n-2})\int_{s_3}\Big[\cdots\Big[(1/\pi_1)\int_{s_n}\pi_0\Big]\cdots\Big]\Big]\Big] .$$

In particular, suppose $s_1 < x \leqq x_2$. Then $z(x) > 0$, because each of the functions π_j is positive on $[a, b]$, and at the jth stage of the indicated iterative procedure used to calculate $z(x)$ the integral function

$$\int_{s_{n-j+1}}\left[(1/\pi_{j-1})\int_{s_{n-j+2}}\left[\cdots\left[(1/\pi_1)\int_{s_n}\pi_0\right]\cdots\right]\right]$$

is necessarily restricted to an interval with left end-point s_{n-j+1} and is therefore positive. However, Theorem 4.1 and the fact that $L_0^* z = (-1)^n$ imply that z cannot vanish at any point other than $t_1^1, t_1^2, \cdots, t_1^n$, and that z must change sign at each of these points. Since $z(x) > 0$ on $(t_1^n, x_2] = (s_1, x_2]$, it follows that $(-1)^n z(x) > 0$ on $[x_1, t_1^1)$. But $z(x_i) = L_0 h_{12}(x_i) = [\sum_{j=0}^n r_j D^j h_{12}](x_i) = [r_n D^n h_{12}](x_i)$ for i in $\{1, 2\}$, and $r_n(x_i) > 0$, so $(-1)^n D^n h_{12}(x_1) > 0$, $D^n h_{12}(x_2) > 0$, and the conclusion follows.

5. **Sub-(L) functions.** We are now prepared to define the notion of a sub-(L) function and to examine some of the properties of functions of this type. *Throughout this section, it is assumed that hypotheses* (H_1) *and* (H_2) *hold.*

A function f which has derivatives of the first $n - 1$ orders on a subinterval I of $[a, b]$ is said to be *sub-(L) on I* if and only if for every pair of points $x_1 < x_2$ in I, if y_{12} is the solution of the boundary-value problem (4.3), then $f(x) \leqq y_{12}(x)$ on $[x_1, x_2]$, and a sub-(L) function f is *strictly sub-(L) on I* if and only if for every pair of points $x_1 < x_2$ in $I, f(x) < y_{12}(x)$ on (x_1, x_2). We have the following characterization of sub-(L) functions.

THEOREM 5.1. *If f is a function in the domain of L on a subinterval I of $[a, b]$, then f is sub-(L) on I if and only if $(-1)^n Lf(t) \leqq 0$ on I. Moreover, if $(-1)^n Lf(t) < 0$ on I, then f is strictly sub-(L) on I.*

Suppose $(-1)^n Lf(t) \leqq 0$ on I. Let x_1 and x_2 be points of I with $x_1 < x_2$, let y_{12} be the solution of (4.3), and let h_{12} be the solution of (4.4). By Theorem 4.3, if $x \in I$ then there is a point t_x in I such that

$$f(x) = y_{12}(x) + h_{12}(x) Lf(t_x) \,. \tag{M}$$

But $(-1)^n Lf(t_x) \leqq 0$ and, by Theorem 4.4, $(-1)^n h_{12}(x) > 0$ on (x_1, x_2), so that if $x_1 < x < x_2$ then $f(x) \leqq y_{12}(x)$. It is also seen that, since $h_{12}(x) > 0$ outside the interval $[x_1, x_2]$,

$$(-1)^n f(x) \leqq (-1)^n y_{12}(x) \qquad \textit{if } x \in I \textit{ and } x \notin [x_1, x_2] \,.$$

Conversely, if f is sub-(L) on I, but there is a point t_0 of I such that $(-1)^n Lf(t_0) > 0$, then there is a nondegenerate subinterval $[x_1, x_2]$ of I on which $(-1)^n Lf(t) > 0$. Applying the mean-value formula (M) on $[x_1, x_2]$, one has $f(x) > y_{12}(x)$ on (x_1, x_2), a contradiction.

The last statement of the theorem clearly follows from formula (M).

In view of the equivalence of $Ly = \varphi$ to the nonhomogeneous first order linear system (3.3) and the classical properties of the Green's matrix for the corresponding incompatible first order system which is equivalent to (4.2), it follows that the solution f of $Ly = \varphi$ which satisfies the boundary conditions (4.1) is given by

$$f(x) = \int_{x_1}^{x_2} g(x, t)\varphi(t)dt ,$$

where the *Green's function* g is real-valued on $[x_1, x_2] \times [x_1, x_2]$ and has the following properties:

(i) *g and the first n partial derivatives with respect to its first argument are continuous.*

(ii) *If $i \in \{2, \cdots, n\}$, then, in the notation of* (3.2), *the mapping $T_i: (x, t) \to \mu_i[g(t)](x)$ is continuous on $[x_1, x_2] \times [x_1, x_2]$.*

(iii) *On each of $\{(x, t): x_1 \leqq x < t \leqq x_2\}$ and $\{(x, t): x_1 \leqq t < x \leqq x_2\}$, the mapping $T_1: (x, t) \to \mu_1[g(t)](x)$ is continuous, and if $x_1 < t < x_2$, then $T_1(t^-, t) - T_1(t^+, t) = (-1)^n$.*

(iv) *If $x_1 < t < x_2$, then on each of the half-open intervals $[x_1, t)$ and $(t, x_2]$ the function $\mu_1[g(t)]$ has a continuous derivative, and $Lg(t) = 0$ on each of these intervals; moreover, $g(t)$ satisfies the boundary conditions* (4.1).

(v) *$g(x, t) \equiv g(t, x)$ on $[x_1, x_2] \times [x_1, x_2]$.*

The following theorem on the Green's function gives a strengthening of the second assertion of Theorem 5.1.

THEOREM 5.2. *If $a \leqq x_1 < x_2 \leqq b$ and g is the Green's function for the incompatible problem* (4.2), *then $(-1)^n g(x, t) \geqq 0$ on $[x_1, x_2] \times [x_1, x_2]$.*

If not, then, since g is continuous, there is a point (x_0, t_0) in $(x_1, x_2) \times (x_1, x_2)$ such that $(-1)^n g(x_0, t_0) < 0$. Using the fact that $g(x, t) = 0$ on the boundary of $[x_1, x_2] \times [x_1, x_2]$, let t_1 denote LUB$\{t: x_1 \leqq t < t_0, g(x_0, t) = 0\}$, and let t_2 denote GLB$\{t: t_0 < t \leqq x_2, g(x_0, t) = 0\}$. Then $t_1 < t_0 < t_2$ and the continuity of g implies that $(-1)^n g(x_0, t) < 0$ on (t_1, t_2) and $g(x_0, t_1) = 0 = g(x_0, t_2)$.

Suppose φ is the function whose value at t is $g(x_0, t)$ for t in $[t_1, t_2]$ and is zero otherwise. Then φ is continuous, and if f is defined on $[x_1, x_2]$ by

$$f(x) = \int_{x_1}^{x_2} g(x, t)\varphi(t)dt ,$$

then $Lf = \varphi$ and, since $(-1)^n \varphi(t) \leqq 0$ on $[x_1, x_2]$, f is sub-(L). But $D^{j-1}f(x_1) = 0 = D^{j-1}f(x_2)$ for j in $\{1, \cdots, n\}$, so, by definition of sub-(L) functions, $f(x) \leqq 0$ on $[x_1, x_2]$. On the other hand,

$$f(x_0) = \int_{t_1}^{t_2} [g(x_0, t)]^2 dt ,$$

which is positive, a contradiction.

THEOREM 5.3. *If f is in the domain of L on a subinterval I of $[a, b]$ and f is sub-(L) on I, then a necessary and sufficient condition that f fail to be strictly sub-(L) on I is that there be a nondegenerate subinterval of I on which $Lf(x) \equiv 0$.*

If f fails to be strictly sub-(L) on I, then there are points $x_1 < x_2$ in I such that if y_{12} is the solution of (4.3), then $f(x) \leqq y_{12}(x)$ on $[x_1, x_2]$ and there is a point x_0 in (x_1, x_2) at which $f(x_0) = y_{12}(x_0)$. If $\varphi = Lf$, then $\varphi = L(f - y_{12})$, and if g is the Green's function for (4.2), then

$$f(x) - y_{12}(x) = \int_{x_1}^{x_2} g(x, t)\varphi(t)dt$$

on $[x_1, x_2]$. But then

$$\int_{x_1}^{x_2} g(x_0, t)\varphi(t)dt = f(x_0) - y_{12}(x_0) = 0 ,$$

and, since $g(x_0, t)$ and φ do not change sign on $[x_1 x_2]$, it follows that $g(x_0, t)\varphi(t) \equiv 0$ on $[x_1, x_2]$. Now, the restriction of $g(x_0, t)$ to $[x_1, x_0]$, using the appropriate one-sided limits at x_0, is a solution of $Ly = 0$. Hence, if $g(x_0, t)$ vanishes on some subinterval of $[x_1, x_0]$, then $g(x_0, t)$ vanishes identically on $[x_1, x_0]$. Since at least the first $n - 1$ derivatives of the function $g(x_0, t)$ are continuous at x_0, is follows that on $[x_0, x_2]$, the function $g(x_0, t)$ is a solution of

$$\begin{aligned} &Ly = 0 \\ &D^{j-1}y(x_0) = 0 = D^{j-1}y(x_2) , \qquad j \text{ in } \{1, \cdots, n\} , \end{aligned}$$

so $g(x_0, t) \equiv 0$ on $[x_0, x_2]$ as well. But then $g(x_0, t) \equiv 0$ on $[x_1, x_2]$, which violates the discontinuity condition which the function $\mu_1[g(x_0)]$ must satisfy at x_0. Correspondingly, the assumption that $g(x_0, t)$ vanishes on some subinterval of $[x_0, x_2]$ leads to a contradiction, so that any subinterval of $[x_1, x_2]$ contains a point t at which $g(x_0, t) \neq 0$, which implies that φ vanishes at this point. Hence, φ is a continuous function whose set of zeroes is dense in $[x_1, x_2]$, so $\varphi(x) \equiv 0$ on $[x_1, x_2]$. This, in turn, implies that $f(x) = y_{12}(x)$ on $[x_1, x_2]$ and $Lf(x) \equiv 0$ on $[x_1, x_2]$.

The sufficiency of the condition is obvious.

It is to be remarked that the result of Theorem 5.3 is weaker than the result that might be expected for sub-(L) functions. In the classical case where $L = D^2$, *any* convex function which fails to be strictly convex must be a solution of $D^2y = 0$ on some interval of its domain, and in [5] Reid generalized this statement exactly for a second-order Euler-Lagrange operator. However, for higher-order operators a generalization stronger than the above theorem is not immediately apparent.

6. Variational properties of sub-(L) function.

In addition to the classes $\Delta^n[c,d]$, $\Delta_0^n[c,d]$, we shall be concerned with the class $\bar{\Delta}_0^n[c,d]$ consisting of those functions w in $\Delta_0^n[c,d]$ for which $w(x) \leqq 0$ on $[c,d]$. If M is any real linear functional on any of these three classes, then M is *positive definite* if and only if $M[w] \geqq 0$ whenever w belongs to the given class with equality holding only if $w = 0$. The next two preliminary results are analogues of those in Reid [5], and the proofs are nearly identical to his.

THEOREM 6.1. *The statement that L is nonoscillatory is equivalent to each of the following conditions*:

(i) (H_1) *holds*;

(ii) *For each subinterval* $[c, d]$ *of* $[a, b]$, *the functional* I_{cd} *is positive definite on* $\Delta_0^n[c, d]$.

Since the system (3.3′) is identically normal and the matrix $B(x) \geqq 0$ on $[a,b]$, it follows from Theorem 5.2 of Reid [6] that the nonoscillation of L is equivalent to each of the conditions (i) and (ii).

THEOREM 6.2. *The condition* (H_1) *implies that if* $[c, d]$ *is a subinterval of* $[a, b]$ *and* $f \in \Delta^n[c, d]$, *then the following conditions are equivalent*:

(i) $I_{cd}[y] \geqq I_{cd}[f]$ *whenever* $y - f \in \bar{\Delta}_0^n[c, d]$.

(ii) *If* J_{cd} *is the bilinear functional defined on* $\Delta^n[c, d] \times \Delta^n[c, d]$ *by*

$$J_{cd}[(y, w)] = \int_c^d \left[\sum_{j=0}^n p_j D^j y D^j w \right],$$

then $J_{cd}[(f, w)] \geqq 0$ *whenever* $w \in \bar{\Delta}_0^n[c, d]$.

If $w \in \Delta_0^n[c,d]$, then $I_{cd}[w] \geqq 0$ by Theorem 6.1. Also, if $w \in \bar{\Delta}_0^n[c,d]$ and t is any positive number, then $tw \in \bar{\Delta}_0^n[c,d]$. The result then follows from the identity

$$I_{cd}[f + tw] = I_{cd}[f] + 2tJ_{cd}[(f, w)] + t^2 I_{cd}[w] .$$

We now obtain a characterization of sub-(L) functions which are in the domain of L in terms of unilateral variational property.

THEOREM 6.3. *If* (H_1) *and* (H_2) *hold and* f *is a function in the domain of* L *on a subinterval* $[c, d]$ *of* $[a, b]$, *then a necessary and sufficient condition for* f *to be sub-*(L) *on* $[c, d]$ *is that*

$$I_{cd}[y] \geqq I_{cd}[f] \text{ whenever } y - f \in \bar{\Delta}_0^n[c, d]. \tag{6.1}$$

If $w \in \Delta_0^n[c, d]$, then

$$J_{cd}[(f, w)] = \int_c^d \left[\sum_{j=0}^n p_j D^j f D^j w\right],$$

which, by definition, means that

$$J_{cd}[(f, w)] = \int_c^d [w(-1)^n Lf] .$$

Therefore, $J_{cd}[(f, w)] \geqq 0$ for every w in $\bar{\Delta}_0^n[c, d]$ if and only if $(-1)^n Lf(x) \leqq 0$ on $[c, d]$. The conclusion then follows from Theorems 5.1 and 6.2.

It would be desirable to remove the condition that f belong to the domain of L from the hypothesis of this theorem. One possibility which might be examined is the simple case where $L = D^{2n}$, for if $f \in \Delta^n[c, d]$, $w \in \Delta_0^n[c, d]$, $\rho_0[f] = p_0 f$, and

$$\rho_i[f] = p_i D^i f - \int_c \rho_{i-1}[f]$$

for i in $\{1, \cdots, n\}$, then

$$J_{cd}[(f, w)] = \int_c^d D^n w p_n[f] ,$$

which is of the form

$$\int_c^d D^n w D^n \varphi ,$$

exactly that which arises in considering the case $L = D^{2n}$. It is to be noted, however, that the "sufficiency" part of Theorem 6.3 does *not* require f to be in the domain of L.

THEOREM 6.4. *If* (H_1) *and* (H_2) *hold and* $f \in \Delta^n[c, d]$, *then* f *is sub-*(L) *on* $[c, d]$ *in case* (6.1) *holds.*

Suppose $c \leqq x_1 < x_2 \leqq d$ and y_{12} is the solution of (4.3). Let t be an arbitrary point in (x_1, x_2), and let w_t be the function whose value at x is zero outside $[x_1, x_2]$ and is $(-1)^{n+1}g(x, t)$ on (x_1, x_2), where g is the Green's function for (4.2). Then $w_t \in \bar{\Delta}_0^n[c, d]$ by Theorem 5.2, so that

$$0 \leqq J_{cd}[f, w_t] = (-1)^{n+1}\left[\sum_{i=0}^{n-1} D^i f \mu_{i+1}[g(t)]\right]_{x_1}^{x_2} - f(t) .$$

But, for arbitrary w in $\Delta_0^n[c, d]$,

$$J_{cd}[y_{12}, w] = \left[\sum_{i=0}^{n-1} D^i w \mu_{i+1}[y_{12}]\right]_c^d = 0 .$$

In particular,

$$0 = J_{cd}[y_{12}, w_t] = (-1)^{n+1}\left[\sum_{i=0}^{n-1} D^i y_{12} \mu_{i+1}[g(t)]\right]_{x_1}^{x_2} - y_{12}(t) ,$$

and, in view of the boundary conditions of (4.3),

$$0 \leqq y_{12}(t) - f(t) .$$

Hence, f is sub-(L) on $[c, d]$.

7. **Strong nonoscillation of L.** Under hypotheses (H_1) and (H_2) we are able to conclude that the null-space of the operator L is a $2n$-*parameter family on* $[a, b]$, i.e., that there is exactly one solution of $Ly = 0$ which assumes $2n$ given values at $2n$ given (distinct) points of $[a, b]$. We first establish the following result, the proof of which is modeled after a proof of Polya [3].

THEOREM 7.1. *Suppose* (H_2) *holds,* $\{z_1, \cdots, z_n\}$ *is the basis for the null-space of* L_0^* *given by* (3.11) *and, for each* k *in* $\{1, \cdots, n\}$, Z_k *is the set of all linear combinations of* $\{z_k, \cdots, z_n\}$. *If* $z \in Z_k$, *then either* $z(x) \equiv 0$ *or else* z *has at most* $n - k$ *zeroes on* $[a, b]$. *In particular, if* $a \leqq t_1 < t_2 < \cdots < t_n \leqq b$, *then the* n*-point boundary-value problem*

$$L_0^* z = 0 , \qquad z(t_i) = 0 , \qquad i \text{ in } \{1, \cdots, n\} , \tag{7.1}$$

is incompatible.

If $z = c_n z_n$ then, since $z_n(x) > 0$ on $[a, b]$, either $z(x) \equiv 0$ or else z vanishes nowhere on $[a, b]$. Assume that $k + 1$ is an index for which the assertion is true and suppose $z \in Z_k$, say $z = \sum_{j=k}^n c_j z_j$. If $c_k = 0$ then $z \in Z_{k+1}$, so either $z(x) \equiv 0$ or else z has less than $n - k$ zeroes on $[a, b]$. If $c_k \neq 0$, then $z(x) \not\equiv 0$ and we may write $z_k = (1/c_k)z - z_0$, where $z_0 = \sum_{j=k+1}^n (c_j/c_k) z_j$. If it were possible that there exist $n - k + 1$ points $t_1, t_2, \cdots, t_{n-k+1}$ at which z vanishes, then $z_k + z_0$ would also vanish at these points and, as in the proof of Theorem 4.1, there would exist a point t intermediate with respect to $\{t_1, t_2, \cdots, t_{n-k+1}\}$ at which $\Gamma_{n-k}[z_k + z_0](t) = 0$. But the proof of Theorem 3.3 shows that $\Gamma_{n-k} z_j = 0$ if $j \geqq k + 1$ and $\Gamma_{n-k} z_k = 1$. In particular, $\Gamma_{n-k}[z_k + z_0](t) = 1$, a contradiction.

THEOREM 7.2. *Under* (H_1) *and* (H_2), *if* $a \leqq x_1 < x_2 < \cdots < x_m \leqq b$ *and, for each* i *in* $\{1, \cdots, m\}$, $\lambda_i \in \{1, 2, \cdots, n+1\}$ *such that* $\sum_{i=1}^m \lambda_i = 2n$, *then the m-point boundary-value problem*

$$(7.2)\qquad \begin{aligned} Ly &= 0\ , \\ D^{j-1}y(x_i) &= 0\ , \qquad i \text{ in } \{1, \cdots, m\},\ j \text{ in } \{1, \cdots, \lambda_i\}\ , \end{aligned}$$

is incompatible.

It will be shown that if y is any function in the domain of the operator L_0 which satisfies the boundary conditions of (7.2), then there exist n points of $[a, b]$ at which $L_0 y$ vanishes.

Let $\nu = \max\{\lambda_1, \lambda_2, \cdots, \lambda_m\}$. If $\nu = 1$, then $m = 2n$, and repeated application of Rolle's theorem using the decomposition (3.9) gives the result. If $\nu > 1$, then for each k in $\{2, \cdots, \nu\}$ let α_k denote the number of points x_i at which $\lambda_i = k$, and for j in $\{1, 2\}$, let $s_{j,k}$ denote the set of integers r with $j \leqq r \leqq \nu$ and $r \neq k$. Now, if i is an index such that $\lambda_i = k$ then, by (3.10), $\Lambda_j y(x_i) = 0$ for j in $\{0, 1, \cdots, k-1\}$, so Rolle's theorem implies that $\Lambda_1 y$ vanishes at $\beta_1 = m - 1 + \sum_{k=2}^{\nu} \alpha_k$ points of $[a, b]$. It will be shown that for each j in $\{1, \cdots, \nu - 1\}$, $\Lambda_j y$ vanishes at

$$\beta_j = m - j + \sum_{k=2}^{j} (k-1)\alpha_k + j \sum_{k=j+1}^{\nu} \alpha_k$$

points of $[a, b]$. Since the assertion is known for $j = 1$, assume that it holds for some j in $\{1, \cdots, \nu - 2\}$. Applying Rolle's theorem, $\Lambda_{j+1} y$ must vanish first of all at $\beta_j - 1$ points, none of which will be an x_i with $\lambda_i \geqq j + 2$. But $\Lambda_{j+1} y$ also vanishes at exactly these points x_i as well, and it follows that

$$\begin{aligned} \beta_{j+1} &= m - j - 1 + \sum_{k=2}^{j+1} (k-1)\alpha_k + j \sum_{k=j+2}^{\nu} \alpha_k + \sum_{k=j+2}^{\nu} \alpha_k \\ &= m - j - 1 + \sum_{k=2}^{j+1} (k-1)\alpha_k + (j+1) \sum_{k=j+2}^{\nu} \alpha_k\ . \end{aligned}$$

In particular, $\Lambda_{\nu-1} y$ vanishes at

$$\beta_{\nu-1} = m - (\nu - 1) + \sum_{k=2}^{\nu} (k-1)\alpha_k$$

points of $[a, b]$. But

$$\alpha_k = \sum_{i=1}^{m} [\textstyle\prod_{s_{1,k}} (\lambda_i - r)/(k - r)]\ ,$$

so

$$(k-1)\alpha_k = \sum_{i=1}^{m} [\textstyle\prod_{s_{2,k}} (\lambda_i - 1)(\lambda_i - r)/(k - r)]\ .$$

Hence,

$$\beta_{\nu-1} = m - (\nu - 1) + \sum_{i=1}^{m} \left\{ \sum_{k=2}^{\nu} [\Pi_{s_2,k} (\lambda_i - r)/(k - r)] \right\} (\lambda_i - 1) .$$

The expression in braces is a polynomial in λ_i of degree at most $\nu - 2$ which has the value 1 for each of the $\nu - 1$ values $\lambda_i = 2, 3, \cdots, \nu$. Hence, this expression is identically 1 in λ_i, and

$$\beta_{\nu-1} = m - (\nu - 1) + \sum_{i=1}^{m} (\lambda_i - 1) = 2n - (\nu - 1) ,$$

i.e., $\Lambda_{\nu-1} y$ vanishes at $2n - (\nu - 1)$ distinct points of $[a, b]$. The same use of Rolle's theorem as that for the case $\nu = 1$ now gives the conclusion that $L_0 y$ must vanish at n distinct points of $[a, b]$.

If y satisfies (7.2), then $z = L_0 y$ satisfies (7.1) for some set $\{t_1, \cdots, t_n\}$ of points in $[a, b]$, so $z = 0$, i.e., $L_0 y = 0$, and, by Theorem II of Polya [3] for the operator L_0, it follows that y must also vanish identically.

In particular, the problem (7.2) with $m = 2n$ is incompatible, and the elementary solvability theorems for vector differential systems imply that the null-space of L is indeed a $2n$-parameter family. Hence, it is possible to examine L-convexity in the sense of Tornheim [7] and Hartman [2], whereby a function f defined on an open subinterval (c, d) of $[a, b]$ is *L-convex* if and only if for every set of $2n$ points $x_1 < x_2 < \cdots < x_{2n}$ of (c, d), if y is the unique function satisfying

$$Ly = 0 ,$$
$$y(x_i) = f(x_i) , \qquad i \text{ in } \{1, \cdots, 2n\} ,$$

then

$$(-1)^i y(x) \leqq (-1)^i f(x) \quad \text{on} \quad (x_i, x_{i+1}) .$$

However, the exact relationship between the two types of convexity remains undecided.

It is also natural to ask about the properties of the operator $\tilde{L} = (-1)^n L_0 L_0^*$. It is easily seen that $\tilde{L} y = 0$ is equivalent to an identically normal system of the type (2.6′) and that if U and V are as specified in (H_1) then $(U^{*-1}; 0)$ satisfies an analogous condition $(\tilde{\mathrm{H}}_1)$ for $\tilde{L}$. Moreover, in the notation of (2.8), if $\{z_1, \cdots, z_n\}$ is contained in the domain of L_0^* then for each k in $\{1, \cdots, n\}$ we define the "generalized Wronskian" $W^*(z_n, z_{n-1}, \cdots, z_{n-k+1})$ to be the determinant of the $k \times k$ matrix $[\nu_{n-i+1}[z_{n-j+1}]]_{i=1}^{k}{}_{j=1}^{k}$. In particular, if $\{z_1, \cdots, z_n\}$ is the basis for the null-space of L_0^* defined by (3.11) and $W_k^* = W^*(z_n, z_{n-1}, \cdots, z_{n-k+1})$, then W_k^* is equal to the lower right principal minor of order k in the matrix U^{*-1}. Hence, if $\mathfrak{U}$ is the adjoint matrix of U^* and $\mathfrak{U}_k$ is the lower right principal minor of order k in the matrix $\mathfrak{U}$, then a well-known formula (see, e.g., Hohn's *Elementary Matrix Algebra*, p. 61) gives

$$W_k^* = \mathfrak{U}_k/(\det U)^k = (\det U)^{k-1} W_{n-k}/(\det U)^k = W_{n-k}/W_n ,$$

which, by hypothesis (H_2), is positive. Thus, we have an analogue ($\tilde{H}_2$) of (H_2). However, it is not evident that properties of convexity, etc., with respect to $\tilde{L}$ shed any light at all on the questions already raised concerning L.

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