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QUASI-ISOMORPHISM FOR INFINITE ABELIAN p-GROUPS

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This paper is concerned with the investigation of two closely related questions. The first question is: What relationships exist between G and nG where G is an Abelian group and n is a positive integer?

It is shown that if G' and H' are Abelian groups, n is a positive integer and $nG' \cong nH'$, then $G \cong H$ where $G' = S \oplus G$ and $H' = T \oplus H$ such that S and T are maximal n-bounded summands of G' and H', respectively. A corollary of this is: Every automorphism of nG can be extended to an automorphism of G.

We define two primary Abelian groups G and H to be quasi-isomorphic if and only if there exists positive integers m and n and subgroups S and T of G and H, respectively, such that $p^nG \subset S$, $p^mH \subset T$ and $S \cong T$, the second question is: What does quasi-isomorphism have to say about primary Abelian groups? It is shown that if two Abelian p-groups G and H are quasi-isomorphic then G is a direct sum of cyclic groups if and only if H is a direct sum of cyclic groups, G is closed if and only if H is closed, and G is a Σ-group if and only if H is a Σ-group.

In this paper the word "group" will mean "Abelian group," and we shall use the notation in [5] except that a direct sum of groups A and B will be denoted by $A \oplus B$. Also if $a \in A$ then $H_A^p(a)$ will denote the p-height of a in A. (If it is clear which group or which prime is referred to then either sub- or super-script may be dropped or both.)

At a symposium on Abelian groups held at New Mexico State University, L. Fuchs asked the question: What does quasi-isomorphism (see Definition 3.2) have to say about primary Abelian groups? A question posed by John M. Irwin that arises in the investigation of this question is: What relationships exist between G and nG where G is an Abelian groups and n is a positive integer? The purpose of this paper is to investigate these two questions.

First, we will begin by considering to what extent nG determines G where G is a group and n is a positive integer. It will be shown that if G' and H' are groups, n is a positive integer, and $nG' \cong nH'$, then $G \cong H$ where $G' = S \oplus G$ and $H' = T \oplus H$ such that S and T

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are maximal n-bounded summands of G' and H', respectively. A corollary to this is: Every automorphism of nG can be extended to an automorphism of G.

In looking at quasi-isomorphism of primary Abelian groups, it is shown that if two Abelian p-groups G and H are quasi-isomorphic then G is a direct sum of cyclic groups if and only if H is a direct sum of cyclic groups, G is closed if and only if H is closed, and G is a Σ-group if and only if H is a Σ-group. Other related results are also obtained.

1. An isomorphism theorem.

LEMMA 1.1. *Let G be a p-group and B' be a basic subgroup of p^nG. Then there exists a basic subgroup B of G such that $p^nB = B'$.*

Proof. Write $B' = \sum_{\lambda \in \Lambda} \{x_\lambda\}$. Now there exists $y_\lambda \in G$, for all $\lambda \in \Lambda$, such that $p^n y_\lambda = x_\lambda$. Let $B_n^* = \sum_{\lambda \in \Lambda} \{y_\lambda\}$. Now B_n^* is pure in G and $[y_\lambda : \lambda \in \Lambda]$ is a maximal pure independent set with respect to the property of B_n^* having no cyclic summand of order $\leqq p^n$. To see the purity of B_n^* first notice that $(B_n^*)[p] = (B')[p]$. Let $x \in (B_n^*)[p]$. Now $H_{B_n^*}(x) = n + H_{p^nG}(x)$ and if $H_{p^nG}(x) = m$, $H_G(x) = m + n$ and $H_{B_n^*}(x) = m + n$. Hence $H_G(x) = H_{B_n^*}(x)$, and B_n is pure in G. That B_n^* is maximal pure as above is clear. Thus B_n^* can be extended to a basic subgroup B of G, and $B = S_n \oplus B_n^*$ where $p^nS_n = 0$ (see p. 97 of [5]). Hence $p^nB = B'$.

Using the above notation note that $G = S_n \oplus G_n$ where $G_n = B_n^* + p^nG$ and B_n^* is basic in G_n. This follows from a theorem of Baer's (Theorem 29.3 in [5]). We shall continue using this notation and refer to this decomposition as Baer's decomposition. From a theorem of Szele's [Theorem 29.4 in [5]) it follows that S_n is a maximal p^n-bounded summand of G. From this it is easy to show that, if H' is a group, then $H' = T \oplus H$ such that T is a maximal n-bounded summand of H'.

THEOREM 1.2. *Let G and H be p-groups such that $p^nG \cong p^nH$ (under an isomorphism ϕ) for some positive integer n. Then $G_n \cong H_n$ according to Baer's decomposition.*

Proof. We may assume that G and H are reduced by Test Problem I and Exercise 9 in [9]. Now $p^nG_n = p^nG \cong p^nH = p^nH_n$. Let $p^n(B_{G_n})$ be a basic subgroup of p^nG such that B_{G_n} is a basic subgroup of G_n, and let B_{H_n} be a basic subgroup of H_n such that $\phi(p^nB_{G_n}) = p^nB_{H_n}$ a basic subgroup of p^nH. This is possible by the above lemma and the fact that $p^nG \cong p^nH$ under ϕ. From the proof of the above lemma it is easily seen that there exists an isomorphism $\bar{\phi}: B_{G_n} \to B_{H_n}$ such that

$\bar{\phi} \mid p^n B_{G_n} = \phi \mid p^n B_{G_n}$. We may write $G_n = B_{G_n} + p^n G_n$ and $H_n = B_{H_n} + p^n H_n$. Define

$$\psi\colon G_n \to H_n\colon g_n = b + p^n g_{n_1} \to \bar{\phi}(b) + \phi(p^n g_{n_1})$$

where $b \in B_{G_n}$ and $g_{n_1} \in G_n$.

Suppose $g_n = b + p^n g_{n_1} = b' + p^n g_{n_2}$, where $b, b' \in B_{G_n}$ and $g_{n_1}, g_{n_2} \in G_n$. Then $b - b' = p^n(g_{n_2} - g_{n_1})$ implies $b - b' \in p^n B_{G_n}$, and $\bar{\phi}(b) - \bar{\phi}(b') = \bar{\phi}(b - b') = \phi(p^n g_{n_2} - p^n g_{n_1}) = \phi(p^n g_{n_2}) - \phi(p^n g_{n_1})$. Therefore

$$\bar{\phi}(b) + \phi(p^n g_{n_1}) = \bar{\phi}(b') + \phi(p^n g_{n_2}) .$$

Hence ψ is well defined. Clearly ψ is a homomorphism.

Suppose $g_n \in G_n$ and $\psi(g_n) = 0$. Now $g_n = b + p^n g_{n_1}$ where $b \in B_{G_n}$ and $g_{n_1} \in G_n$, and $\psi(g_n) = \psi(b + p^n g_n) = \bar{\phi}(b) + \phi(p^n g_n) = 0$. Hence $\bar{\phi}(b) = -\phi(p^n g_{n_1})$ and $b \in p^n G_n$. Thus $\bar{\phi}(b) = \phi(b) = \phi(-p^n g_{n_1})$ and $\phi(b + p^n g_n) = 0$. Therefore $g_n = b + p^n g_{n_1} = 0$ and ψ is one-to-one. If $h_n \in H_n$ then $h_n = b + p^n h_{n_1}$ for some $b \in B_{H_n}$ and $h_{n_1} \in H_n$. Since ϕ and $\bar{\phi}$ are onto, ψ is onto.

COROLLARY 1.3. *Let G be a p-group. Let B and B' be basic subgroups of G such that $G_n = B_n^* + p^n G$ and $G'_n = (B')_n^* + p^n G$. Then $G'_n \cong G_n$.*

COROLLARY 1.4. *Let G and H be torsion groups such that $nG \cong nH$ for some positive integer n. Then $G \oplus B_1 \cong H \oplus B_2$ where B_1 and B_2 are groups of bounded order bounded by n.*

COROLLARY 1.5. *Let G' and H' be torsion groups and n a positive integer. Write $G' = S \oplus G$ and $H' = T \oplus H$ where S and T are maximal n bounded summands of G' and H', respectively. Suppose that $nG' = nH'$. Then $G \cong H$.*

COROLLARY 1.6. *Let G and H be p-groups such that $p^m G \cong p^n H$, $n > m$. Then $p^{n-m} H_m \cong G_m$.*

Proof. Now $p^m(p^{n-m}H) \cong p^m G$ implies $(p^{n-m}H)_m \cong G_m$ by 1.2. By Baer's Theorem $H = S_m \oplus H_m$ where $H_m = B_m^* + p^n H$. Thus $p^{n-m}H = p^{n-m}S_m \oplus (p^{n-m}B_m^* + p^{n-m}(p^m H))$. Also

$$p^{n-m}H = p^{n-m}S_m \oplus (p^{n-m}B_m^* + p^m(p^{n-m}H))$$

since $p^{n-m}(S_m \oplus B_m^*)$ is a basic subgroup of $p^{n-m}H$. Thus $(p^{n-m}H)_m \cong p^{n-m}(H_m)$ and we have that $G_m \cong p^{n-m}H_m$.

A generalization of Theorem 1.2 is the following:

THEOREM 1.7. *Let G' and H' be groups such that $nG' \cong nH'$, for some positive integer n. Write $G' = S \oplus G$ and $H' = T \oplus H$ such that S and T are maximal n-bounded summands of G' and H', respectively. Then $G \cong H$.*

Proof. If G' and H' are torsion groups, we are done by 1.5. Thus suppose that G' and H' are not torsion groups. It suffices to prove the theorem for $n = p$, a fixed but arbitrary prime. To see this write $n = p_1^{k_1} \cdot p_2^{k_2} \cdots p_m^{k_m}$ such that p_i's are distinct primes. Let n_i be chosen such that $p_i n_i = n$. Observe that if $p_i(n_{ij}G) \cong p_i(n_i H)$ implies that $n_i G \cong n_i H$, then by finite induction $G \cong H$.

Suppose that $pG' \cong pH'$ under the isomorphism σ. Write $G' = S \oplus G$ and $H' = T \oplus H$ such that S and T are maximal p-bounded summands of G' and H', respectively. Let G_t and H_t be the torsion subgroups of G and H, respectively. Then $pG_t \cong pH_t$ under the isomorphism $\sigma \mid pG_t$. By 1.5, $G_t \cong H_t$ under an isomorphism σ_t such that $\sigma_t \mid pG_t = \sigma \mid pG_t$. Define a map

$$\phi: G_t + pG \rightarrow H_t + pH: \phi(t + g) = \sigma_t(t) + \sigma(g)$$

where $t \in G_t$ and $g \in pG$. Note that ϕ is an isomorphism since $\sigma \mid G_t \cap pG = \sigma_t \mid G_t \cap pG$ and $\sigma(G_t \cap pG) = H_t \cap pH$. (The proof of this is similar to that in Theorem 1.2.)

Next we will show that there exists a pair (S', ϕ') such that $S' = \{G_t, pG, x\}$ with $x \in G$ and $x \notin G_t + pG$, ϕ' is an isomorphism from S' to a subgroup of H, and $\phi' \mid G_t + pG = \phi$. To this end let $x \in G$ such that $x \notin G_t + pG$. Then x is torsion free. Let $y \in H$ such that $py = \phi(px)$. Define

$$\phi': \{G_t, pG, x\} \rightarrow \{H_t, pH, y\}: \phi'(t + g + nx) = \phi(t + g) + ny$$

such that $(n, p) = 1$ or $n = 0$, $t \in G_t$ and $g \in G$. Note that $H^p(nx + t) = 0$ for all $t \in G_t$ and $(n, p) = 1$. If there exists $z \in G$ such that $pz = nx + t$ then $nx = pz - t \in G_t + pG$, a contradiction to the choice of x. Thus $H^p(t + g + nx) = 0$ for $(n, p) = 1, t \in G_t$ and $g \in pG$. Suppose that $g' + nx = g'' + mx$ with $g', g'' \in G_t + pG$. Then $\phi'(g' + nx) = \phi(g') + ny$ and $\phi'(g'' + mx) = \phi(g'') + my$. Now $(n - m)x = g'' - g' \in pG$ since $H^p(kx + t) = 0$, $(k, p) = 1$, for all $t \in G_t$. For if p does not divide $(n - m)$, then $H^p(x + t) > 0$ for some $t \in G_t$ since $g'' - g' = g - t$ such that $g \in pG$ and $t \in G_t$. Thus $n - m = pn_1$, and

$$\phi(g'' - g') = \phi((n - m)x) = \phi(n_1 px) = n_1 py = (n - m)y .$$

Hence $\phi(g') + ny = \phi(g'') + my$, and the map is well defined. Now ϕ' is clearly a homomorphism onto $\{H_t, pH, y\}$. If $\phi'(z) = 0$ for some

$z \in \{G_t, pG, x\}$ then $pz = 0$ since $\phi' \mid pG$ is one-to-one. Thus $z \in G_t$. But $\phi' \mid G_t$ is one-to-one and hence $z = 0$. Thus we have an extension (S', ϕ') of $(G_t + pG, \phi)$.

Next let $\mathscr{S}$ be the set of all pairs (S_α, ϕ_α) such that S_α is a subgroup of G containing $G_t + pG$ and ϕ_α is an extension of ϕ. Partially order $\mathscr{S}$ as follows: If $(S_\alpha, \phi_\alpha), (S_\beta, \phi_\beta) \in \mathscr{S}$, then $(S_\alpha, \phi_\alpha) \geqq (S_\beta, \phi_\beta)$ if and only if $S_\alpha \supset S_\beta$ and ϕ_α is an extension of ϕ_β. Now $\mathscr{S} \neq 0$ as shown above, and every chain $\mathscr{C}$ has a least upper bound in $\mathscr{S}$. To see this let $\mathscr{C} = [(S_\alpha, \phi_\alpha): \alpha \in \Lambda]$. Let $S_c = \bigcup_{\alpha \in \Lambda} S$ and ϕ_c be defined by $\phi_c(s_\alpha) = \phi_\beta(s_\alpha)$ for $\beta \geqq \alpha$, $s_\alpha \in S_\alpha$.

Clearly (S_c, ϕ_c) is the least upper bound and $(S_c, \phi_c) \in \mathscr{S}$. Therefore by Zorn's Lemma there exists a maximal element (S_M, ϕ_M). Now $S_M = G$, for otherwise there exists $x \in G$ such that $x \notin S_M$, and we may extend ϕ_M to $\{S_M, x\}$ as before. Thus we have an isomorphism ϕ_M from G into H. If $y \in H$ and $x \in G$ such that $px = \phi^{-1}(py)$ then $\phi_M(x) - y = t \in H_t$ which implies that $y = \phi_m(x) - t \in H$. Thus ϕ_M is onto and G and H are isomorphic.

COROLLARY 1.8. *Let G' be a group and n a positive integer. Then every automorphism of nG' can be extended to an automorphism of G'.*

This is actually a corollary to the proof of Theorem 1.7. For if we write $G' = S \oplus G$, where S is a maximal n-bounded summand of G', every automorphism of nG' can be extended to an automorphism of G, as the proof of Theorem 1.7 indicates. This together with any automorphism of S gives the desired automorphism of G'.

COROLLARY 1.9. *Let G and H be groups and n a positive integer. Suppose that $nG \cong nH$ and the maximal n-bounded summands of G and H are isomorphic. Then $G \cong H$.*

COROLLARY 1.10. *Let G' be a group and n a positive integer. The only pure subgroups between G' and nG' are groups of the form $S \oplus G$ where S is a pure subgroup of G' bounded by n and $G' = S' \oplus G$ such that S' is a maximal n-bounded summand of G'.*

Proof. Let H' be a pure subgroup of G' such that $G' \supset H' \supset nG'$. Then $G'/H' \cong T$ a group of bounded order bounded by n. By Theorem 5 in [9], $G' = H' \oplus T'$ such that $T' \cong T$. Thus $nG' = nH' \oplus nT' = nH'$. By 1.7, $H \cong G$ where $H' = K \oplus H$ with K a maximal n-bounded summand of H'. Since H is pure in G' and G'/H is bounded by n, H is a summand of G' and $G' = S' \oplus H$. Therefore $H' \cong S \oplus G$ where S is a pure subgroup of G' bounded by n.

COROLLARY 1.11. *Let G be a group and n a positive integer. The pure subgroups between G and nG are all isomorphic up to summands of bounded order, bounded by n.*

COROLLARY 1.12. *Let G be a group and suppose that* $G = S \oplus G' = T \oplus G''$ *where S and T are maximal n-bounded summands of G. Then* $G' \cong G''$.

2. **Some properties of G and nG.** In this section we will be concerned mainly with the question: If P is a property of a group, does G have property P if and only if nG has property P where G is a group and n is a positive integer? This question is of interest in itself, but it is also of interest in looking at the question of Fuchs: What does quasi-isomorphism have to say about primary Abelian groups? We shall begin by proving a decomposition theorem.

THEOREM 2.1. *Let G be a group and n a positive integer such that* $nG = \sum_{\lambda \in \Lambda} G'_\lambda$. *Then* $G = \sum_{\lambda \in \Lambda} G_\lambda$ *such that* $nG_\lambda = G'_\lambda$.

Proof. Let $\alpha \in \Lambda$. Set $H_\alpha = G/\sum_{\Lambda'} G'_\lambda$, where $\Lambda' = \Lambda - \alpha$, for all $\alpha \in \Lambda$, and observe that $nH_\alpha \cong G'_\alpha$. Now $H_\lambda = S^\lambda_n \oplus (H_\lambda)_n$ where S^λ_n is a maximal n-bounded summand of H_λ. Set $\hat{G} = \sum_\Lambda (H_\lambda)_n \oplus S_n$ (external direct sum) where S_n is a maximal n-bounded summand of G. Now $n\hat{G} \cong nG$. By 1.9, $\hat{G} = G$. Therefore $G = \sum_\Lambda G_\lambda$ where $G_\lambda \cong (H_\lambda)_n$ for all $\lambda \in \Lambda$ except $\beta \in \Lambda$, and $G_\beta = (H_\beta)_n \oplus S_n$. Also $nG_\lambda \cong G'_\lambda$.

Now $nG = \sum_\Lambda nG_\lambda = \Sigma G'_\lambda$. Let ϕ be an automorphism of nG such that $\phi(nG_\lambda) = G'_\lambda$. By 1.8, we can extend ϕ to an automorphism of G, say ψ. Thus we have $G = \Sigma\psi(G_\lambda)$ such that $n\psi(G_\lambda) = \psi(nG_\lambda) = G'_\lambda$.

THEOREM 2.2. *Let G be a group. Let H be a pure subgroup of nG, n a positive integer. Then there exists a pure subgroup K of G such that* $nK = H$ *and* $K[p] = H[p]$ *for each prime p.*

We will first prove two lemmas.

LEMMA 2.3. *Let G be a p-group. Let H′ be a pure subgroup of* p^nG. *Then there exists a pure subgroup H of G such that* $p^nH = H'$ *and* $H[p] = H'[p]$.

Proof. Let $X = [y \in H' : H_{H'}(y) = 0]$. For each $y \in X$ let $x_y \in G$ such that $p^n x_y = y$. Let $\hat{H} = \{[x_y : y \in X], H'\}$. Now

$$p^n\hat{H} = \{[p^n x_y : y \in X], p^n H'\} \subset H' .$$

If $x \in H'$ then either $H_{H'}(x) = \infty$ or $H_{H'}(x) < \infty$. If $H_{H'}(x) = \infty$ then $x \in p^n H'$ and $x \in p^n \hat{H}$. If $H_{H'}(x) = k < \infty$ then there exists $z \in H'$ such that $p^k z = x$, and there exists $x_z \in X$ such that $p^n x_z = z$. Thus $p^{k+n} x_z = x$ and $x \in p^n \hat{H}$. Therefore $H' \subset p^n \hat{H}$, and $p^n \hat{H} = H'$. Now write $\hat{H} = S_n \oplus H_n$ according to Baer's decomposition. Let $H = \hat{H}_n$. Clearly $p^n H = H'$. Also $H[p] = H'[p]$ and H is pure in G. Now $H[p] = H'[p]$ since $H = H_n$ and thus $H[p] = (p^n H)[p] = H'[p]$. To see that H is pure in G suppose not; i.e., suppose that there exists $y \in H[p]$ such that $H_H(y) < H_G(y)$. Since $H[p] = H'[p]$ and $H' \subset H$ we may assume that $H_G(y) = k < \infty$. Now $H_{H'}(y) = k - n, (k > n)$. Thus $H_H(y) = H_{\hat{H}}(y) = k$, a contradiction. Hence H is pure.

LEMMA 2.4. *Let G be a torsion group. Let H be a pure subgroup of nG, n a positive integer. Then there exists a pure subgroup K of G such that $nK = H$ and $K[p] = H[p]$ for each prime p.*

Proof. Write $n = p_1^{k_1} \cdots p_m^{k_m}$, $p_1, \cdots, p_m$ distinct primes. Now by Theorem 1 in [9], $G = \sum_p G_p$, p a prime and G_p the p-primary component of G. Likewise $H = \sum_p H_p$ where H_p is a pure subgroup of nG_p. (Note that if $(p, p_i) = 1$ for $i = 1, \cdots, m$ then $nG_p = G_p$.) If $(p, p_i) = 1, i = 1, \cdots, m$, let $K_p = H_p$. If $(p, p_i) \neq 1$ for some $i = 1, \cdots, m$, let K_p be a pure subgroup of G_p such that $nK_p = H_p$ and $K_p[p] = H_p[p]$. Such a K_p exists by 2.3. Now define $K = \sum_p K_p$. Clearly K is pure, $K[p] = H[p]$ for all primes p and $nK = H$.

Now we are ready to prove Theorem 2.2.

Proof. If H is a torsion group we are done by 2.4. Thus suppose that H is not a torsion group. Let G_t and H_t be the torsion subgroups of G and H, respectively. Then H_t is pure in $nG_t = (nG)_t$, and by 2.4 there exists $\bar{K}_t$, a pure subgroup of G_t, such that $n\bar{K}_t = H_t$ and $\bar{K}_t[p] = H_t[p]$ for all primes p. Now let $U = [x \in H: x \text{ is torsion free}]$. Let $V = [x \in U: H_H^{p_i}(x) = 0 \text{ for } i = 1, \cdots, m]$. For each $x \in V$ let $y_x \in G$ such that $ny_x = x$. Let $W = [y_x: x \in V]$. Define $\hat{K} = \{\bar{K}_t, H, \{W\}\}$. Now $\hat{K} = S_n \oplus K$ where S_n is a maximal n-bounded summand of $\hat{K}$, and K is the desired group. To see this let K_t be the torsion subgroup of K. Now

$$\hat{K}_t = S_n \oplus K_t = S_n \oplus \bar{K}_t .$$

Thus $nK_t = n\bar{K}_t = H_t$ and $K_t[p] = H_t[p]$ for all primes p. Hence K_t is pure in G_t. Thus to check the purity of K, we need only check torsion free elements in K. Let $x \in K$ such that $o(x) = \infty$, and suppose that there exists $y \in G$ such that $p^r y = x$. (We need only check divisibility for a power of an arbitrary prime p by p. 76 in [5].) We will show that there exists $z' \in K$ such that $p^r z' = x$. Now there exist nonnegative integers $j_1 \leqq k_1, \cdots, j_m \leqq k_m$ such that $n_j = p_1^{j_1} \cdots p_m^{j_m}$ and n_j is the least positive integer such that $n_j x \in H$. We consider two cases. First suppose that $(p, p_i) = 1$ for all $i = 1, \cdots, m$. Then there exists $z \in K$ such that $p^r n_j z = n_j x$. Thus $p^r z - x = k_t \in K_t$ and

$$H^p_{G_t}(k_t) \geqq \min(H^p_G(p^k z), H^p_G(x)) \geqq k \ .$$

Thus since K_t is pure in G_t there exists $k'_t \in K_t$ such that $p^r k'_t = k_t$. Hence $p^r(z - k'_t) = x$ and $z' = z - k'_t \in K$. Next suppose that $(p, p_i) \neq 1$ for some $i = 1, \cdots, m$, say $i = i_0$. Then $H^p_G(n_j x) \geqq j_{i_0} + r$. If $x \notin H$ then $H^p_G(n_j x) \leqq k_{i_0}$ and hence there exists $z \in K$ such that $p^r n_j z = n^j x$. Thus $p^r z - x = k_t \in K_t$, and as before there exists $k'_t \in K_t$ such that $p^r(z - k'_t) = x$ and $z' = z = k'_t \in K$. If $x \in H$, then $H^p_G(x) = H^p_H(x) + k_{i_0} < r$, and hence again there exists $z' \in K$ such that $p^r z' = x$. Thus K is a pure subgroup of G with the desired properties.

It is easily seen that another proof of Theorem 2.1 can be obtained by using Theorem 2.2.

One might note at this point that if n is a positive integer and K is pure in G then nK is pure in nG. Also $K^1 = \bigcap_n n!\, K$ is pure in $G^1 = \bigcap_n n!\, G$ (see p. 452 in [7]). We have just shown that if H is pure in nG then there exists K in G such that K is pure in G and $nK = H$. The question then arises: If H is a pure subgroup of G^1, does there exist a pure subgroup K of G such that $K^1 = H$? The answer to this question is in the affirmative as will be seen in the next theorem. First we need a lemma.

LEMMA 2.5. *Let G be a group and K a pure subgroup of G. Let E_G be the divisible hull of G and $E_K \subset E_G$ the divisible hull of K. Then $E_K \cap G = K$.*

Proof. Clearly $E_K \cap G \supset K$. Let $0 \neq x \in E_K \cap G$. Then $\{x\} \cap K \neq 0$ (see Lemma 20.3, p. 66 in [5]). Let n be the smallest integer such that $nx \in K$. Now since K is pure, there exists $y \in K$ such that $ny = nx$. Observe that $y \in E_K$ and thus $x - y \in E_K$. We shall show that $\{x - y\} \cap K = 0$, thus $\{x - y\} = 0$ and thus $x = y \in K$. Suppose there exists m such that $0 \neq m(x - y) \in K$. Then clearly $(m, n) = i < n$ and there exist integers s and t such that $ms + nt = i$. Now $mx \in K$ since $mx - my, my \in K$. Also $msx + nsx = ix$, and since $msx, nsx \in K$, we have $ix \in K$. But $i < n$, and this contradicts the fact that n was the smallest integer such that $nx \in K$. Thus $\{x - y\} \cap K = 0$, and we have $x \in K$. Therefore $E_K \cap G = K$.

In [6] a high subgroup is defined to be a subgroup H of a group G

maximal with respect to disjointness from G^1.

THEOREM 2.6. *Let G be a group with $G^1 = \bigcap_n n!\, G \neq 0$. Let H be a high subgroup of G. Let K be a pure subgroup of G^1. Then there exists a pure subgroup M of G such that $M^1 = K$ and H is high in M.*

Proof. Let ϕ be the natural homomorphism from G to G/H. Then $\phi \mid G^1$ is an isomorphism. Thus $\phi(K)$ is pure in $\phi(G^1)$. Let E be the divisible hull of $\phi(G^1)$ in G/H (note that $E = G/H$), and let D be the divisible hull of $\phi(K)$ contained in E. Then $D \cap \phi(G^1) = \phi(K)$ by 2.5. Now define $M = \phi^{-1}(D)$. Observe that H is pure in M, $M/H = D$, and $M/H \subset G/H$ as a pure subgroup. Thus by Lemma 2 in [9], M is pure in G. Now $G^1 \supset M^1 \supset K$ by construction, and $M^1 \subset K$ since $\phi(M^1) \subset D \cap \phi(G^1) = \phi(K)$. Hence $M^1 = K$. Also H is high in M since $H \cap M^1 = 0$ and $\{H, x\} \cap M^1 \neq 0$ for any $x \in M \backslash H$. The latter statement is true since if there exists an $x \in M \backslash H$ such that $\{H, x\} \cap M^1 = 0$, this would imply that $\{H, x\} \cap G^1 = 0$ which would contradict the hypothesis that H is high in G.

We will now show that several properties are possessed by a group G if and only if they are possessed by p^nG. We will only consider primary groups.

THEOREM 2.7. *Let G be a reduced p-group. Then G is closed if and only if p^nG is closed.*

Proof. Suppose G is closed. Then $G = \bar{B}$ for some basic subgroup B of G. Now $B = \sum_{n=1}^{\infty} B_n$, $p^n \Pi B_n = \Pi p^n B$, and thus $p^n(\bar{B}) = \overline{p^n B}$. Therefore p^nG is closed.

Suppose p^nG is closed. If B is a basic subgroup of G then p^nB is a basic subgroup of p^nG. Let $\bar{B}$ be a closed subgroup with basic subgroup B and identify G with its pure subgroup between B and $\bar{B}$ (see p. 112 in [5]). Now $p^n\bar{B} \cong p^nG$ since p^nG and $p^n\bar{B}$ are closed and have the same basic subgroup B. Also $(\bar{B})_n \cong G_n$ by Theorem 1.2. Thus $G \cong \bar{B}$ since they contain the same basic subgroup. Therefore G is closed.

DEFINITION 2.8. A Σ-group is any group G all of whose high subgroups are direct sums of cyclic groups. (see [6]).

THEOREM 2.9. *Let G be a reduced p-group. Then G is a Σ-group if and only if p^nG is a Σ-group.*

Proof. Theorem 11, p. 1370 in [8], states that "Every subgroup L of a torsion Σ-group G with $L^1 = L \cap G^1$ is a Σ-group". Thus if G is a Σ-group then p^nG is a Σ-group since $(p^nG)^1 = G^1$.

If p^nG is a Σ-group and H is high in G then p^nH is high in p^nG (see p. 1368 in [8]). Thus p^nH is a direct sum of cyclic groups, and hence H is a direct sum of cyclic groups. Therefore G is a Σ-group.

DEFINITION 2.10. A group G is a direct sum of countable groups if and only if $G = \sum_{\lambda \in \Lambda} G_\lambda$ such that $|G_\lambda| \leqq \aleph_0$ for all $\lambda \in \Lambda$.

THEOREM 2.11. *A group G is a direct sum of countable groups if and only if p^nG is a direct sum of countable groups.*

Proof. If G is a direct sum of countable groups then clearly p^nG is.

Now suppose that p^nG is a direct sum of countable groups. Write $G = S_n \oplus G_n$ according to Baer's decomposition. Then $p^nG = p^nG_n = \sum_{\lambda \in \Lambda} G'_\lambda$ such that $|G'_\lambda| \leqq \aleph_0$. By Theorem 2.1, we may write $G_n = \sum_{\lambda \in \Lambda} G_\lambda$ such that $p^nG_\lambda = G'_\lambda$. Now $G_\lambda[p] = G'[p]$ and thus $r(G_\lambda) = r(G_\lambda[p]) = r(G'_\lambda[p]) = r(G'_\lambda)$. Also $|G_\lambda| = r(G_\lambda)\cdot\aleph_0$ and since $r(G_\lambda) = r(G'_\lambda) \leqq \aleph_0$ we have that $|G_\lambda| \leqq \aleph_0$ (see pp. 32–33 in [5]). Thus G_n is a direct sum of countable groups. Since S_n is bounded, S_n is a direct sum of cyclic groups and hence a direct sum of countable groups.

THEOREM 2.12. *A p-group G is a direct sum of closed groups if and only if p^nG is a direct sum of closed groups.*

Proof. Suppose that G is a direct sum of closed groups. Then $G = \sum_{\lambda \in \Lambda} G_\lambda$ such that G_λ is closed for each $\lambda \in \Lambda$ and $p^nG = \sum_{\lambda \in \Lambda} p^nG_\lambda$. By 2.7, p^nG_λ is closed for each $\lambda \in \Lambda$. Thus p^nG is a direct sum of closed groups.

Now suppose that p^nG is a direct sum of closed groups. Then $p^nG = \sum_{\lambda \in \Lambda} G'_\lambda$ such that G'_λ is closed for all $\lambda \in \Lambda$. By 2.1, $G = \sum_{\lambda \in \Lambda} G_\lambda$ such that $p^nG_\lambda = G'_\lambda$. Thus by 2.7, G is closed for each $\lambda \in \Lambda$.

DEFINITION 2.13. A group G is essentially indecomposable if $G = A \oplus B$ implies that A or B is finite.

THEOREM 2.14. *Suppose that the first n Ulm invariants of a p-group G are finite, n a positive integer. Then G is essentially indecomposable if and only if p^nG is essentially indecomposable.*

Proof. The proof follows immediately from 2.1.

DEFINITION 4.14. A group G is "Fuchs 5" if and only if every infinite (pure) subgroup of G is contained in a summand of the same cardinality.

THEOREM 2.15. *A p-group G is a Fuchs* 5 *group if and only if p^nG is a Fuchs* 5 *group.*

Proof. First assume p^nG is Fuchs 5. Let H be an infinite unbounded pure subgroup of G. Then p^nH is a pure subgroup of p^nG. Since p^nG is Fuchs 5, p^nH is contained in a subgroup K' of p^nG such that K' is a summand of p^nG and $|p^nH| = |K'|$. Thus $p^nG = K' \oplus L'$. Now $G_n = K \oplus L$ (where $G = S_n^G \oplus G_n$ according to Baer's decomposition) such that $p^nK = K'$ and $p^nL = L'$ (by 2.1). Write $H = S_n^H \oplus H_n$ according to Baer's decomposition. Then p^nH_n is a pure subgroup of p^nK which is a pure subgroup of p^nG. Let B_{H_n} be a basic subgroup of H_n. Then $p^nB_{H_n}$ is a basic subgroup of p^nH_n and can be extended to a basic subgroup of p^nK, say $p^nB = p^nB' \oplus p^nB_{H_n}$ where $p^nB' = \Sigma\{x_i\}$. For each x_i there exists $y_i \in K$ such that $p^ny_i = x_i$ and $H_K(y_i) = 0$. Define $B' = \Sigma\{y_i\}$. Define groups $B = B' \oplus B_{H_n}$ and $M = \{B, p^nK\}$. Notice that B is a direct sum of cyclic groups, B is pure in M and $M/B \cong (B + p^nK)/B \cong p^nK/(B \cap p^nK) \cong p^nK/p^nB$ a divisible group since p^nB is a basic subgroup of p^nK. Hence B is a basic subgroup of M. Now $M \cong K$, $H_n \subset M$, and $G = S_n^G \oplus M \oplus L$. To see that $M \cong K$ notice that p^nB is a basic subgroup of p^nK and thus $p^nM \cong \{p^nB, p^{2n}K\} = p^nK$. By Theorem 1.2, if $p^nM \cong p^nK$ then $M_n \cong K_n$ (where $M = S_n^M \oplus M_n$ and $K = S_n^K \oplus K_n$ according to Baer's decomposition), and since $M = M_n$ and $K = K_n$, $M \cong K$. ($M = M_n$ since B is a basic subgroup of M and B is isomorphic to a basic subgroup of K.) Also $H_n \subset M$ since $H_n = \{B_{H_n}, p^nH_n\}$, $B_{H_n} \subset B$ and $p^nH_n \subset p^nK$. To see that $G = S_n^G \oplus M \oplus L$ we first observe that $M[p] = (p^nM)[p] = (p^nK)[p] = K[p]$ and thus $M \cap L = 0$. Thus $M \oplus L$ is a direct sum and hence $S_n^G \oplus (M \oplus L)$ is a direct sum since $M \oplus L \cong K \oplus L$. Clearly $S_n^G \oplus M \oplus L \subset G$. Now $G_n = K \oplus L = \{B_K, p^nK\} \oplus L$ where $B_K = B' \oplus B''$ such that $B'' = \Sigma\{w_i\}$ and $B_{H_n} = \Sigma\{z_i\}$ with $p^nw_i = p^nz_i$ (i.e., $p^nB'' = p^nB_{H_n}$). Thus to show that $G \subset S_n^G \oplus M \oplus L$, it suffices to show that each $w_i \in S_n^G \oplus M \oplus L$. Now $p^nw_i = p^nz_i$ and thus $w_i - z_i = s + k + t \in S_n^G \oplus K \oplus L$ with $s \in S_n^G$, $k \in K$, $t \in L$, and $o(k), o(t) \leqq p^n$. But B'' may as well have been chosen such that its ith generator was $w_i - k$, and thus we may assume that $w_i - z_i = t + s \in L \oplus S_n^G$. Hence $w_i = z_i + t + s \in S_n^G \oplus M \oplus L$. Therefore $G \subset S_n^G \oplus M \oplus L$ and $G = S_n^G \oplus M \oplus L$.

Now $|p^nH_n| = |p^nM|$, $(p^nH)[p] = H_n[p]$ and $(p^nM)[p] = M[p]$. Thus $|H_n| = |M|$. Each G_n is an absolute summand of G and we may write $G = S_n^G \oplus G_n$ such that S_n^H is a summand of S_n^G. Thus $H = S_n^H \oplus H_n \subset S_n^H \oplus M$, a summand of G and clearly $|S_n^H \oplus M| = |H|$. Therefore G is Fuchs 5.

Next assume G is Fuchs 5. Let H be a subgroup of p^nG. Then H is a subgroup of G and is contained in a subgroup K of G such that K is a summand of G and $|H| = |K|$. Thus $G = K \oplus L$ and $p^nG = p^nK \oplus p^nL$. Now $H \subset p^nG$ and hence $H \subset p^nG \cap K = p^nK$ by the purity of K in G. Thus $|H| \leqq |p^nK| \leqq |K| = |H|$. Therefore p^nG is Fuchs 5.

DEFINITION 2.16. A group G is a Crawley group if G contains no proper isomorphic subgroups.

The existence of such groups has been shown by Peter Crawley in [4]. For he has constructed a group C between $B = \sum_{i=1}^{\infty}(Cp^i)$ and $\bar{B} = T(\prod_{i=1}^{\infty} C(p^i))$ (the torsion subgroup of the complete direct sum of the $C(p^i)$ for $i = 1, \cdots$) which has no proper isomorphic subgroups. This group can be chosen such that $r(\bar{B}/C) = 1$. This fact was first observed by R. A. Beaumont and R. S. Pierce in [3]. This group is also essentially indecomposable. The fact that if $r(\bar{B}/C) = 1$ then C is essentially indecomposable was first proved (as follows) by John M. Irwin: Suppose $C = H \oplus K$. Then $B = \bar{B}_1 \oplus \bar{B}_2$ where $H \subset \bar{B}_1$ and $K \subset \bar{B}_2$. Thus either $H = \bar{B}_1$ or $K = \bar{B}_2$ since $r(\bar{B}/C) = 1$. Suppose $H = \bar{B}_1$. Then there exists a copy of $\bar{B} \subset \bar{B}_1$ and thus a copy of C in $\bar{B}_1$.

It seems that this class of groups will be quite important in the study of p-groups.

THEOREM 2.17. *If C is a Crawley group then p^nC is a Crawley group. If p^nC is a Crawley group and C is essentially indecomposable then C is a Crawley group.*

Proof. Suppose first that C is a Crawley group. Suppose that there exists a group $L \subsetneqq p^nC$ such that $L \cong p^nC$. Let $U = [x \in L : H_L(x) = 0]$. Then for each $x \in U$ there exists $y \in C$ such that $p^ny = x$. For each x let $y_x \in C$ such that $p^ny_x = x$. Let $V = [y_x : x \in U]$. Define $C_L = \{V, L\}$. Now $p^nC_L = L$. Let $(C_L)_n$ be a summand of C_L according to Baer's decomposition and let $G' = S_n \oplus (C_L)_n$ where $B = S_n \oplus B_n^*, B$ a basic subgroup of C and S_n a maximal p^n-bounded summand of B. Now $C' \subsetneqq C$ and $p^nC' = L \cong p^nC$. Thus by 1.9, $C' \cong C$. But this contradicts the fact that C is Crawley.

Suppose p^nC is Crawley. Then if C is essentially indecomposable then C is Crawley. For if there exists $L \subsetneqq C$ such that $C \cong L$, then $p^nC \cong p^nL \subsetneqq p^nC$ which would contradict the fact that p^nC is Crawley.

COROLLARY 2.18. *There are at least $\aleph_0$ nonisomorphic Crawley* *groups C between B and $\bar{B}$.*

Proof. Observe that $\bar{B} \cong p^n\bar{B}$ and $B \cong p^nB$ for all n. Let ϕ be an isomorphism between $\bar{B}$ and $p^n\bar{B}$. Let C be a Crawley subgroup of $\bar{B}$. Then p^nC is a Crawley subgroup of $p^n\bar{B}$, and p^nC is not isomorphic to C. Thus $\phi(p^nC)$ is a Crawley subgroup of $\bar{B}$ which is not isomorphic to C.

3. **Quasi-isomorphic p-groups.** In a recent paper by R. A. Beaumont and R. S. Pierce (see [2]), it was shown that two countable primary groups are quasi-isomorphic if and only if their basic subgroups are quasi-isomorphic and their subgroup of elements of infinite height are isomorphic. Also if two primary groups are quasi-isomorphic their basic subgroups are quasi-isomorphic. In [3] they have considered quasi-isomorphism in relation to direct sums of cyclic groups. We will extend these results to closed p-groups.

In considering quasi-isomorphism, it is of interest to investigate what properties of primary Abelian groups are preserved under quasi-isomorphism. It will be shown that if G and H are quasi-isomorphic primary groups, then the statement that G has property P if and only if H has property P is equivalent to

(1) property P is preserved under isomorphism,

(2) G has property P if and only if p^nG has property P and

(3) groups between G and p^nG have property P if G does. This reduces this problem to considering G, p^nG, and groups between G and p^nG.

DEFINITION 3.2. Let G and H be p-groups. Then $G \doteq H$ (quasi-isomorphic) if there are subgroups $S \subset G$, $T \subset H$ and positive integers m and n such that $p^mG \subset S$, $p^nH \subset T$, and $S \cong T$.

The following theorem (among other things) shows that if two p-groups are quasi-isomorphic, then their subgroups of elements of infinite height are isomorphic.

THEOREM 3.2. *Let G and H be p-groups. If G and H are quasi-isomorphic, then G/G^1 is quasi-isomorphic to H/H^1 and $G^1 \cong H^1$.*

Proof. Now G and H quasi-isomorphic implies that for some positive integers m and n there exists subgroups S and T of G and H, respectively, such that $G \supset S \supset p^mG$, $H \supset T \supset p^nH$ and $S \cong T$. Now clearly $G^1 \cong S^1$, $H^1 \cong T^1$ and $S^1 \cong T^1$. Thus $G^1 \cong H^1$. Now $S/S^1 \cong T/T^1$. Thus $p^m(G/G^1) \cong p^mG/G^1 \subset S/G^1 \subset G/G^1$ and $p^n(H/H^1) \cong p^nH/H^1 \subset T/H^1 \subset H/H^1$. Hence H/H^1 is quasi-isomorphic to G/G^1.

The converse of the above theorem is not true as can be seen from an example on pp. 134–135 in [5].

PROPOSITION 3.3. Let p be a property of p-groups. Then the following statements are equivalent:

(1) Let G and H be quasi-isomorphic p-groups. Then G has P if and only if H has P.

(2) Property P is such that for any p-group L:

(a) L has property P if and only if p^nL has property P for all positive integers n.

(b) Whenever L has property P and S is a subgroup of L such that $L \supset S \supset p^nL$ then S has property P, and

(c) Property P is preserved under isomorphism.

Proof. First 2 implies 1: If G and H are quasi-isomorphic then for some positive integers m and n there exist subgroups S and T of G and H, respectively, such that $G \supset S \supset p^mG$, $H \supset T \supset p^nH$, and $S \cong T$. If G has property P then S has property P by (b). Thus T has property P by (c), and since $p^nS \subset p^nH \subset S$, p^nH has property P by (b), and H has property P by (a). By symmetry we have 2 implies 1.

Next 1 implies 2: Since G is quasi-isomorphic to p^nG, we have (a). Also G is quasi-isomorphic to any subgroup S such that $G \supset S \supset p^nG$, thus we have (b). Clearly (c) holds.

Using the above proposition, we can show that if G and H are quasi-isomorphic p-groups, then G is a direct sum of cyclic groups if and only if H is a direct sum of cyclic groups, G is a closed p-group if and only if H is a closed p-group, and G is a Σ-group if and only if H is a Σ-group.

THEOREM 3.4. *Let G and H be quasi-isomorphic p-groups. Then G is a direct sum of cyclic groups if and only if H is a direct sum of cyclic groups.*

Proof. Now G is a direct sum of cyclic groups if and only if p^nG is, and any group between G and p^nG is a direct sum of cyclic groups (see p. 46 in [5]). Then by Proposition 3.3 the theorem is proved.

LEMMA 3.5. *Let G be a reduced p-group. If G is a Σ-group and S is a subgroup of G such that $G \supset S \supset p^nG$, then S is a Σ-group.*

Proof. Now $S^1 = G^1$. Thus, apply the theorem stated in the proof of Theorem 2.9.

THEOREM 3.6. *Let G and H be quasi-isomorphic p-groups. Then G is a Σ-group if and only if H is a Σ-group.*

Proof. Apply Theorem 2.9, Lemma 3.5 and Proposition 3.3.

LEMMA 3.7. *Let G be a closed p-group and $G \supset S \supset p^n G$. Then S is closed.*

Proof. Let $g_1, \cdots, g_n, \cdots$ be a Cauchy sequence in S. Then $g_1, \cdots, g_n, \cdots$ is a Cauchy sequence in G. Since G is closed this sequence converges to some $g \in G$. Also $g - g_m \in p^n G \subset S$ for $m > n$, and since $g_m \in S$ we have $g \in S$.

THEOREM 3.8. *Let G and H be quasi-isomorphic p-groups. Then G is closed if and only if H is closed.*

Proof. An application of Proposition 3.3, Theorem 2.7 and Lemma 3.7 proves the theorem.

An important problem along these lines that seems to be a very difficult one is the following: If G and H are quasi-isomorphic p-groups, then is it true that G is a direct sum of countable groups if and only if H is a direct sum of countable groups? By Proposition 3.3 and Theorem 2.11 this problem is reduced to the following: If G is a direct sum of countable p-groups and S is a subgroup of G such that $p^n G \subset S$, is S a direct sum of countable groups? We are able to answer two special cases of this question in the following two theorems.

THEOREM 3.9. *Let G be a direct sum of countable p-groups. Let K be a subgroup of G such that $G \supset K \supset p^n G$ and $K/p^n G$ is countable. Then K is a direct sum of countable groups.*

Proof. Write $K/p^n G = \sum_{j \in I} \{k_j + p^n G\}$ and $p^n G = \sum_{\lambda \in \Lambda} p^n G_\lambda$ (where $G = \Sigma G_\lambda$) such that $|G_\lambda| = \aleph_0$. Then $K = \{\{k_j\}_{j \in I}, \{p^n G\}_{\lambda \in \Lambda}\} = \{\{k_j\}_{j \in I}, p^n G\}$. Now $k_j \in G$ and hence $k_j = \sum_{i=1}^m g_{\lambda_i}$ such that $g_{\lambda_i} \in G_{\lambda_i}$. Let $\Lambda' = [\lambda \in \Lambda$: for some $g_\lambda \in G_\lambda$, g_λ is a representative in some $k_j]$. Let $\Lambda'' = \Lambda \backslash \Lambda'$. Then $K = \{\{k_j\}_{j \in I}, \{p^n G_\lambda\}_{\lambda \in \Lambda'}\} \oplus \sum_{\lambda \in \Lambda''} p^n G_\lambda$. Thus K is a direct sum of countable groups.

THEOREM 3.10. *Let G be a direct sum of countable p-groups, and let K be a subgroup of countable index. Then K is a direct sum of countable p-groups.*

Proof. Write $G = \sum_{\lambda \in \Lambda} G_\lambda$. Now $G_\lambda \subset K$ for all but a countable number of $\lambda \in \Lambda$ since K is of countable index in G. Let $\Lambda' = [\lambda \in \Lambda: G_\lambda \subset K]$. Let $K_1 = \sum_{\lambda \in \Lambda'} G_\lambda$ and $K_2 = \{[k \in K: k \in \sum_{\lambda \in \Lambda \backslash \Lambda'} G_\lambda]\}$.

Then $K = K_1 \oplus K_2$ such that K_2 is countable and K_1 is a direct sum of countable groups.

The following theorem extends Beaumont and Pierce's results in [1] and [2] to closed p-groups.

THEOREM 3.11. *Let $\bar{B}$ and $\bar{C}$ be closed p-groups with basic subgroups B and C, respectively. Then $\bar{B} \doteq \bar{C}$ if and only if $B \doteq C$.*

Proof. If $\bar{B} \doteq \bar{C}$ then $B \doteq C$ by [2]. Thus suppose that $B \doteq C$. Then there exist subgroups S and T of B and C, respectively, such that $S \supset p^n B$, $T \supset p^n C$ and $S \cong T$. Thus $\bar{B} \supset \bar{S} \supset p^n \bar{B}$, $\bar{C} \supset \bar{T} \supset p^n \bar{C}$ and $\bar{S} \cong \bar{T}$ since closed subgroups are completely determined by their basic subgroups (see p. 115 in [5]). Thus $\bar{B} \doteq \bar{C}$.

4. Special cases of quasi-isomorphism.

In this section we will impose some restrictions on the definition of quasi-isomorphism, and in some cases we will be able to determine by just how much two quasi-isomorphic groups with these restrictions differ.

DEFINITION 4.1. $G \stackrel{\frown}{\cong} H$ (S.B. quasi-isomorphic, i.e., in the sense of Schroeder-Bernstein) if there exist subgroups $S \subset G$, $T \subset H$, and positive integers m and n such that $G \cong T$, $H \cong S$, $p^m G \subset S$, and $p^n H \subset T$.

DEFINITION 4.2. Two p-groups G and H are purely quasi-isomorphic if for some positive integers m and n there exist pure subgroups S and T of G and H, respectively, such that $G \supset S \supset p^m G$, $H \supset T \supset p^n H$ and $S \cong T$.

DEFINITION 4.3. $G \doteq \oplus H$ (summand quasi-isomorphic) if there are subgroups $S \subset G$, $T \subset H$, and positive integers m and n such that $S \cong T$, $G = S \oplus G_1$, $H = T \oplus H_1$, $p^m G_1 = 0$, and $p^n H_1 = 0$.

DEFINITION 4.4. $G \stackrel{..}{\cong} H$ (strongly quasi-isomorphic) if there are subgroups $S \subset G$, $T \subset H$ such that $S \cong T$, $[G : S] < \infty$, and $[H : T] < \infty$.

DEFINITION 4.5. $G \stackrel{..}{\cong} \oplus H$ if there exists subgroups $S \subset G$, $T \subset H$ such that, $G = S \oplus G_1$, $H = T \oplus H_1$ and G_1 and H_1 are finite.

DEFINITION 4.6. Two groups are strongly S.B. quasi-isomorphic if there exist subgroups S and T of G and H, respectively, such that $[G : S] < \infty$, $[H : T] < \infty$, $S \cong H$ and $T \cong G$.

DEFINITION 4.7. Two p-groups G and H are purely S.B. quasi-isomorphic if for some positive integers m and n there exists pure subgroups S and T of G and H, respectively, such that $G \supset S \supset p^m G$, $H \supset T \supset p^n H$, $G \cong T$ and $H \cong S$.

Each of the above definitions yields an equivalence relation. Definitions 4.2 and 4.3 are equivalent. It will be shown that Definitions 4.4 and 4.5 are equivalent, that Definition 4.6 is equivalent to $G_n \cong H_n$ for some positive integer n according to Baer's decomposition and that Definition 4.7 implies the groups are isomorphic. With this it will be clear that each relation (excluding Definition 4.1) is no weaker than the preceding one. Examples will be given to show that Definition 4.1 and 4.2 are stronger than Definition 3.1, and except for equivalent definitions, each relation from Definition 4.2 to Definition 4.7 is stronger than the preceding ones.

PROPOSITION 4.8. $G \doteqdot H$ if and only if $G \doteqdot \oplus H$.

Proof. That $G \doteqdot \oplus H$ implies $G \doteqdot H$ is clear. Thus suppose $G \doteqdot H$. Let $S_1 \subset G$, $T_1 \subset H$ such that $S_1 \cong T_1$ and G/S_1 and H/T_1 are finite. A lemma of R. S. Pierce says: Let S_1 be a subgroup of a reduced p-group G of unbounded order such that the index $[G: S_1] < \infty$. Then there exists $S_2 \subset S_1$ such that $[G: S_2] < \infty$ and $G = S_2 \oplus L$. Thus there exist subgroups $S_2 \subset S_1$, and $T_2 \subset T_1$ such that S_2 is a direct summand of finite index in G and T_2 is a direct summand of finite index in H. Let ϕ be an isomorphism of S_1 onto T_1. Then $S_2 \cap \phi^{-1}(T_2)$ has finite index in S_1. Again by the above lemma, there exists a subgroup S of $S_2 \cap \phi^{-1}(T_2)$ such that $S_1 = S \oplus L$ where L is finite. Now $S_2 = S \oplus (S_2 \cap L)$, where $S_2 \cap L$ is finite. Thus $G = S \oplus (S_2 \cap L) \oplus M$, where $(S_2 \cap L) \oplus M$ is finite. Let $T = \phi(S)$. Then $T_1 = T \oplus \phi(L)$, where $T \subset T_2$. Consequently $T_2 = T \oplus (\phi(L) \cap T_2)$, and $H = T \oplus (\phi(L) \cap T_2) \oplus N$, where $(\phi(L) \cap T_2) \oplus N$ is finite. Since $S \cong T$, it follows that $G \doteqdot \oplus H$.

The following theorem shows that if two groups are strongly S.B. quasi-isomorphic then they only differ (up to isomorphism) by summands of bounded order.

THEOREM 4.9. *Let G and H be strongly S.B. quasi-isomorphic p-groups. Then there exists a positive integer n such that $G_n \cong H_n$ according to Baer's decomposition.*

Proof. Now G and H being strongly S.B. quasi-isomorphic implies that there exist subgroups S and T of G and H, respectively, such that $[G: S] < \infty$, $[H: T] < \infty$, $G \cong T$ and $H \cong S$. By the lemma stated in the proof of 4.8, there exist subgroups $S_1 \subset S$ and $T_1 \subset T$ such that S_1 and T_1 are pure in G and H, respectively, $[G: S_1] < \infty$ and $[H: T_1] < \infty$. Thus $G = S_1 \oplus \tilde{S}_1$ where $\tilde{S}_1 \cong G/S_1$ and S_1 is finite, and $H = T_1 \oplus \tilde{T}_1$

where $\tilde{T}_1 \cong H/T_1$ and T_1 is finite. Choose n such that $p^n(\tilde{S}_1) = p^n(\tilde{T}_1) = 0$. Now $p^nG = p^nS_1$ and $p^nH = p^nT_1$. Thus by Theorem 1.2, $G_n \cong (S_1)_n$ and $H_n \cong (T_1)_n$. Also $p^nS = p^nS_1$ and $p^nT = p^nT_1$, thus $S_n \cong (S_1)_n$ and $T_n \cong (T_1)_n$. Hence by the hypothesis we have that $G_n \cong T_n \cong (T_1)_n \cong H_n$.

COROLLARY 4.10. *If G and H (as in Theorem 4.13) have isomorphic basic subgroups, then $G \cong H$.*

In the next theorem we will show that if G and H satisfy Definition 4.7, then $G \cong H$. First we need a lemma.

LEMMA 4.11. *Let G and H be p-groups such that $G \cong H \oplus B_1$, $H \cong G \oplus B_2$ where B_1 and B_2 are groups of bounded order. Then $G \cong H$.*

Proof. Now $G \cong G \oplus B_1 \oplus B_2$ where $B_1 \oplus B_2$ is of bounded order. Thus if we write $G = S_n \oplus G_n$ according to Baer's decomposition, where $p^n(B_1 \oplus B_2) = 0$, it is clear that $S_n \cong S_n \oplus B_1 \oplus B_2 \cong S_n \oplus B_2$ by Ulm's Theorem. Thus $H \cong G \oplus B_2 \cong G_n \oplus (S_n \oplus B_2) \cong G_n \oplus S_n \cong G$.

THEOREM 4.12. *If G and H are purely S.B. quasi-isomorphic p-groups, then $G \cong H$.*

Proof. If S is a pure subgroup of G such that G/S is bounded, then S is a summand of G by Theorem 5 in [9]. Thus $G = S \oplus B_1$ where B_1 is of bounded order. Also $H = T \oplus B_2$ where B_2 is of bounded order. Thus $G \cong H \oplus B_1$ and $H \cong G \oplus B_2$ and by 4.11, $G \cong H$.

Now to see that Definitions 4.1 and 4.2 are stronger than Definition 3.1, let $G = C(p^2)$ and $H = C(p)$. Then $G \doteq H$ but G and H do not satisfy either Definitions 4.1 or 4.2.

To see that Definition 4.4 is stronger than Definition 4.3, let $G = \sum_{i=1}^{\infty} C(p^i)$ and $H = G \oplus \sum_{\aleph_0} C(p)$. Now clearly $G \doteq \oplus H$, and by Ulm's Theorem and 4.8, it is clear that G and H are not strongly quasi-isomorphic.

Next we show that Definition 4.6 is stronger than Definition 4.5. Let $G = C(p) \oplus C(p)$ and $H = C(p)$. Now clearly $G \ddot{\cong} \oplus H$, but G and H do not satisfy Definition 4.6.

Finally to see that Definition 4.7 is stronger than Definition 4.6, let $G = \sum_{\aleph_0} C(p^2)$ and $H = G \oplus C(p)$. Then clearly G and H satisfy Definition 4.6 but not Definition 4.7.

5. Some related problems. Given a p-group G and a subgroup H containing p^nG, n a positive integer, the question arises: If B is a basic subgroup of G, does there exist a basic subgroup B' of H such that $B \supset B' \supset p^n B$? We will give an answer to a very special case of this question in a corollary to the following theorem.

THEOREM 5.1. *Let G be a p-group and H a subgroup of G containing $p^n G$. If N is a high subgroup of G, then there exists a high subgroup M of H such that $N \supset M \supset p^n N$.*

Proof. Let N be a high subgroup of G. Now $p^n N$ is a high subgroup of $p^n G$ (see p. 1380 in [6]), $p^n N \subset H$ and $p^n N \cap H^1 = 0$. Thus let M be the maximal subgroup of H such that $N \supset M \supset p^n N$ and $M \cap H^1 = 0$. Then M is high in H. To see this suppose not, i.e., suppose there exists $x \in H, x \notin M$, such that $\{x, M\} \cap H^1 = 0$. Since N is high in $G, \{x, N\} \cap H^1 \neq 0$. There exists $y \in N$ such that $p^k x + y = h_1 \in H^1 (h_1 \neq 0)$. Now $p^k x, h_1 \in H$ and hence $y \in H$. Since $y \in H \cap N, y \in M$. Hence $\{x, M\} \cap H^1 \neq 0$, a contradiction.

COROLLARY 5.2. *Let G be a p-group and H a subgroup of G containing $p^n G$. If G is a Σ-group and B is a basic subgroup of G which is also a high subgroup of G, then there exists a basic subgroup B' of H such that $B \supset B' \supset p^n B$. Here B' is a high subgroup of H.*

The general question seems to be a little more elusive. The following two theorems are results related to this problem.

THEOREM 5.3. *Let G be a reduced p-group and H a subgroup of G such that $H \supset p^n G$. Let B' be a basic subgroup of H. Then there exists a basic subgroup B of G such that $B \supset B'$.*

Proof. We may write $B' = \bigcup_{m=1}^{\infty} S_m$, $S_m = \sum_{i=1}^{m} B_i$, and $B' = \sum_{i=1}^{\infty} B_i$ such that $B_i = \Sigma C(p^i)$. Now $H = S_m \oplus H_m$ where $H_m = \{\sum_{i=m+1}^{\infty} B_i, p^m H\}$. Thus $S_m \cap H_m = 0$. Since $H_m[p] = (p^m H)[p]$, $S_m \cap p^m H = 0$. Also since $p^n G \subset H$, $S_m \cap p^{n+m} G = 0$. Thus S_m is contained in a maximal p^{n+m} bounded summand of G, and therefore the height in G of any element of S_m is bounded by $n + m$. By Kovács' Theorem (p. 99 in [5]), B' can be extended to a basic subgroup B of G. Hence $B' \subset B$ a basic subgroup of G.

THEOREM 5.4. *Let G be a reduced p-group. Write $G = S_n \oplus G_n$ according to Baer's decomposition. Let T be a subgroup of G_n such that $T \supset p^n G$. Then there exists a basic subgroup B_T of T such that $B_T = \sum_{i=n+1}^{\infty} L_i$ with $H_{G_n}(x) = i - 1$ for all $x \in L_i[p]$.*

Proof. To prove the theorem we will construct a basic subgroup B_T of T such that $B_T = \bigcup_{i=n+1}^{\infty} S_i$ where if $x \in S_{i+1}[p]$ then $H_{G_n}(x) \leqq i$

and $S_{i+1} = S_i \oplus L_{i+1}$ such that if $x \in L_{i+1}[p]$, $H_{G_n}(x) = i$. Note that B_T will have the desired properties.

Let $\mathscr{S}_{n+1} = [L: L$ is a summand of T and if $x \in L[p]$, $H_G(x) = n]$. If $\mathscr{S}_{n+1} = 0$ let $S_{n+1} = 0$ and if $\mathscr{S}_{n+1} \neq 0$ let S_{n+1} be a maximal element of $\mathscr{S}_{n+1}$. (Clearly such a maximal element exists by Zorn's Lemma.) Assuming that S_i has been defined, define S_{i+1} as follows: Let $\mathscr{S}_{i+1} = [L: L$ is a summand of T; $S_i \subset L$; if $x \in L$, $H_{G_n}(x) \leqq i$; and $L = S_i \oplus \hat{L}$ such that if $y \in \hat{L}[p]$ then $H_{G_n}(y) = i]$. Let S_{i+1} be a maximal element of $\mathscr{S}_{i+1}$ which we will now show exists. Clearly some $\mathscr{S}_i \neq 0$. Thus partially order $\mathscr{S}_{i+1}$ by set inclusion and let $L_1 \subset L_2 \cdots$ be a chain in $\mathscr{S}_{i+1}$. Put $L = \bigcup_{j=1}^{\infty} L_j$. Then L is pure in T since each L_j is pure in T. Also if $x \in L$, then $x \in L_j$ for some j which implies that $H_{G_n}(x) \leqq i$. Hence L is of bounded order (with order bound $\leqq p^{i+1}$) and thus L is a summand of T. Clearly $L \supset S_i$. Note that L_j is a summand of L_{j+1} for all $j = 1, 2, \cdots$. Now $L_j = S_{n+1} \oplus \hat{L}_j$ such that for $x \in \hat{L}_j[p]$, $H_{G_n}(x) = i$, and $L_{j+1} = S_{n+1} \oplus L_{j+1}$ such that for $x \in \hat{L}_{j+1}[p]$, $H_{G_n}(x) = i$. To show that $L = \bigcup_{j=1}^{\infty} L_j \in \mathscr{S}_{i+1}$ we need only show that $\hat{L}_{j+1}$ can be chosen such that it contains $\hat{L}_j$. Write $\hat{L}_j = \Sigma\{x_\alpha\}$ such that $o(x_\alpha) = n_\alpha + 1$. Now $p^{n_\alpha} x_\alpha \in \hat{L}_{j+1}[p]$ since otherwise $H_{G_n}(p^{n_\alpha} x_\alpha) < i$. If $x_\alpha \in \hat{L}_{j+1}$ we are done. So suppose that $x_\alpha \notin L_{j+1}$. Since $x_\alpha \in L_{j+1}$, $x_\alpha = s + y_\alpha$ where $s \in S_{n+1}$, $y_\alpha \in \hat{L}_{j=1}$ and $p^{n_\alpha} x_\alpha = p^{n_\alpha} y_\alpha$. By the purity of $\hat{L}_j$ in L_{j+1}, $\hat{L}_{j+1} = \{y_\alpha\} + K$, and thus we may rewrite L_{j+1} as $L_{j+1} = S_{n+1} \oplus (\{s + y_\alpha\} \oplus K)$ where our new $\hat{L}_{j+1} = \{s + y_\alpha\} \oplus K$ and hence contains $x_\alpha = s + y_\alpha$. Since we may do this for any $x_\alpha \in \hat{L}_j$, we can choose $\hat{L}_{j+1}$ to contain $\hat{L}_j$. Therefore $L = \bigcup_{j=1}^{\infty} L_j = S_i \oplus \bigcup_{j=1}^{\infty} \hat{L}_j$, and L has the desired properties.

Next set $B_T = \bigcup_{i=n+1}^{\infty} S_i$, and note that B_T is pure in T and B_T is a direct sum of cyclic groups (see Theorem 11.1 in [5]). We will now show that T/B_T is divisible which will imply that B_T is a basic subgroup of T. To do this we will show that for all $x \in T[p]$ such that $x \notin B_T[p]$, $H_{T/B_T}(x + B_T) = \infty$. This is sufficient since, by Lemma 1 in [9], if $x + B_T \in (T/B_T)[p]$, there exists $x' \in T[p]$ such that $x' + B_T = x + B_T$. Let $x \in T[p]$ such that $x \notin B_T[p]$, and suppose that $H_{G_n}(x) = m$. Suppose that $H_T(x) = k$, and let $z \in T$ such that $p^k z = x$. Now $x \in S_{m+1}$ which implies that either $S_{m+1} \oplus \{z\}$ is not pure in T or that $H_{G_n}(x+y) > m$ for some $y \in S_{m+1}[p]$ (by the maximality of S_{m+1}). If $S_{m+1} \oplus \{z\}$ is not pure in T then there exists $y_1 \in S_{m+1}[p]$ such that $H_T(y_1 + x) = k_1 > k$. Let $z_1 \in T$ such that $p^{k_1} z_1 = y_1 + x$. Now $H_{G_n}(x + y_1) \leqq m$, and $x + y_1 \notin S_{m+1}$ so that again either $S_{m+1} \oplus \{z_1\}$ is not pure in T or $H_{G_n}(x + y_1 + y) > m$ for some $y \in S_{m+1}[p]$. If $S_{m+1} \oplus \{z_1\}$ is not pure in T then there exists $y_2 \in S_{m+1}$ such that $H_T(x + y_1 + y_2) = k_2 > k_1 > k$. Thus in either case there exists $y \in S_{m+1}[p]$ such that $H_{G_n}(x + y) = m_1 > m$. Now clearly $x + y \notin S_{m_1+1}$, and by a similar argument there exist a $y^1 \in S_{m_1+1}$ such that $H_{G_n}(x + y + y^1) = m_2 > m_1$. Continuing by induction we see that the height of $x + B_T \in T/B_T$ is infinite.

It seems that an answer to the general question (asked at the beginning of this section) would give useful information about the structure of infinite p-groups.

Another question that arises is the following: If G and H are p-groups such that $G \doteqdot H$ and $G/G^1 \cong H/H^1$, then is it true that $G \cong H$? It would be interesting to know the answer to this question at least in the case G/G^1 is a direct sum of cyclic groups.

It would also be of interest to know: If G is a direct sum of closed p-groups, is a group S between G and $p^n G$ also a direct sum of closed p-groups?

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