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FACTORIZATIONS OF p-SOLVABLE GROUPS

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The object of this paper is to put in relief one of the ideas which has been very helpful in studying simple groups, viz. using factorizations of p-solvable groups to obtain information about the subgroups of a simple group which contain a given S_p-subgroup. Since the idea is so simple, it seems to deserve a simple exposition.

The group $\mathbf{J}(\mathfrak{X})$ was introduced in [3]. In this paper, $\mathbf{J}(\mathfrak{X})$ is again used, together with a similarly defined group, to obtain factorizations of some p-solvable groups which are of relevance in the study of simple groups.

As in [3], $m(\mathfrak{X})$ denotes the minimal number of generators of the finite group $\mathfrak{X}$, and $d(\mathfrak{X}) = \max\{m(\mathfrak{A})\}$, $\mathfrak{A}$ ranging over all the abelian subgroups of $\mathfrak{X}$. For each nonnegative integer n, let $\mathbf{J}_n(\mathfrak{X}) = \langle \mathfrak{A} \mid \mathfrak{A}$ is an abelian subgroup of $\mathfrak{X}$ with $m(\mathfrak{A}) \geqq d(\mathfrak{X}) - n\rangle$. Thus $\mathbf{J}_0(\mathfrak{X}) = \mathbf{J}(\mathfrak{X})$ and $\mathbf{J}_k(\mathfrak{X}) = \mathfrak{X}$ whenever $k \geqq d(\mathfrak{X}) - 1$. Also $\mathbf{J}_n(\mathfrak{X}) \subseteqq \mathbf{J}_{n+1}(\mathfrak{X})$ for $n = 0, 1, \cdots$.

THEOREM 1. *Suppose $\mathfrak{G}$ is a p-solvable finite group, p is a prime, and $\mathfrak{G}_p$ is a S_p-subgroup of $\mathfrak{G}$. Suppose also that $\mathbf{O}_{p'}(\mathfrak{G}) = 1$ and that one of the following holds*:

(a) $p \geqq 5$.

(b) $p = 3$ *and $SL(2, 3)$ is not involved in* $\mathfrak{G}$.

(c) $p = 2$ *and $SL(2, 2)$ is not involved in* $\mathfrak{G}$.

Let $\mathfrak{H} = \bigcap_{G \in \mathfrak{G}} \mathbf{C}_{\mathfrak{G}}(\mathbf{Z}(\mathfrak{G}_p))^G$. *Then* $\mathfrak{G} = \mathfrak{H} \cdot \mathbf{N}_{\mathfrak{G}}(\mathbf{J}(\mathfrak{G}_p))$ *and if* $p \geqq 5$, *then* $\mathfrak{G} = \mathfrak{H} \cdot \mathbf{N}_{\mathfrak{G}}(\mathbf{J}_1(\mathfrak{G}_p))$. *In particular,* $\mathfrak{G} = \mathbf{C}_{\mathfrak{G}}(\mathbf{Z}(\mathfrak{G}_p)) \cdot \mathbf{N}_{\mathfrak{G}}(\mathbf{J}(\mathfrak{G}_p))$.

Proof. Let $\mathfrak{W}_1 = \mathbf{Z}(\mathfrak{G}_p)^{\mathfrak{G}}$, $\mathfrak{W} = \Omega_1(\mathfrak{W}_1)$. Then $\mathfrak{H} = \mathbf{C}_{\mathfrak{G}}(\mathfrak{W}_1)$ and $\mathfrak{H} = \mathbf{O}_p(\mathfrak{G} \bmod \mathfrak{H})$. If $p \geqq 5$, then since $\mathbf{J}(\mathfrak{G}_p)$ char $\mathbf{J}_1(\mathfrak{G}_p)$, it suffices to show that $\mathbf{J}_1(\mathfrak{G}_p) \subseteqq \mathfrak{H}$, while if $p \leqq 3$, it suffices to show that $\mathbf{J}(\mathfrak{G}_p) \subseteqq \mathfrak{H}$.

Suppose the theorem is false and $\mathfrak{G}$ is a minimal counterexample. Let $\mathfrak{A}$ be an abelian subgroup of $\mathfrak{G}_p$, $\mathfrak{A} \not\subseteqq \mathfrak{H}$, and $m(\mathfrak{A}) \geqq d(\mathfrak{G}_p) - \delta$, where $\delta = 0$ if $p \leqq 3$ and $\delta = 1$ if $p \geqq 5$. Let $\mathfrak{K} = \mathbf{O}_{p'}(\mathfrak{G} \bmod \mathfrak{H})$, $\mathfrak{L} = \mathfrak{K}\mathfrak{A}$. Since $\mathfrak{G}_p \cap \mathfrak{L}$ is a S_p-subgroup of $\mathfrak{L}$, it follows that the theorem is violated in $\mathfrak{L}$, so by induction, $\mathfrak{L} = \mathfrak{G}$. Minimality of $\mathfrak{G}$ forces $\mathfrak{A}/\mathfrak{A} \cap \mathfrak{H}$ to be cyclic and forces $\mathfrak{K}/\mathfrak{H}$ to be a special q-group. On the other hand, since $m(\mathfrak{A}) \geqq d(\mathfrak{G}_p) - \delta$, it follows that $|\mathfrak{W} : \mathfrak{W} \cap \mathfrak{A}| \leqq p^{1+\delta}$. If $p \geqq 5$, Theorem B of Hall-Higman [2] yields a contradiction, while if $p \leqq 3$,

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(b) or (c) yields a contradiction, as in [3]. The proof is complete.

REMARKS. If the condition $\mathbf{O}_{p'}(\mathfrak{G}) = 1$ is dropped, then $\mathfrak{G} = \mathbf{O}_{p'}(\mathfrak{G})\mathbf{C}_{\mathfrak{G}}(\mathbf{Z}(\mathfrak{G}_p))\mathbf{N}_{\mathfrak{G}}(\mathbf{J}(\mathfrak{G}_p))$. This is so since $\mathbf{A}(\overline{\mathfrak{G}}_p) = \overline{\mathbf{A}(\mathfrak{G}_p)}$ where $\mathbf{A}$ is any one of the operations, $\mathbf{Z}$, $\mathbf{J}$, $\mathbf{CZ}$, $\mathbf{NJ}$ and — is any epimorphism of $\mathfrak{G}$ with ker (—) a p'-group.

It would appear that the hypothesis of p-solvability in Theorem 1 is not the proper one and that some more general family of groups will admit exploitable factorizations. However, our meagre knowledge of simple groups makes it impossible at present to guess the shape of the factorization.

THEOREM 2. *Suppose $\mathfrak{G}$ is a finite group, p is a prime, $\mathfrak{G}_p$ is a S_p-subgroup of $\mathfrak{G}$ and $p \geqq 5$. Suppose also that the following hold:*

(a) *1 is the only p-signalizer of $\mathfrak{G}$.*[1]

(b) $\mathbf{C}_{\mathfrak{G}}(\mathbf{Z}(\mathfrak{G}_p))$, $\mathbf{N}_{\mathfrak{G}}(\mathbf{J}(\mathfrak{G}_p))$, *and* $\mathbf{N}_{\mathfrak{G}}(\mathbf{Z}(\mathbf{J}_1(\mathfrak{G}_p)))$ *are p-solvable. Then* $\mathbf{C}_{\mathfrak{G}}(\mathbf{Z}(\mathfrak{G}_p)) \cdot \mathbf{N}_{\mathfrak{G}}(\mathbf{J}(\mathfrak{G}_p))$ *is a subgroup of $\mathfrak{G}$ which contains every p-solvable subgroup of $\mathfrak{G}$ which contains $\mathfrak{G}_p$.*

Proof. Let $\mathfrak{N}_1 = \mathbf{C}_{\mathfrak{G}}(\mathbf{Z}(\mathfrak{G}_p))$, $\mathfrak{N}_2 = \mathbf{N}_{\mathfrak{G}}(\mathbf{J}(\mathfrak{G}_p))$, $\mathfrak{N}_3 = \mathbf{N}_{\mathfrak{G}}(\mathbf{Z}(\mathbf{J}_1(\mathfrak{G}_p)))$, $\mathfrak{N}_{ij} = \mathfrak{N}_i \cap \mathfrak{N}_j$. By Theorem 1 with $\mathfrak{N}_2$ in the role of $\mathfrak{G}$, we have $\mathfrak{N}_2 = \mathfrak{N}_{21}\mathfrak{N}_{23}$; and similarly, $\mathfrak{N}_3 = \mathfrak{N}_{31}\mathfrak{N}_{32}$. The factorization $\mathfrak{N}_1 = \mathfrak{N}_{12}\mathfrak{N}_{13}$ is easily obtained, as in Lemmas 24.4 and 7.7 of [1], for example, so by Lemma 8.6 of [1], $\mathfrak{N}_1\mathfrak{N}_2$ is a subgroup of $\mathfrak{G}$. If $\mathfrak{K}$ is a p-solvable subgroup of $\mathfrak{G}$ which contains $\mathfrak{G}_p$, then by Theorem 1, $\mathfrak{K} = (\mathfrak{K} \cap \mathfrak{N}_1)(\mathfrak{K} \cap \mathfrak{N}_2) \subseteqq \mathfrak{N}_1\mathfrak{N}_2$. The proof is complete.

REMARK. It is clear that Theorem 2 may be used to shorten some of the proofs in [1] which deal with π_4.

REFERENCES

1. W. Feit and J. G. Thompson, *Solvability of groups of odd order*, Pacific J. Math. **13**, (1963).
2. P. Hall and G. Higman, *The p-length of a p-soluble group, and reduction theorems for Burnside's problem*, Proc. London Math. Soc. (3), **7** (1956), 1-42.
3. J. G. Thompson, *Normal p-complements for finite groups*, Jour. of Alg., vol. 1, no. 1, pp. 43-46.

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[1] The subgroup $\mathfrak{A}$ of $\mathfrak{G}$ is a p-signalizer if and only if $|\mathfrak{A}|$ and $|\mathfrak{G}:\mathbf{N}_{\mathfrak{G}}(\mathfrak{A})|$ are prime to p.