PACIFIC JOURNAL OF MATHEMATICS
Vol. 16, No. 2, 1966

ON TWO-SIDED H*-ALGEBRAS

PARFENY P. SAWOROTNOW

We call a Banach algebra A, whose norm is a Hilbert space norm, a *two-sided H^*-algebra* if for each $x \in A$ there are elements x^l, x^r in A such that $(xy, z) = (y, x^l z)$ and $(yx, z) = (y, zx^r)$ for all $y, z \in A$. A two-sided H^*-algebra is called *discrete* is each right ideal R such that $\{x^r \mid x \in R\} = \{x^l \mid x \in R\}$ contains an idempotent e such that $e^r = e^l = e$. The purpose of this paper is to obtain a structural characterization of those two-sided H^*-algebras M which consist of complex matrices $x = (x_{ij} \mid i, j \in J)$ (J is any index set) for which

$$\sum_{i,j} t_i \mid x_{ij} \mid^2 t_j$$

converges. Here t_i is real and $1 \leqq t_i \leqq a$ for all $i \in J$ and some real a. The inner product in M is

$$(x, y) = \sum_{i,j} t_i x_{ij} \bar{y}_{ij} t_j$$

and

$$x^r_{ij} = (t_i/t_j)\bar{x}_{ji}\,, \qquad x^l_{ij} = (t_j/t_i)\bar{x}_{ji}\,.$$

Then every algebra M is discrete simple and proper ($Mx = 0$ implies $x = 0$). Conversely every discrete simple and proper two-sided H^*-algebra is isomorphic to some algebra M. An incidental result is that the radical of a two-sided H^*-algebra is the right (left) annihilator of the algebra.

In this paper we will refer to such an algebra M above as a *canonical* algebra. We studied two-sided H^*-algebras (and more general algebras) in two previous papers [4, 5]. When $x^r = x^l$ for all x in A we have the H^*-algebras of Ambrose [1] and if we omit x^l we have the right H^*-algebra of Smiley [6]. Incidentally, in [4, Theorem 2] we proved that a proper right H^*-algebra is a two-sided H^*-algebra. So most of the theory of this paper applies to a right H^*-Algebra.

Our proof of the main result (Theorem 4) uses the technique of Ambrose [1] and the lemmas about existence of minimal two-sided projections (Theorem 3 and Lemma 6).

The author is very grateful to the referee for his suggestions for the improvement of the paper.

2. A general theorem. The following theorem may be of an independent interest (compare with § 2 in [1]).

THEOREM 1. *The radical $\mathfrak{R}$ of each two-sided H^*-algebra A*

Received July 2, 1964.

coincides with both the right and left annihilator of the algebra.

Proof. $Ax = 0$ gives $(xy, z) = (x, zy^r) = (z^l x, y^r) = 0$ for all $y, z \in A$ so that $xA = 0$. Thus $r(A)$, the right annihilator or A, and $l(A)$ coincide. Now consider $B = r(A)^p$ which is easily seen to be a two-sided H^*-algebra which is proper in the sense that $r(B) = l(B) = 0$. The proof of Theorem 3.1 of [1] shows that each nonzero ideal of B contains a nonzero idempotent (see also [3], page 101). This means that $B \cap \mathfrak{R} = (0)$ since radical cannot contain idempotents [2, page 309]; thus $\mathfrak{R} = r(A) = l(A)$.

COROLLARY. *The following conditions are equivalent in any two-sided H^*-algebra (each one of these conditions can be used to define a proper algebra):*

(i) $r(A) = 0$
(ii) $l(A) = 0$
(iii) x^r *is unique for each* $x \in A$
(iv) x^l *is unique for each* $x \in A$
(v) A *is semi-simple.*

Proof. Equivalence of (i) and (iii) ((ii) and (iv)) can be established as in the proof of Theorem 2.1 of [1].

3. **Invariant ideals.** Unless otherwise stated A will denote a simple proper two-sided complex H^*-algebra. Note that both involutions ($x \to x^r$ and $x \to x^l$) in A are continuous (This follows from the closed graph theorem).

LEMMA 1. *If $x, y \in A$ then $(x, y) = (y^l, x^r) = (y^r, x^l)$.*

Proof. The set I of linear combinations of products of members of A is dense in A (because I is a two-sided ideal). If $x = uv$ for some $u, v \in A$ then $(x, y) = (uv, y) = (u, yv^r) = (y^l u, v^r) = (y^l, v^r u^r) = (y^l, x^r)$. Hence $(x, y) = (y^l, x^r)$ (and similarly $(x, y) = (y^r, x^l)$) holds if $x \in I$. The lemma now follows from the continuity of the involutions.

COROLLARY. *If S is any subset of A, then $S^{rp} = S^{pl}$ and $S^{lp} = S^{pr}$ (as in [4] S^p denoted the set of elements of A orthogonal to S and S^r (S^l) denotes the image of S under the involution $x \to x^r$ ($x \to x^l$)).*

LEMMA 2. *If B is a closed right (left) ideal of A, then $l(B) = B^{rp} = B^{pl}$ ($r(B) = B^{lp} = B^{pr}$).*

Proof. From $(B^{rp}B, A) = (B^{rp}, AB^r) = A^lB^{rp}, B^r) = (B^{rp}, B^r) = 0$ we conclude that $B^{rp}B = 0$. Thus $B^{rp} \subset l(B)$. If $xB = 0$, then $0 = (xB, A) = (x, AB^r) = (A^lx, B^r) = (Ax, B^r)$, $Ax \subset B^{rp}$ and $x \in B^{rp}$ by Lemma 1 of [6]. This simple means that $l(B) \subset B^{rp}$.

DEFINITION. An ideal I in A is said to be *invariant* if $I^r = I^l$.

LEMMA 3. *A closed (right, left) ideal I in A is invariant if and only if I^p is invariant.*

Proof. Direct verification: $I^{pl} = I^{rp} = I^{lp} = I^{pr}$.

COROLLARY. *A closed right (left) ideal R (L) is invariant if and only if $l(R^p) = l(R)^p$ ($r(L^p) = r(L)^p$).*

DEFINITION. An idempotent in A which is both left and right self-adjoint will be called a *two-sided projection.*

LEMMA 4. *If $e \in A$ is a left projection and eA is invariant, then e is a two-sided projection.*

Proof. From $Ae = Ae^r$ we have $ee^r = e$ which shows that $e^r = e$ also.

THEOREM 2. *A proper two-sided H^*-algebra A is an H^*-algebra if and only if each closed right (left) ideal of A is invariant.*

Proof. In view of the first structure theorem (Theorem 1 in [4] we may assume (without loss of generality) that A is simple. Now the condition of the theorem implies that each left projection is a right projection (Lemma 4) an vice-versa. From this it is not difficult to show that both involutions coincide. This could be done either by proving the second structure theorem (Theorem 4.3 of [1]) or by showing that the set S of all linear combinations of products of projections is dense in A (using the arguments in proofs of Lemma 8 in [4] and Theorem 1 in [5] one can show that S is a two-sided ideal).

4. Finite-dimensional algebras.

LEMMA 5. *For each right projection f in A there exist a left projection $e \in A$ such that $(e, f - e) = 0$ and $ef = e$, $fe = f$. If f is minimal then e is minimal also. A similar statement holds for a left projection.*

Proof. Consider the closed right ideal $R = \{x - fx \mid x \in A\} = r(f)$ and write $f = e + u$ with $e \in R^p$, $u \in R$. Then by Lemma 2 in [4] e is a left projection such that $R^p = eA$ and $R = r(e) = \{x \in A \mid ex = 0\}$. Also $(e, f - e) = (e, u) = 0$, $ef = e(e + u) = e$ and $fe = f(f - u) = f$. If f is minimal then minimality of e follows from the fact that $Af = Ae$.

REMARK. The algebra A in Lemma 5 does not have to be finite-dimensional.

THEOREM 3. *Every finite-dimensional proper two-sided H^*-algebra A contains a minimal two-sided projection.*

Proof. We may assume that A is simple. By Lemma 5 there exists a sequence $\{f_1, f_2, \cdots, f_n, \cdots\}$ of minimal right projections and a sequence $\{e_1, e_2, \cdots, e_n, \cdots\}$ of minimal left projections such that $\|f_n\|^2 = \|e_n\|^2 + \|f_n - e_n\|^2$, $\|e_n\|^2 = \|f_{n+1}\|^2 + \|e_n - f_{n+1}\|^2$ (and $e_n f_n = e_n$, $f_n e_n = f_n$, $e_n f_{n+1} = f_{n+1}$, $f_{n+1} e_n = e_n$) Also $\|f_n\| \leqq \|f_1\| \geqq \|e_n\|$ for each n. By the Bolzano-Weierstrass theorem there exists a subsequence $\{g_k\}$ of $\{f_n\}$ (for simplicity we write g_k instead of f_{n_k}) and some $g \in A$ such that $g = \lim g_k$. Then g is right self-adjoint and idempotent. From

$$\|f_1\|^2 = \|f_1 - e_1\|^2 + \|e_1 - f_2\|^2 + \|f_2 - e_2\|^2 + \cdots$$
$$+ \|f_n - e_n\|^2 + \|e_n - f_{n+1}\|^2 + \|f_{n+1}\|^2$$

and $\|f_{n+1}\| \geqq \|f_{n+p}\| \geqq \|g\|$ it follows that $\|f_n - e_n\| \to 0$. Therefore $g = \lim_k e_{n_k}$ also and so g is left self-adjoint.

It remains to show that g is minimal. If $x \in A$ then for each k there exists a complex number λ_k such that $g_k x g_k = \lambda_k g_k$ ([4], page 52 and [1], page 380). Then $\lambda_k g_k$ tends to gxg. From $|\lambda_k| \leqq |\lambda_k| \cdot \|g_k\| = \|g_k x g_k\| \leqq \|g_k\|^2 \|x\| \leqq \|g_1\|^2 \|x\|$ it follows that λ_k has a subsequence converging to some complex number λ. Then $gxg = \lambda g$ and so gAg is isomorphic to the complex number field, from which we may conclude that g is minimal.

Later (corollary to Theorem 4) we will see that each finite-dimensional proper simple two-sided H^*-algebra is isomorphic to a canonical algebra M. In fact each such an algebra is discrete in the sense of the next definition.

5. Discrete algebras.

DEFINITION. A two-sided H^*-algebra A is said to be *discrete* if each invariant ideal in A contains an invariant ideal of the form eA where e is a left projection.

Because of Lemma 4 this definition is equivalent to the corresponding definition in the introduction.

LEMMA 6. *Each invariant closed right ideal R in a discrete two-sided H^*-algebra A contains a minimal two-sided projection.*

Proof. By Lemma 4 R contains a two-sided projection e. The set eAe is a finite-dimensional proper two-sided H^*-algebra included in R. The lemma now follows from Theorem 3.

COROLLARY. *Each discrete proper two-sided H^*-algebra A contains a (maximal) family $\{g_i\}$ of mutually orthogonal minimal two-sided projections such that $A = \sum_i g_iA = \sum_i Ag_i = \sum_{i,j} g_iAg_j$.*

THEOREM 4. *Each simple discrete proper two-sided H^*-algebra A is isomorphic to a canonical algebra.*

Proof. Consider the family $\{g_i\}$ of the last corollary and select $g_{ij} \in g_iAg_j$ such that $g^l_{ij} = g_{ji}$, $g_{ij}g_{jk} = g_{ik}$ and $g_{ii} = g_i$ for each i, j, k (as in [1], page 381). Then the g_{ij}'s are mutually orthogonal. We set $t_i = \| g_i \|$; then $1 \leqq t_i$ for each i and also $\| g_{ji} \|^2 = (g_{ji}, g_{ji}) = \| g_i \|^2 = t_i^2$ for each j (and a fixed i). Also one can show that $g^r_{ij} = t_i^{-2}t_j^2 g_{ji}$ (note that $(g_{ij}, g_{ij}) = (g_{ij}g^r_{ij}, g_{ii}) = (g^r_{ij}, g_{ji})$ and that g^r_{ij} is a scalar multiple of g_{ji}). Let $e_{ij} = t_i^{1/2}t_j^{-1/2}g_{ij}$, then $(e_{ij}, e_{ij}) = t_it_j$, $e^l_{ij} = (t_i/t_j)e_{ji}$ and $e^r_{ij} = (t_j/t_i)e_{ji}$. The theorem now is easy to complete (see for example the proof of Theorem 4.3 in [1]). Boundedness of the set $\{t_i\}$ follows from continuity of the right involutions: take a fixed k and consider $x_i = g^r_{ik}$, then $\| x_i \| = t_i^{-2}t_k^2 \| g_{ki} \| = t_i^{-1}t_k^2$ and $\| x^r_i \| = t_k$.

COROLLARY. *Each finite-dimensional proper simple two-sided H^*-algebra is isomorphic to a canonical algebra M for some finite set J.*

6. Remark on the algebra M. To complete the paper we show that the canonical algebra M in the introduction is discrete. For each k let e_k be the matrix $x_{ij} = \delta^k_i\delta^k_j$ (δ^k_i, δ^k_j are Kronecker deltas). Then $\{e_k\}$ is a maximal family of mutually orthogonal minimal two-sided projections in M. Let R be an invariant closed right ideal in M. Let e in $\{e_k\}$ be such that $eR \neq 0$. Let $R_1 = (eM)^p = r(e)$; then $R_2 = R \cap (R \cap R_1)^p$ is an invariant closed nonzero right ideal (note that $R_2 = 0$ would imply $R \subset R_1 = r(e)$ since R_2 is the orthogonal complement of $R \cap R_1$ relatively to R).

Suppose that R_2 is not minimal. Let e_1, e_2 be two orthogonal left projections in R_2. Let $x = \lambda e_1 + \mu e_2$ (λ, μ are scalars) be such that $(x, e) = 0$. If $xe = 0$ then $ex^l = 0$ and so $R_1 \cap R_2 \neq 0$ (note that $x^l = \bar{\lambda} e_1 + \bar{\mu} e_2$ belongs to R_2). If $xe \neq 0$ then xeM contains a left projection e_3 ([4], Lemma 5), $e_3 = xey$ for some $y \in M$. Then $(e_3, e) = (xey, e) = (x, ey^r e) = 0$ (since $ey^r e$ is a scalar multiple of e) from which it follows that $e_3 e = 0$ ($(e_3 e, e_3 e) = (e_3, e) = 0$). But then $ee_3 = 0$ since e_3 and e are both left self-adjoint. So we see that also in this case there exists a nonzero element z in $R_2 \cap R_1$. But this implies $z \in R \cap R_1$ and $z \in (R \cap R_1)^p$, which is impossible.

Thus R_2 is minimal and so it is of the form $R_2 = gM$ for some (minimal) left projection g.

BIBLIOGRAPHY

1. W. Ambrose, *Structure theorems for a special class of Banach algebras*, Trans. Amer. Math. Soc. **57** (1945), 364-386.

2. N. Jacobson, *The radical and semi-simplicity for arbitrary rings*, Amer. J. Math. **67** (1945), 300-320.

3. L. H. Loomis, *An introduction to abstract harmonic analysis*, New York, 1953.

4. P. P. Saworotnow, *On a generalization of the notion of an H*-algebra*, Proc. Amer. Math. Soc. **8** (1957), 49-55.

5. ———, *On the imbedding of a right H*-algebra into Ambrose's H*-algebra*, Proc. Amer. Math. Soc. **8** (1957), 56-62.

6. M. F. Smiley, *Right H*-algebras*, Proc. Amer. Math. Soc. **4** (1953), 1-4.

THE CATHOLIC UNIVERSITY OF AMERICA