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ON THE REGIONS BOUNDED BY HOMOTOPIC CURVES

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We give a short proof of the following theorem of H. I. Levine [2]:

Let $\mathscr{A}$ and $\mathscr{B}$ be nonintersecting simple closed curves on an orientable surface S (compact or not).

(i) If $\mathscr{A}$ is homotopic to zero, then $\mathscr{A}$ bounds a closed disk in S.

(ii) If $\mathscr{A}$ and $\mathscr{B}$ are freely homotopic but not homotopic to zero then $\mathscr{A} \cup \mathscr{B}$ bounds a closed cylinder in S.

The proof is based on some elementary properties of covering surfaces and the Jordan-Schönflies Theorem for planar surfaces and is as follows. Let H be the cyclic subgroup of $\pi_1(S)$ generated by $\mathscr{A}$. By a standard construction (see [1]) we may form a covering surface $\tilde{S} \xrightarrow{\pi} S$ of S having the following properties: (1) $\pi_1(\tilde{S}) \approx H$; (2) we distinguish a point $O \in S$ and a point $\tilde{O} \in \tilde{S}$ lying over O—then if $\mathscr{D}$ is a closed curve in S through O and $\tilde{\mathscr{D}}$ is the lift of $\mathscr{D}$ to $\tilde{S}$ *which passes through* $\tilde{O}$, the curve $\tilde{\mathscr{D}}$ is closed in $\tilde{S}$ if and only if $\mathscr{D} \in H$.

There exist *closed* curves $\tilde{\mathscr{A}}_1$ and $\tilde{\mathscr{B}}_1$ in $\tilde{S}$ which are lifts of $\mathscr{A}$ and $\mathscr{B}$. To see this assume that $O \in \mathscr{A}$ so by (2) $\mathscr{A}$ lifts to a closed curve through $\tilde{O}$. Since $\mathscr{A}$ and $\mathscr{B}$ are freely homotopic there is an arc γ from $\mathscr{A}$ to $\mathscr{B}$ such that $\mathscr{A} \sim \gamma\mathscr{B}\gamma^{-1}$. Then $\gamma\mathscr{B}\gamma^{-1}$ has at least one lift to a closed curve in $\tilde{S}$; hence so does $\mathscr{B}$.

LEMMA. *If $\tilde{R}$ is a compact region in $\tilde{S}$, π is one-to-one on $\partial\tilde{R}$, and π (int $\tilde{R}$) $\cap\, \pi(\partial\tilde{R}) = \emptyset$, then π maps $\tilde{R}$ homeomorphically onto $\pi(\tilde{R})$.*

Proof. The hypotheses imply that the region $\tilde{R}_0 \equiv \text{int}\,\tilde{R}$ is an unlimited covering surface of $\pi(\tilde{R}_0)$. We show that every point of $\pi(\tilde{R}_0)$ is covered exactly once. For suppose $\tilde{p}_1, \tilde{p}_2$ are two points over $p \in \pi(\tilde{R}_0)$. Let γ be an arc in $\pi(\tilde{R}_0)$ from p to $\pi(\partial\tilde{R})$ and let $\tilde{\gamma}_1, \tilde{\gamma}_2$, be arcs over γ from $\tilde{p}_1$ and $\tilde{p}_2$. Then by our hypotheses $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ must intersect on $\partial\tilde{R}$, and the trivial curve $\gamma - \gamma$ in S lifts to the curve $\tilde{\gamma}_1 - \tilde{\gamma}_2$ in $\tilde{S}$, a contradiction.

To prove (i) above, we may assume that $\tilde{S}$, the universal covering surface of S, is a disk. Since $\mathscr{A} \sim 1$, every lift of $\mathscr{A}$ to $\tilde{S}$ is a

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closed curve in $\tilde{S}$ (monodromy theorem). The points in $\tilde{S}$ over a point $p \in S$ have no limit point in $\tilde{S}$; hence there must exist at least one arc $\tilde{\mathscr{A}}_0$ lying above $\mathscr{A}$ and bounding a region $\tilde{R}$ whose interior contains no points lying over $\mathscr{A}$. $\tilde{R}$ is a topological disk (Jordan-Schönflies), and the above lemma implies that $\mathscr{A}$ bounds the region $\pi(\tilde{R})$ which is homeomorphic to $\tilde{R}$. This proves (i).

The proof of (ii) is achieved through the following steps.

1°. $\tilde{S}$ is a cylinder which we realize as the open plane region bounded by the circles $C_1 = \{|z| = 1\}$ and $C_2 = \{|z| = 2\}$. Indeed, $\pi_1(\tilde{S})$ is cyclic, $\tilde{S}$ is orientable, and hence $\tilde{S}$ has genus zero ($H_1(\tilde{S})$ has no nonzero intersections). Thus $\tilde{S}$ may be imbedded in $\mathbf{R}^2$; since $H_1(\tilde{S})$ is cyclic and not zero, $\tilde{S}$ is a cylinder.

2°. If $\mathscr{A} \nsim 1$ in S, then $\mathscr{A}^n \nsim 1$ for all n. For $\mathscr{A}$ lifts to a *closed* curve $\tilde{\mathscr{A}}_1 \nsim 1$ in the cylinder $\tilde{S}$ (otherwise $\mathscr{A} \sim 1$); if $\mathscr{A}^n \sim 1$, then $\tilde{\mathscr{A}}_1^n \sim 1$ which is impossible since the cylinder has no torsion.

3°. If $\tilde{\mathscr{A}}$ is any lift of $\mathscr{A}$ in $\tilde{S}$ then either $\tilde{\mathscr{A}}$ is a simple closed curve which separates C_1 and C_2 or the corresponding lift $\tilde{\mathscr{A}^n}$ of $\mathscr{A}^n$ is not closed in $\tilde{S}$ for any n. Any pair of closed lifts of $\mathscr{A}$ and/or $\mathscr{B}$ lie one inside the other with respect to the fixed imbedding of $\tilde{S} \subset \mathbf{R}^2$.

For the proof, suppose that for some n, $\tilde{\mathscr{A}^n}$ is a simple closed curve. Then by the Jordan-Schönflies Theorem, $\tilde{\mathscr{A}^n}$ is homotopic either to $\tilde{\mathscr{A}}_1$ (the fixed closed lift of $\mathscr{A}$ in $\tilde{S}$) or to 1. Since freely homotopic curves in $\tilde{S}$ project to freely homotopic curves in S, we then have $\mathscr{A}^{n-1} \sim 1$ or $\mathscr{A}^n \sim 1$ respectively; by 2°, we see that $n = 1$ and $\tilde{\mathscr{A}} \nsim 1$. $\tilde{\mathscr{A}}$ separates C_1 and C_2 since, for any connected set Γ such that $\Gamma \cap C_1$, $\Gamma \cap C_2 \neq \emptyset$, $S - \Gamma$ is simply connected. Given any pair of disjoint simple closed curves which separate C_1 and C_2, the one which first meets the cross cut $\{1 \leqq x \leqq 2\}$ lies inside the other one.

4°. There are at most a finite number of lifts (closed or not) of $\mathscr{A}$ and $\mathscr{B}$ between $\tilde{\mathscr{A}}_1$ and $\tilde{\mathscr{B}}_1$ in $\tilde{S}$. For the set of lifts of any point of S can have no accumulation point in the compact region bounded by $\tilde{\mathscr{A}}_1$ and $\tilde{\mathscr{B}}_1$.

To finish the proof of (ii), we observe that by 3° and 4° any lift of $\mathscr{A}$ or $\mathscr{B}$ which lies between $\tilde{\mathscr{A}}_1$ and $\tilde{\mathscr{B}}_1$ in $\tilde{S}$ must be closed, and there are at most a finite number of such lifts. Hence there must be an adjacent pair $\tilde{\mathscr{A}}_p, \tilde{\mathscr{B}}_q$. The compact region $\tilde{R}$ bounded by $\tilde{\mathscr{A}}_p$ and $\tilde{\mathscr{B}}_q$ is a topological cylinder (Jordan-Schönflies). The above lemma implies that π maps $\tilde{R}$ homeomorphically onto $\pi(\tilde{R})$, and the proof is complete.

REMARK. In the case (i) an alternative proof can be given based on the fact that the cover transformations of $\tilde{S}$ over S have no fixed points. In fact it follows easily from the Brouwer Theorem and the fact that the cover transformations are homeomorphisms that the compact regions bounded by the curves $\tilde{\mathscr{A}}_i$ over $\mathscr{A}$ are disjoint. The proof of (i) then follows by applying the above lemma to any one of these regions.

Finally we wish to point out that the following somewhat more general theorem can be proved by methods entirely analogous to those used above.

THEOREM. *Let $\mathscr{A}$ and $\mathscr{B}$ be nonintersecting simple closed curves in an orientable surface S (which we represent as a Riemann surface) such that $\mathscr{A}^n \sim \mathscr{B}^m$ for some positive integers m, n, but $\mathscr{A} \not\sim 1$. Then $m = n$, $\mathscr{A} \sim \mathscr{B}$, and $\mathscr{A} - \mathscr{B}$ is the boundary of a closed cylinder in S.*

We give a very brief outline of the proof. Let $\tilde{S}$ be the covering surface of S generated by $\mathscr{A}^n$. It follows from the unique continuation property that two closed lifts of $\mathscr{A}^n$ cannot intersect in $\tilde{S}$ unless they coincide. Hence exactly as above we can find a compact cylinder $\tilde{R}$ in $\tilde{S}$ bounded by a lift $\tilde{\mathscr{A}}^n$ of $\mathscr{A}^n$ and a lift $\tilde{\mathscr{B}}^m$ or $\mathscr{B}^m$ such that no point in the interior of $\tilde{R}$ lies over $\mathscr{A}$ or $\mathscr{B}$. Setting $R = \pi(\tilde{R})$, fairly straightforward topological reasoning shows that R is bounded by $\mathscr{A}$ and $\mathscr{B}$ and that the interior of $\tilde{R}$ is a smooth unlimited covering surface of $\operatorname{int}(R)$, which each point in $\operatorname{int}(R)$ covered the same number of times. In particular, this shows that $m = n$. Since the interior of $\tilde{R}$ is an n-sheeted covering surface of $\operatorname{int}(R)$, the fundamental group of $\operatorname{int}(\tilde{R})$ is naturally isomorphic to a subgroup of index n of the group $\pi_1(\operatorname{int}(R))$. Since $\pi_1(\operatorname{int}(\tilde{R}))$ is infinite cyclic and $\pi_1(\operatorname{int}(R))$ has no torsion, it follows that $\pi_1(\operatorname{int}(R))$ is a cyclic group and hence that $\operatorname{int}(R)$ is a topological cylinder. Finally, the fact that R is a closed cylinder bounded by the Jordan curves $\mathscr{A}$ and $\mathscr{B}$ may be proved by fairly standard conformal mapping arguments.

BIBLIOGRAPHY

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