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DIFFERENTIABILITY OF SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS IN HILBERT SPACE

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Consider the differential equation

$$\frac{1}{i}\frac{du}{dt} - A(t)u = f(t) \ (a < t < b) \tag{1.1}$$

where $u(t)$, $f(t)$ are elements of a Hilbert space E and $A(t)$ is a closed linear operator in E with a domain $D(A)$ independent of t and dense in E. Denote by $C^m(a, b)$ the set of functions $v(t)$ with values in E which have m strongly continuous derivatives in (a, b). Introducing the norm

$$|v|_m = \left\{\sum_{j=0}^{m}\int_a^b |v^{(j)}(t)|^2\, dt\right\}^{1/2} \tag{1.2}$$

where $|v(t)|$ is the E-norm of $v(t)$, we denote by $H^m(a, b)$ the completion with respect to the norm (1.2) of the subset of functions in $C^m(a, b)$ whose norm is finite. Set $H^m = H^m(-\infty, \infty)$ and denote by H_0^m the subset of functions in H^m which have compact support. The solutions $u(t)$ of (1.1) are understood in the sense that $u(t) \in H^1(a', b')$ for any $a < a' < b' < b$.

THEOREM 1. **Assume that, for each $a < t < b$, the resolvent $R(\lambda, A(t)) = (\lambda - A(t))^{-1}$ of $A(t)$ exists for all real λ, $|\lambda| \geqq N(t)$, and that**

$$|R(\lambda, A(t))| \leqq \frac{C(t)}{|\lambda|} \text{ if } \lambda \text{ real, } |\lambda| \geqq N(t)\,, \tag{1.3}$$

where $N(t)$, $C(t)$ are constants. Assume next that for each $s \in (a, b)$, $A^{-1}(s)$ exists and

(1.4) $A(t)A^{-1}(s)$ **has m uniformly continuous t-derivatives,**

for $a < t < b$, where m is any integer $\geqq 1$. If u is a solution of (1.1) and if $f \in H^m(a, b)$, then $u \in H^{m+1}(a', b')$ for any $a < a' < b' < b$.

THEOREM 2. **If the assumptions of Theorem 1 hold with $m = \infty$, if $A(t)A^{-1}(s)$ is analytic in $t(a < t < b)$ for each $s \in (a, b)$, and if $f(t)$ is analytic in (a, b), then $u(t)$ is also analytic in (a, b).**

In case E is a Banach space, an analogue of Theorem 1 was proved by Sobolevski [3] and Tanabe [4] and an analogue of Theorem 2 was proved by Sobolevski [3] and Komatzu [2], but all these authors

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assume a stronger condition on the resolvent, namely, they assume that (1.3) holds for all complex λ with $Im(\lambda) \geqq 0$. On the other hand analogs of Theorems 1, 2 were proved by Agmon and Nirenberg [1] (for E a Banach space) under weaker bounds on $R(\lambda, A)$, but only in the case where $A(t) \equiv A$ is independent of t. It was shown in [1] that the condition (1.3) is necessary if $u \in C^{m+1}(a, b)$ whenever $f \in C^m(a, b)$.

Before proving Theorem 1 we wish to observe that (1.4) implies that

(1.5) $\quad A(s)A^{-1}(t)$ has m uniformly continuous t-derivatives.

Indeed, setting $B(t) = A(t)A^{-1}(s)$ and multiplying both sides of $B(t+h) - B(t) = B(t, h)h$ (here $\| B(t, h) \|$ is bounded independently of h, $|h|$ small) by $B^{-1}(t), B^{-1}(t+h)$, we find that $\| B^{-1}(t) \|$ is locally bounded. We further find that $B^{-1}(t)$ is continuous in t and also differentiable, and $(B^{-1}(t))' = B^{-1}(t)B'(t)B^{-1}(t)$; (1.5) now easily follows.

Writing $A(t)A^{-1}(s) = A(t)A^{-1}(\bar{s})\,[A(\bar{s})A^{-1}(s)]$ we see that if (1.4) holds for one particular $s = \bar{s}$ and if $A(\bar{s})A^{-1}(s)$ is a bounded operator for each s, then (1.4) holds.

2. Proof of Theorem 1.

Consider first the case $A(t) \equiv A$.

Lemma 1. *If $f \in H_0^m (m \geqq 0)$, $u \in H_0^1$ and (1.1) holds for $-\infty < t < \infty$, then $u \in H_0^{m+1}$ and*

$$|u|_{m+1} \leqq C(|f|_m + |u|_0) \tag{2.1}$$

where C depends only on A, m.

Proof. Taking the Fourier transform of (1.1) we get $(\lambda - A)\hat{u}(\lambda) = \hat{f}(\lambda)$, hence

$$\begin{aligned}\sqrt{2\pi}\; u(t) &= \int_{-N}^{N} e^{i\lambda t}\hat{u}(\lambda)d\lambda + \int_{-\infty}^{-N} e^{i\lambda t}R(\lambda, A)\hat{f}(\lambda)d\lambda + \int_{N}^{\infty} e^{i\lambda t}R(\lambda, A)\hat{f}(\lambda)d\lambda \\ &\equiv u_1 + u_2 + u_3 .\end{aligned}$$

By Schwarz's inequality and Plancherel's theorem,

$$|u_1|_{m+1}^2 \leqq C\int_{-N}^{N} |\hat{u}(\lambda)|^2 d\lambda \leqq C|u|_0^2$$

where various constants depending only on A, m are denoted by C. Next, if f is sufficiently smooth then

$$u_2^{(j)}(t) = \int_{-\infty}^{-N} e^{i\lambda t}(i\lambda)^j R(\lambda, A)\hat{f}(\lambda)d\lambda \quad (0 \leqq j \leqq m+1) ,$$

so that by Plancherel's theorem and (1.3),

$$| u_2 |_{m+1}^2 \leqq C \sum_{j=1}^{m+1} \int_{-\infty}^{-N} | \lambda^{j-1} \hat{f}(\lambda) |^2 d\lambda \leqq C | f |_m^2 .$$

If now f is only assumed to belong to H_0^m, then the inequality $| u_2 |_{m+1}^2 \leqq C | f |_m^2$ follows by approximating f by sufficiently smooth functions (for instance, by employing mollifiers and using the fact that "weak" derivatives are also "strong" derivatives). Since a similar inequality holds for u_3, $u \in H_0^{m+1}$ and (2.1) holds.

From (2.1), (1.3) we get

$$| Au |_m \leqq C(| f |_m + | u |_0) . \tag{2.2}$$

LEMMA 2. *Let the assumptions of Theorem 1 hold for $(a, b) = (-\infty, \infty)$, let the derivatives in (1.4) be uniformly bounded in t, and let $\| B(t) \| < \delta$ where $B(t) = [A(t) - A(s)]A^{-1}(s)$. If u is a solution of (1.1) in $(-\infty, \infty)$, if $f \in H_0^m (m \geqq 0)$, $u \in H_0^1$, $A(s)u \in H_0^m$, and if δ is sufficiently small (depending only on $A(s), m$), then $u \in H_0^{m+1}$ and*

$$| u |_{m+1} \leqq C(| f |_m + | u |_0) . \tag{2.3}$$

Proof. u satisfies

$$\frac{1}{i} \frac{du}{dt} - A(s)u = B(t)A(s)u(t) + f(t) , \tag{2.4}$$

from which it follows that $u \in H_0^{m+1}$. Applying (2.2) with $m = 0$ and taking $\delta < 1/2C$(C as in (2.2)) we get $| A(s)u |_0 \leqq C(| f |_0 + | u |_0)$. Next applying (2.2) with $m = 1$ and using the last inequality we find that $| A(s)u |_1 \leqq C(| f |_1 + | u |_0)$.

Proceeding step by step one gets

$$| A(s)u |_m \leqq C(| f |_m + | u |_0) . \tag{2.5}$$

(2.3) follows from (2.4), (2.5).

Setting $v_h(t) = [v(t + h) - v(t)]/h$, we have the following

LEMMA 3. *Let $u \in H_0^0$, $u \in H^{m+1} (m \geqq 0)$ if and only if $u_h \in H^m$ for all h sufficiently small and $| u_h |_m \leqq M$, and, in that case, $| u |_{m+1} \leqq CM$ and $| u_h |_m \leqq C | u |_{m+1}$.*

The lemma is well known in the special case where $u(t)$ is a complex-valued function. The proof in the present more general case can be given analogously, or also by expanding $u(t)$ in terms of a fixed orthonormal basis of E and applying the special case to each component.

LEMMA 4. *Lemma 2 holds even if the assumption that $A(s)u \in H^m$ is dropped.*

Proof. Taking finite differences in (1.1) we get

$$\frac{1}{i}\frac{du_h}{dt} - A(t)u_h = [A_h(t)A^{-1}(s)]A(s)u(t+h) + f_h(t) \equiv \varphi(t; h) .$$

Since $A(t)u \in H^\circ$ the same is true of $A(s)u$ (using (1.5)) and of $A(s)u_h$. Lemma 2 can then be applied to u_h with $m = 0$. We find (using Lemma 3) that $|u_h|_1 \leqq C$; hence, by Lemma 3, $u \in H^2$. Then $A(t)u \in H^1$ and we can proceed to apply Lemma 2 to u_h with $m = 1$. Thus, $u \in H^3$, etc.

Let $\zeta(t)$ be a C^∞ function satisfying: $\zeta(t) = 1$ if $|t - s| < \varepsilon$, $\zeta(t) = 0$ if $|t - s| > 2\varepsilon$, where ε is sufficiently small. $v = \zeta u$ satisfies

$$\frac{1}{i}\frac{dv}{dt} - A(t)v = \zeta f + i\zeta' u .$$

Applying Lemma 4 with $m = 1$ we find that $u \in H^2(s - \varepsilon, s + \varepsilon)$. Similarly, by considering $v_1 = \zeta_1 u$ where $\zeta_1(t) = \zeta(2t - s)$ and applying to it Lemma 4 with $m = 2$, we find that $u \in H^3(s - (1/2)\varepsilon, s + (1/2)\varepsilon)$. Proceeding in this manner, step by step, we find that $u \in H^{m+1}(s - \varepsilon_1, s + \varepsilon_1)$ for some $\varepsilon_1 > 0$. Since s is an arbitrary point in (a, b), the proof of Theorem 1 is complete.

REMARK. If $u \in H^{m+1}(a, b)$ then $u(t)$ is equal almost everywhere to (and therefore can be identified with) a function in $C^m(a, b)$.

3. Proof of Theorem 2. It suffices to prove analyticity in a small interval (a', b'). Furthermore, it suffices to show that for some fixed $s \in (a', b')$,

$$(3.1) \qquad |A(s)u|_{m-1,\delta} + |u|_{m,\delta} \leqq \frac{H_0 H^m}{\delta^m} m! \qquad \left(m = 0, 1, \cdots;\ 0 < \delta < \frac{b' - a'}{2}\right)$$

where $|u|_{m,\delta} = \left[\int_{a'+\delta}^{b'-\delta} |u^{(m)}(t)|^2 dt\right]^{1/2}$. The proof is by induction on m. To pass from m to $m + 1$ we differentiate (1.1) m times and thus obtain

$$\frac{1}{i}\frac{du^{(m)}}{dt} - A(t)u^{(m)} = \sum_{j=0}^{m-1} \binom{m}{j} [A^{(m-j)}(t)A^{-1}(s)]A(s)u^{(j)}(t) + f^{(m)}(t) \equiv \varphi_m .$$

Let $\zeta(t)$ be a smooth function satisfying: $\zeta(t) = 1$ if $a' + \delta < t < b' - \delta$, $\zeta(t) = 0$ if $a' < t < a' + \delta'$ or if $b' - \delta' < t < b'$, and $|\zeta'(t)| \leqq C/(\delta - \delta')$. $v = \zeta u^{(m)}$ satisfies

$$\frac{1}{i}\frac{dv}{dt} - A(t)v = \zeta\varphi_m + i\zeta' u^{(m)} .$$

If $b' - a'$ is sufficiently small then we can apply (2.3), (2.5) (with $m = 0$) and thus obtain, if $\delta = \delta'(1 + 1/m)$ and if H is sufficiently large (independently of m, δ),

$$| A(s)u |_{m,\delta} + | u |_{m+1,\delta} \leqq C \frac{H_0 H^m}{\delta^{m+1}} (m+1)! \leqq \frac{H_0 H^{m+1}}{\delta^{m+1}} (m+1)! ;$$

use has been made of the inequalities

$$| A^{(n)}(t)A^{-1}(s) |_0 + | f^{(n)} |_0 \leqq (\text{const.})^{n+1} n! .$$

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