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PSEUDOCOMPACTNESS AND UNIFORM CONTINUITY IN TOPOLOGICAL GROUPS

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This work contains a number of theorems about pseudocompact groups. Our first and most useful theorem allows us to decide whether or not a given (totally bounded) group is pseudocompact on the basis of how the group sits in its Weil completion. A corollary, which permits us to answer a question posed by Irving Glicksberg (Trans. Amer. Math. Soc. 90 (1959), 369–382) is: The product of any set of pseudocompact groups is pseudocompact. Following James Kister (Proc. Amer. Math. Soc. 13 (1962), 37–40) we say that a topological group G has property U provided that each continuous function mapping G into the real line is uniformly continuous. We prove that each pseudocompact group has property U.

Sections 2 and 3 are devoted to solving the following two problems: (a) In order that a group have property U, is it sufficient that each bounded continuous real-valued function on it be uniformly continuous? (b) Must a nondiscrete group with property U be pseudocompact? Theorem 2.8 answers (a) affirmatively. Question (b), the genesis of this paper, was posed by Kister (loc. cit.). For a large class of groups the question has an affirmative answer (see 3.1); but in 3.2 we offer an example (a Lindelöf space) showing that in general the answer is negative.

Much of the content of this paper is summarized by Theorem 4.1, in which we list a number of properties equivalent to pseudocompactness for topological groups. We conclude with an example of a metrizable, non totally bounded Abelian group on which each uniformly continuous real-valued function is bounded.

Conventions and definitions. All topological groups considered here are assumed to be Hausdorff. The algebraic structure of the groups we consider is virtually immaterial; in particular, our groups are permitted to be non-Abelian.

A topological group G is said to be totally bounded if, for each neighborhood U of the identity, a finite number of translates of U covers G. It has been shown in [10] by Weil that each totally bounded group is a dense topological subgroup of a compact group and that this compactification is unique to within a topological isomorphism leaving G fixed pointwise. We refer to this compactification of G as the Weil

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completion of G, and we reserve the symbol $\bar{G}$ to denote it.

Kister's property U was defined in the summary above. In the same vein, we say that a topological group has property BU if each bounded continuous real-valued function on G is uniformly continuous. The uniform structure on G referred to implicitly in the definitions of properties U and BU should be taken to be either the left uniform structure, defined as in 4.11 of [7], or the right uniform structure. It often happens that these structures do not coincide, and in this case there is a left uniformly continuous real-valued function on G which is not right uniformly continuous. Nevertheless it is easy to see that every [bounded] continuous real-valued function on G is left uniformly continuous if and only if every [bounded] continuous real-valued function on G is right uniformly continuous. Hence the definitions of properties U and BU are unambiguous.

Our topological vocabulary is that of the Gillman-Jerison text [5]. The following definition, which is useful in § 2, is in consonance with 4J of [5]: A topological space is a P-space provided that each of its G_δ subsets is open.

1. Pseudocompact groups. The Weil completion of a topological group plays a fundamental rôle in many of the arguments which follow. Our first result shows that each pseudocompact group admits such a completion.

THEOREM 1.1. *Each pseudocompact group is totally bounded.*

Proof. If the topological group G is not totally bounded, then there is a neighborhood U of the identity e in G and a sequence $\{x_k\}$ of points in G for which

$$x_k \notin \bigcup_{n<k} x_n U$$

for all k. We choose a symmetric neighborhood V of e for which $V^4 \subset U$, and we select for each positive integer k a nonnegative continuous function f_k on G such that

$$f_k(x_k) = k \text{ and } f_k \equiv 0 \text{ off } x_k V .$$

Using the local finiteness of the sequence $\{x_k V\}$, it is easy to check that the real-valued function f defined on G by the relation

$$f(x) = \sum_{k=1}^{\infty} f_k(x)$$

is continuous. Since f is unbounded, the group G is not pseudocompact.

Discussion. Although pseudocompact groups have not (so far as we can determine) been in their own right an object of detailed study, various authors have considered an example in one connection or another. If $\{X_\alpha\}_{\alpha \in A}$ is a set of separable metric spaces — which we here take to be topological groups — then the set

$$Y = \{x \in \Pi X_\alpha : x_\alpha \text{ is the identity in } X_\alpha \text{ for all but countably many } \alpha \text{ in } A\}$$

is an example of what Corson in [3] calls a Σ-space. Corson shows in his Theorem 2 that each continuous real-valued function on Y admits a continuous extension to ΠX_α. It follows that if each X_α is compact, then Y is pseudocompact and ΠX_α is the Stone-Čech compactification of Y. This and other interesting results were obtained (also in the product-space context) by Glicksberg in [6]. Kister examined in [8] the case in which each X_α is a compact topological group.

Like every pseudocompact space, the Σ-space Y defined above meets each nonempty G_δ subset of its Stone-Čech compactification. The appropriate group-theoretic analogue of this topological characterization of pseudocompactness is given in the following theorem. The reader will notice instantly that this theorem yields information about the Stone-Čech compactification of a pseudocompact group; we shall incorporate this observation into Theorem 4.1.

The Baire sets in a topological space X are those subsets of X belonging to the smallest σ-algebra containing all zero-sets in X.

THEOREM 1.2. *Let G be a totally bounded group and let*

$$\mathscr{N} = \{N : N \text{ is a closed, normal subgroup of } \bar{G} \text{ and } N \text{ is a } G_\delta \text{ set in } \bar{G}\}.$$

Then the following assertions are equivalent:

(a) *G is pseudocompact;*

(b) *each translate of each element of $\mathscr{N}$ meets G;*

(c) *each nonempty Baire subset of $\bar{G}$ meets G;*

(d) *each nonempty G_δ subset of $\bar{G}$ meets G;*

(e) *each continuous real-valued function on G admits a continuous extension to $\bar{G}$.*

Proof. (a) $\Rightarrow$ (b). If (b) fails, then $x_0 N \cap G = \varnothing$ for some x_0 in $\bar{G}$ and some N in $\mathscr{N}$. Since N is clearly not open, the quotient group $\bar{G}/N$ is infinite. Like any compact, first countable group, $\bar{G}/N$ is metrizable. Choosing an unbounded real-valued continuous function f on $\bar{G}/N \backslash \{x_0 N\}$ and defining g on $\bar{G}\backslash x_0 N$ by the relation

$$g(x) = f(xN),$$

we see that g is unbounded and continuous. The restriction of g to G is unbounded, and hence (a) fails.

(b) $\Rightarrow$ (c). This implication follows trivially from the following fact, a special case of Lemma 2.4 of [9]: If E is a Baire subset of $\bar{G}$, then $E = EN$ for some N in $\mathscr{N}$.

(c) $\Rightarrow$ (d). Since $\bar{G}$ is completely regular, each nonempty G_δ subset of $\bar{G}$ contains a nonempty zero-set of $\bar{G}$.

(d) $\Rightarrow$ (b). This is clear.

(b) $\Rightarrow$ (e). Let f be a real-valued continuous function on G, and let $\mathscr{B}$ be a countable base for the topology on the line. For each B in $\mathscr{B}$ there is clearly an open subset U_B of $\bar{G}$ for which

$$f^{-1}(B) = U_B \cap G .$$

By 1.6 and 2.4 of [9], there is an element N_B of $\mathscr{N}$ for which

$$cl_{\bar{G}} U_B = N_B \cdot cl_{\bar{G}} U_B .$$

Setting $N = \bigcap_{B \in \mathscr{B}} N_B$, we clearly have $N \in \mathscr{N}$ and $cl_{\bar{G}} U_B = N \cdot cl_{\bar{G}} U_B$ for each B in $\mathscr{B}$.

We next prove:

(*) If $x_1 \in G$, $x_2 \in G$, and $x_1^{-1}x_2 \in N$, then $f(x_1) = f(x_2)$.

If (*) fails, we can find neighborhoods B_1 and B_2 of $f(x_1)$ and $f(x_2)$ respectively such that $B_1 \in \mathscr{B}$, $B_2 \in \mathscr{B}$, and $clB_1 \cap clB_2 = \varnothing$. Since f is continuous on G, we have

$$cl_G f^{-1}(B_1) \cap cl_G f^{-1}(B_2) = \varnothing ,$$

i.e.,

$$cl_G(U_{B_1} \cap G) \cap cl_G(U_{B_2} \cap G) = \varnothing .$$

Now $x_1 \in N \cdot cl_G(U_{B_1} \cap G)$; hence

$$x_2 \in N \cdot cl_G(U_{B_1} \cap G) \subset N \cdot cl_{\bar{G}}(U_{B_1}) = cl_{\bar{G}}(U_{B_1}) ,$$

so that $x_2 \in cl_G(U_{B_1} \cap G)$. Of course $x_2 \in cl_G(U_{B_2} \cap G)$, and this contradiction completes the proof of (*).

With (*) and hypothesis (b) at our disposal, it is easy to define an extension $\bar{f}$ of f: given x_0 in $\bar{G}$, we choose any x in $x_0 N \cap G$ and set $\bar{f}(x_0) = f(x)$.

To check the continuity of $\bar{f}$ at an arbitrary point x_0 in $\bar{G}$, we choose $\varepsilon > 0$. We will produce a neighborhood U of the identity in $\bar{G}$ with the property that $|\bar{f}(x_0) - \bar{f}(y_0)| < \varepsilon$ whenever $y_0 \in x_0 U$. Indeed, choose $x \in x_0 N \cap G$ and let V be a neighborhood of the identity in $\bar{G}$ such that

$$|f(x) - f(y)| < \varepsilon \text{ whenever } y \in xV \cap G .$$

Now let U be any neighborhood of the identity in $\bar{G}$ for which $U^2 \subset V$. It is easy to see (directly, or by 8.7 of [7]) that there is an M in $\mathscr{N}$ such that $M \subset U \cap N$. Now for any point y_0 in $x_0 U$ there is (again by hypothesis (b)) a point z in $(xx_0^{-1}y_0 M) \cap G$. Since $z \in xx_0^{-1}y_0 N \subset Ny_0 N = y_0 N$, we have $\bar{f}(y_0) = f(z)$. And since $z \in xx_0^{-1}y_0 M \subset xUM \subset xU^2 \subset xV$, it follows that $|\bar{f}(x_0) - \bar{f}(y_0)| = |f(x) - f(z)| < \varepsilon$. Hence U is as desired and $\bar{f}$ is continuous at x_0.

(e) $\Rightarrow$ (a). Since every continuous real-valued function with domain $\bar{G}$ is bounded, this implication is obvious.

1.3. *Discussion.* If $(G_0, \mathscr{T})$ is a compact group and $\mathscr{N}$ denotes the family of subgroups of G_0 defined as in the hypothesis of Theorem 1.2, then the collection of translates of elements of $\mathscr{N}$ clearly constitutes a base for a P-space topology $\mathscr{P}$ on G_0. Since any G_δ set in G_0 that contains the identity must contain a member of $\mathscr{N}$, $\mathscr{P}$ is the smallest P-space topology containing $\mathscr{T}$. In fact,

$$\mathscr{P} = \{U : U \text{ is a countable intersection of } \mathscr{T}\text{-open subsets of } G_0\} .$$

Using these observations and 1.2, we have the following fact: A (dense) subgroup G of G_0 is pseudocompact if and only if G is $\mathscr{P}$-dense in G_0.

Gillman and Jerison present in 9.15 of [5] an example (due to Novák-Terasaka) of a pseudocompact space X for which $X \times X$ is not pseudocompact. In the positive direction, a number of authors (see especially [6] and [4]) have given various conditions on a family of pseudocompact spaces sufficient to ensure that the product be pseudocompact.

THEOREM 1.4. *The product of any set of pseudocompact groups is pseudocompact.*

Proof. Let the set A index the family $\{G_\alpha\}_{\alpha \in A}$ of pseudocompact groups, and let

$$G = \prod_{\alpha \in A} G_\alpha .$$

The uniqueness aspect of Weil's theorem assures us that the compact group $\prod_{\alpha \in A} \overline{G_\alpha}$ is (homeomorphic with) $\bar{G}$. According to 1.2, then, we need only show that each nonempty G_δ subset of $\prod_{\alpha \in A} \overline{G_\alpha}$ hits G.

Let U be such a set, say $U = \bigcap_{n=1}^{\infty} U_n$ where each U_n is a basic set of the form

$$U_n = \prod_{\alpha \in A} U_{n,\alpha} ;$$

here each $U_{n,\alpha}$ is open in $\overline{G_\alpha}$, and for each n we have $U_{n,\alpha} = \overline{G_\alpha}$ for all but finitely many α in A. Let

$$V_\alpha = \bigcap_{n=1}^{\infty} U_{n,\alpha} .$$

Then V_α is a nonempty G_δ set in $\overline{G_\alpha}$, and thus by 1.2 there is a point x_α in $V_\alpha \cap G_\alpha$. Evidently the point of $\bar{G}$ whose α coordinate is x_α lies in $U \cap G$.

In what follows we will consider at length Kister's question "Must a nondiscrete group with property U be pseudocompact?" We now quickly handle the converse question.

THEOREM 1.5. *Every pseudocompact group has property U.*

Proof. If f is a continuous real-valued function on the pseudocompact group G, then by 1.1 and (a) $\Rightarrow$ (e) of 1.2, f admits a continuous extension $\bar{f}$ on $\bar{G}$. Since $\bar{f}$ is uniformly continuous on $\bar{G}$, it follows that f is uniformly continuous on G.

2. **Property BU implies property U.** This theorem is proved in 2.8. Our key lemma is 2.2.

LEMMA 2.1. *If the topological group G is not a P-space, then some nonempty G_δ subset H of G has no interior. The set H may be chosen to be a closed subgroup.*

Proof. There is a sequence $\{V_k\}$ of neighborhoods of e for which $e \notin \operatorname{int} \bigcap_{k=1}^{\infty} V_k$. Selecting a sequence $\{U_k\}$ of symmetric neighborhoods of e such that $U_{k+1}^2 \subset U_k \cap V_k$ and defining $H = \bigcap_{k=1}^{\infty} U_k$, we see (directly, or from 5.6 of [7]) that the G_δ set H is a closed subgroup of G. Being a subgroup that is not open, H has no interior.

THEOREM 2.2. *If the topological group G has property BU, then G is totally bounded or G is a P-space.*

Proof. Suppose the conclusion fails. Since G is not a P-space, there is a sequence $\{U_k\}$ of neighborhoods of e for which $\operatorname{int} \bigcap_{k=1}^{\infty} U_k = \varnothing$. Since G is not totally bounded, there is, just as in the proof of 1.1, a neighborhood V of e and a sequence $\{x_k\}$ of points in G such that the sequence $\{x_k V\}$ is locally finite and pairwise disjoint.

For each integer k there is a continuous function f_k on G for which $f_k(x_k) = 1, f_k \equiv 0$ off $x_k(V \cap U_k)$, and $0 \leqq f_k \leqq 1$. The function $f = \sum_{k=1}^{\infty} f_k$ is bounded and continuous on G, and hence is (left) uniformly continuous. Thus there is a neighborhood W of e for which $|f(x) - f(y)| < 1$ whenever $x^{-1}y \in W$. We may take $W \subset V$. Since int $W \neq \emptyset$, we cannot have $W \subset \bigcap_{k=1}^{\infty} U_k$. Thus there is an integer m and a point p for which $p \in W \backslash U_m$. Now $x_m^{-1}(x_m p) \in W$, so that $|1 - f(x_m p)| = |f(x_m) - f(x_m p)| < 1$ and we have $f(x_m p) \neq 0$. Thus $x_m p \in \bigcup_k x_k(V \cap U_k)$. Since $x_m V \cap x_k V = \emptyset$ whenever $k \neq m$, we must have $x_m p \in x_m(V \cap U_m)$. But then $p \in U_m$, a contradiction completing the proof.

Our next result, used in the proof of 2.4, is given here in considerable generality because of its application in connection with Example 3.2.

THEOREM 2.3. *Let the topological group G be a P-space. Then the following are equivalent:*

(a) *G has property U;*

(b) *G has property BU;*

(c) *the characteristic function of every open-and-closed subset of G is uniformly continuous.*

Proof. Only the implication (c) $\Rightarrow$ (a) requires proof. Given a continuous real-valued function f on G, we note that for each rational pair (a, b), with $a \leqq b$, the set $f^{-1}([a, b])$ is closed; being a G_δ set in G, this set is also open. Since the characteristic function $\psi_{f^{-1}([a, b])}$ is left uniformly continuous, there is a neighborhood $U_{a,b}$ of e such that $x^{-1}y \in U_{a,b}$ implies $|\psi_{f^{-1}([a, b])}(x) - \psi_{f^{-1}([a, b])}(y)| < 1$. That is, $x^{-1}y \in U_{a,b}$ implies that $x \in f^{-1}([a, b])$ if and only if $y \in f^{-1}([a, b])$. Let $U = \cap \{U_{a,b} : a, b$ rational and $a \leqq b\}$; then U is a neighborhood of e since G is a P-space. To establish the left uniform continuity of f it will clearly suffice to show that $f(x) = f(y)$ whenever $x^{-1}y \in U$. Suppose then that $x^{-1}y \in U$ and that $f(x) = p$. For appropriate sequences $\{a_k\}$ and $\{b_k\}$ of rational numbers, we have $\{p\} = \bigcap_k [a_k, b_k]$. Then $x \in f^{-1}([a_k, b_k])$ for all k. Since $x^{-1}y \in U_{a_k, b_k}$ for all k, we have $y \in \bigcap_k f^{-1}([a_k, b_k]) = f^{-1}(\{p\})$ and $f(y) = p = f(x)$.

COROLLARY 2.4. *If the topological group G has property BU and is not totally bounded, then G has property U.*

Proof. By Theorem 2.2, G is a P-space. The result now follows from 2.3.

Corollary 2.4 gives an affirmative answer to problem (a) of the introduction for groups which are not totally bounded. The trick which handles the totally bounded situation consists, roughly speaking, in reducing to the metrizable case (where the proof is easy).

LEMMA 2.5. *If a topological group G is metrizable and has property BU, then G is compact or discrete.*

Proof. This is immediate from Atsuji's Theorem 1 in [1]. For a direct proof (by contradiction), assume otherwise and note that by Theorem 2.2, G must be totally bounded. Since G is not compact, G is not complete. Hence there is a nonconvergent Cauchy sequence $\{x_k\}$ in G. By Tietze's theorem the function mapping x_k to $(-1)^k$ can be extended to a real-valued continuous function bounded on G, and this bounded function is obviously not uniformly continuous.

LEMMA 2.6. *If G is a topological group with property BU, and if H is a closed normal subgroup of G, then G/H has property BU.*

Proof. Let f be a bounded continuous real-valued function on G/H, and let $\varepsilon > 0$. Denoting by π the natural projection of G onto G/H, we note that $f \circ \pi$ is left uniformly continuous on G. Hence there is a neighborhood V of e for which

$$|f \circ \pi(x) - f \circ \pi(y)| < \varepsilon \text{ whenever } x^{-1}y \in V .$$

Of course $\pi(V)$ is a neighborhood of H in G/H. Now suppose that $(xH)^{-1}(yH) \in \pi(V)$. Then $x^{-1}yH = vH$ for some $v \in V$, so that $x^{-1}yh = v$ for some $h \in H$. Then $x^{-1}(yh) \in V$ and therefore

$$\begin{aligned} |f(xH) - f(yH)| &= |f \circ \pi(x) - f \circ \pi(y)| \\ &= |f \circ \pi(x) - f \circ \pi(yh)| < \varepsilon . \end{aligned}$$

That is, f is left uniformly continuous.

THEOREM 2.7. *Let G be a totally bounded group with property BU. Then G is pseudocompact.*

Proof. If G is not pseudocompact, then according to 1.2 there is a point p in $\bar{G}$ and a closed normal subgroup N of $\bar{G}$ such that $G \cap pN = \varnothing$ and $\bar{G}/N$ is metrizable. Since $pN \notin GN/N$ and GN/N is the continuous image of GN under the natural projection, GN/N is a dense proper subgroup of $\bar{G}/N$. Since a discrete subgroup of a topological group is closed (see 5.10 of [7]), it follows that GN/N is a nondiscrete, noncompact metrizable group.

It is clear that any group, one of whose dense subgroups has property BU, must itself have property BU. In particular the group GN, in which G is dense, has property BU. Hence GN/N has property BU by 2.6, and GN/N does not have property BU by 2.5. This contradiction completes the proof.

THEOREM 2.8. *A topological group has property BU if and only if it has property U.*

Proof. Use 2.4 and 2.7.

3. **Kister's question.** We first give a partial affirmative answer to the question posed by Kister in [8].

THEOREM 3.1. *If the topological group G has property BU and is not a P-space, then G is pseudocompact.*

Proof. The group G is totally bounded by 2.2, and hence is pseudocompact by 2.7.

EXAMPLE 3.2. We now give an example of a nondiscrete topological Abelian group that is a P-space and has property U. Such a group is clearly not pseudocompact: every pseudocompact P-space is finite. Hence this example shows that Kister's question mentioned in the summary has a negative answer.

Let A be an index set of cardinality $\aleph_1$ and let G consist of all elements x in $\prod_{\alpha\in A}\{1, -1\}_\alpha$ such that $x_\alpha = 1$ for all but finitely many coordinates α. Let Ω be the first uncountable ordinal and well-order A according to the order—type Ω: $A = \{\alpha : \alpha < \Omega\}$. For $\alpha \in A$, let

$$H_\alpha = \{x \in G : x_\beta = 1 \text{ for all } \beta < \alpha\} .$$

We decree that the subgroups H_α and each of their translates be open and thereby obtain a basis for a topology under which G is a topological group. Clearly G is a P-space and G is not discrete.

We shall show that G has property U. By Theorem 2.3 we need show only that the characteristic function ψ_W of an open-and-closed set W is uniformly continuous. For $\alpha \in A$, let $W_\alpha = \bigcup \{xH_\alpha : xH_\alpha \subset W\}$. Evidently $\{W_\alpha\}_{\alpha<\Omega}$ is a nondecreasing family of open-and-closed sets, and $\bigcup_{\alpha<\Omega} W_\alpha = W$. Since $\psi_{W_\alpha}(x) = \psi_{W_\alpha}(y)$ whenever $x^{-1}y \in H_\alpha$, the characteristic function of each W_α is uniformly continuous. Hence it suffices to show that $W = W_\alpha$ for some α.

Assume that $W \neq W_\alpha$ for all α, and let

$$V_\alpha = \bigcup \{xH_\alpha : xH_\alpha \cap W \neq \emptyset \text{ and } xH_\alpha \cap (G\backslash W) \neq \emptyset\} .$$

It is easy to see that each V_α is nonvoid and that $V_\alpha \supset V_\gamma$ whenever $\alpha < \gamma < \Omega$. It suffices now to prove that $\bigcap_{\alpha<\Omega} V_\alpha$ is nonvoid, since any element in this intersection belongs to the closures of both W and $G\backslash W$, contrary to the supposition that W is open-and-closed.

We prove that $\bigcap_{\alpha<\Omega} V_\alpha$ is nonvoid. For x in G and α in A, we define $N(x, \alpha)$ to be the number of elements in the finite set $\{\beta \in A : \beta < \alpha$ and $x_\beta = -1\}$. For $\alpha \in A$, we define

$$n_\alpha = \inf_{x\in V_\alpha} N(x, \alpha) \text{ and } J_\alpha = \{x \in V_\alpha : N(x, \alpha) = n_\alpha\} .$$

Clearly $\emptyset \neq J_\alpha \subset V_\alpha$ for all α. The integer-valued transfinite sequence $\{n_\alpha\}_{\alpha<\Omega}$ is nondecreasing because for $\alpha < \gamma$ we have

$$n_\alpha = \inf_{x\in V_\alpha} N(x, \alpha) \leqq \inf_{x\in V_\alpha} N(x, \gamma) \leqq \inf_{x\in V_\gamma} N(x, \gamma) = n_\gamma .$$

It follows that the sequence $\{n_\alpha\}_{\alpha<\Omega}$ is eventually constant. There is, then, an integer n_0 and an α_0 in A for which $n_\alpha = n_0$ whenever $\alpha \geqq \alpha_0$. We next show that $\{J_\alpha\}_{\alpha\geqq\alpha_0}$ is a nonincreasing family of sets. Suppose that $\alpha_0 \leqq \alpha < \gamma$ and $z \in J_\gamma$. Then $z \in J_\gamma \subset V_\gamma \subset V_\alpha$. Since also

$$n_\alpha \leqq N(z, \alpha) \leqq N(z, \gamma) = n_\gamma = n_0 = n_\alpha ,$$

we see that $z \in J_\alpha$. Now let Y consist of all elements y of G such that $y_\beta = 1$ for all $\beta \geqq \alpha_0$. Then Y is a countable set. Assume now that $\bigcap_{\alpha<\Omega} V_\alpha = \emptyset$, so that $\bigcap_{\alpha\geq\alpha_0} J_\alpha = \emptyset$. Then for each y in Y there is an $\alpha_y \geqq \alpha_0$ such that $y \notin J_{\alpha_y}$. Selecting γ_0 in A larger than each α_y, we find that $Y \cap J_{\gamma_0} = \emptyset$. Now choose z in J_{γ_0}. Then z also belongs to J_{α_0}, so that $N(z, \gamma_0) = N(z, \alpha_0) = n_0$. Hence $z_\beta = 1$ for $\alpha_0 \leqq \beta < \gamma_0$. Define w so that $w_\beta = z_\beta$ for $\beta < \gamma_0$ and $w_\beta = 1$ for $\beta \geqq \gamma_0$. Clearly w belongs to Y. Since $z \in J_{\gamma_0} \subset V_{\gamma_0}$, we have $w \in zH_{\gamma_0} \subset V_{\gamma_0}$. Also $N(w, \gamma_0) = N(z, \gamma_0) = n_0$, so that $w \in J_{\gamma_0}$. That is, w belongs to $Y \cap J_{\gamma_0}$, contrary to the relation $Y \cap J_{\gamma_0} = \emptyset$. Thus $\bigcap_{\alpha<\Omega} V_\alpha \neq \emptyset$, and we conclude that G has property U.

REMARK. It may be interesting to note that the group discussed above is Lindelöf (and hence normal). To see this, assume that $\mathscr{U}$ is a cover of G by basic open sets, and that $\mathscr{U}$ admits no countable subcover. For $\alpha \in A$, let $\mathscr{U}_\alpha$ consist of all elements of $\mathscr{U}$ which are translates of some H_β where $\beta \leqq \alpha$. Since each $\mathscr{U}_\alpha$ is countable, no $\mathscr{U}_\alpha$ is a cover for G. Let $U_\alpha = \bigcup \mathscr{U}_\alpha$. Then $\{U_\alpha\}_{\alpha<\Omega}$ is a nondecreasing sequence of proper subsets of G, and each U_α is a union of cosets of H_α. Let $V_\alpha = G\backslash U_\alpha$. As in Example 3.2 above, we have $\bigcap_{\alpha<\Omega} V_\alpha \neq \emptyset$: hence $\mathscr{U}$ does not cover G.

One may wonder whether Example 3.2 is typical of topological groups that are P-spaces: Do all topological groups that are P-spaces satisfy property U? The next theorem and the examples following it make Example 3.2 appear atypical.

THEOREM 3.3. *Let G be a nondiscrete topological group. If G admits a base $\mathscr{H}$ at the identity consisting of open subgroups such that card $(G/K) \geqq$ card $\mathscr{H}$ for some K in $\mathscr{H}$, then G does not have property U.*

Proof. We may clearly suppose that $H \subset K$ for all H in $\mathscr{H}$. By the cardinality hypothesis, there exists a subset $\{x_H\}_{H\in\mathscr{H}}$ of G, indexed by $\mathscr{H}$, such that $\{x_H K\}_{H \in \mathscr{H}}$ is a family of distinct cosets of K. Let $W = \bigcup_{H \in \mathscr{H}} x_H H$. Clearly W is open, and W is closed because

$$G \backslash W = \Big(G \backslash \bigcup_{H \in \mathscr{H}} x_H K\Big) \cup \bigcup_{H \in \mathscr{H}} (x_H K \backslash x_H H) .$$

Therefore ψ_W is continuous; we next show that ψ_W is not left uniformly continuous. Indeed, suppose that there exists an $H_0 \in \mathscr{H}$ such that $x^{-1}y \in H_0$ implies $x \in W$ if and only if $y \in W$. Since G is nondiscrete, there exists an H in $\mathscr{H}$ such that $H \subset H_0$ and $H \neq H_0$. If y is chosen so that $y \in x_H H_0 \backslash x_H H$, then $y \in x_H K \backslash x_H H \subset G \backslash W$. Since $x_H^{-1} y \in H_0$, we also have $x_H \in G \backslash W$. This contradicts the fact that $x_H \in x_H H \subset W$.

EXAMPLES 3.4. Let μ be a cardinal number less than the first strongly inaccessible cardinal[1]. Let G be the algebraic group $\{1, -1\}^\mu = \prod_{\alpha \in A} \{1, -1\}_\alpha$, where the index set A is ordered according to the least ordinal having cardinality μ. Let the subgroups

$$H_\alpha = \{x \in G : x_\beta = 1 \text{ for all } \beta < \alpha\}$$

and all their translates be a basis for a topology on G.

If ν denotes the smallest cardinal number which is the cardinal number of some cofinal subset of A, then evidently ν is the minimal cardinality of a base at the identity of G. If μ is chosen so that $\nu > \aleph_0$, then the nondiscrete topology imposed upon G is clearly a P-space topology, and under the condition $\nu > \aleph_0$ we can show that G does not have property U.

To do this, suppose first that $2^\kappa < \nu$ whenever $\kappa < \nu$. Then (from 12.4–12.6 of [5]) there is a set $\{\nu_\lambda\}_{\lambda \in \Lambda}$ of cardinal numbers such that card $\Lambda < \nu$, $\nu_\lambda < \nu$ for each λ in Λ, and $\sup \nu_\lambda = \nu$. Since there is then a cofinal set $\{\alpha_\lambda : \lambda \in \Lambda\}$ in A indexed by Λ, contrary to the minimality of ν, we conclude that $2^\kappa \geqq \nu$ for some $\kappa < \nu$. Now let $\mathscr{H}$ be a basis of open subgroups at the identity for which card $\mathscr{H} = \nu$, and choose $\beta \in A$ so that $H_\beta \in \mathscr{H}$ and card $\{\alpha \in A : \alpha < \beta\} \geqq \kappa$. Then

$$\text{card}\,(G/H) \geqq 2^\kappa \geqq \nu ,$$

so that G does not have property U by 3.3.

4. Related concepts. Much of our earlier work is summarized in the following theorem. The symbol βG denotes the Stone-Čech compactification of the (completely regular) space G; it is, to within a homeomorphism leaving G fixed pointwise, the only compactification

[1] A cardinal number is said to be strongly inaccessible if it is an uncountable cardinal whose set of predecessors is closed under the standard operations of cardinal arithmetic. It is not known whether any strongly inaccessible cardinal number exists.

of G to which each bounded continuous real-valued function on G admits a continuous extension. The amusing suggestion that G might induce a topological group structure on βG is not original with us: The appearance of this phenomenon was explicitly pointed out in [6] by Glicksberg in connection with the Corson Σ-space mentioned earlier.

The implication (b) $\Rightarrow$ (g) of 4.1 below was given in [6], and Glicksberg asked whether or not the implication (a) $\Rightarrow$ (g) is valid. Our proof of its validity does not depend upon the results of [6].

If the identity in a topological group G admits a neighborhood U which is bounded (in the sense that for each nonempty open subset V of G there is a finite set F such that $U \subset FV$), then G is said to be locally bounded.

We remark finally that additional conditions equivalent to those listed below may be obtained by replacing the expression "G has property U" when it appears by the expression "G has property BU."

THEOREM 4.1. *For a topological group G, conditions* (a) *through* (g) *are equivalent, and each implies* (h). *If in addition G is nondiscrete, then all eight conditions are equivalent.*

(a) *G is pseudocompact;*

(b) *$G \times G$ is pseudocompact;*

(c) *G is pseudocompact and has property U;*

(d) *G is totally bounded and has property U;*

(e) *G is totally bounded and $\beta G = \bar{G}$;*

(f) *βG admits a topological group structure relative to which the inclusion mapping of G into βG is a topological isomorphism;*

(g) *every continuous real-valued function on G is almost periodic;*

(h) *G is locally bounded and has property U.*

Proof. Theorem 1.4 gives the implication (a) $\Rightarrow$ (b), and the converse follows from the fact that the continuous image of a pseudocompact space is pseudocompact. The implications (a) $\Rightarrow$ (c), (c) $\Rightarrow$ (d), and (d) $\Rightarrow$ (a) are 1.5, 1.1, and 2.7 respectively, while the implication (a) $\Rightarrow$ (e) follows from 1.1 and the implication (a) $\Rightarrow$ (e) of 1.2. That (e) $\Rightarrow$ (f) is obvious, and the implication (f) $\Rightarrow$ (d) follows from 2.8.

We have shown so far that the first six conditions listed are equivalent.

To deduce (g), suppose that (a) and (e) hold and let f be any continuous real-valued function on G. Being bounded, f admits a continuous real-valued extension to βG. A routine computation, based on the fact that every continuous real-valued function on the compact group βG is almost periodic on βG, shows that f is almost periodic on G.

To see that (g) implies (a), let f be any continuous real-valued function on G and let F be a finite subset of G with the property that for each x in G there exists y in F such that $|f(xz) - f(yz)| < 1$ whenever $z \in G$. Then for each x in G we have

$$|f(x)| = |f(xe)| < \max_{y \in F} |f(y)| + 1 .$$

Since the implication (d) $\Rightarrow$ (h) is obvious, we may complete the proof by supposing that G is nondiscrete and deducing (d) from (h). If (h) holds but (d) fails, then G is a P-space by 2.2. Let U be a bounded neighborhood of e and let $\{x_k\}$ be an infinite set of distinct points in U. For each pair (m, n) of distinct positive integers there is a neighborhood $V_{m,n}$ of the identity such that $x_m \notin x_n V_{m,n}$. Choosing a symmetric neighborhood V of the identity such that

$$V^2 \subset \bigcap_{(m,n)} V_{m,n} ;$$

we see easily that no set of the form xV can contain more than one of the points x_k. Thus there exists no finite subset F of G for which $U \subset FV$.

In the discussion and example which follow we will say that a uniform space on which each real-valued uniformly continuous function is bounded has property *UB*. Clearly any totally bounded uniform space has property *UB*, and Atsuji gives in [1] an example of a connected metric space that is not totally bounded but which has property *UB*. Further metric examples are given in exercises 15.D and 15.L of [5]. Although Atsuji in Theorem 7 of [2] characterizes uniform spaces with property *UB* by means of a chainability condition, the following question has not so far as we can determine been treated in the literature: Must a topological group with property *UB* be totally bounded? We now answer this question in the negative.

EXAMPLE 4.2. Let T denote the circle group and let G be the algebraic group $T^{\aleph_0} = \prod_{k=1}^{\infty} T_k$. Defining

$$d(x, y) = \sup_k |x_k - y_k|$$

for each pair of points x, y in G, we obtain a metric topology on G under which G is a topological group. To see that G has property *UB*, let f be a uniformly continuous real-valued function on G and find $\delta > 0$ such that $|f(x) - f(y)| < 1$ whenever $d(x, y) < \delta$. Choose an integer m so that, given any point t in T, there is a sequence $1 = t^0, t^1, \cdots, t^m = t$ in T such that $|t^{j+1} - t^j| < \delta/2$ for $0 \leqq j \leqq m - 1$. We will show that $|f(x)| \leqq |f(e)| + m$ for all x in G. For a fixed x in G, select for each integer $k > 0$ a sequence $1 = x_k^0, x_k^1, \cdots, x_k^m = x_k$ in T such that $|x_k^{j+1} - x_k^j| < \delta/2$. The finite sequence $x^0, x^1, \cdots, x^m$ in G has the property that

$$d(x^{j+1}, x^j) \leqq \delta/2 < \delta \quad \text{for} \quad 0 \leqq j \leqq m - 1 .$$

Hence $|f(x^{j+1}) - f(x^j)| < 1$ for $0 \leqq j \leqq m - 1$, so that $|f(x) - f(e)| \leqq m$. Thus G has property UB.

To see that G is not totally bounded, let W be the open set $\{x \in G : d(x, e) < 1/2\}$. Regarding G as the usual compact topological group $T^{\aleph_0}$ with its Haar measure, we see that the G_δ set W has Haar measure 0. It follows that no finite number of translates of W can cover G.

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