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ON A PROBLEM OF J. F. RITT

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In the Ritt algebra $R\{u, v\} = R(u_0, v_0, u_1, v_1, u_2, v_2, \cdots)$ where the derivation is such that $y_i' = y_{i+1}$ for $y = u$ or v, consider the differential ideal $\Omega = [uv] = ((uv), (uv)_1, (uv)_2, \cdots)$. Let $P = u_{i_1} \cdots u_{i_m} v_{j_1} \cdots v_{j_n}$ be a power product in u, v and their derivatives. For sufficiently large q, it is known that $P^q \equiv 0\ [uv]$. Power products of the form $u_i v_j$ are of particular interest; one of J. F. Ritt's unsolved problems is to find the smallest q such that $(u_i v_j)^q \equiv 0\ [uv]$. The purpose of this paper is to solve this problem in the special case $i = 1$. The main theorem is: The smallest q such that $(u_1 v_j)^q \equiv 0\ [uv]$ is $2 + j$. Part of the solution involves generalizing some results of D. G. Mead and part is an application of the well-known reduction process of H. Levi.

Basic Notions. H. Levi's reduction process [1] starts with replacing a factor $u_a v_b$ of $P = u_{i_1} \cdots u_{i_m} v_{j_1} \cdots v_{j_n} = u_a v_b Q$ by the other terms in $(uv)_{a+b}$. Thus we obtain the congruence

$$P \equiv - \sum_{\substack{i=0 \\ i \neq a}}^{a+b} \frac{\binom{a+b}{i}}{\binom{a+b}{a}} u_i v_{a+b-i} Q \ [uv] \, .$$

Repeated substitutions of this kind eventually permit P to be congruent to a linear combination of terms which are not in $[uv]$, unless all of the coefficients are zero. The unwieldy coefficients involved in this reduction process are simplified by M-congruences which may be introduced in the following way: Let S be the set

$$(a_0, a_1, a_2, \cdots;\ b_0, b_1, b_2, \cdots;\ c_0, c_1, c_2, \cdots)$$

and A the free algebra on S over R. Define an (algebra) homomorphism $h: A \to R\{u, v\}$ by $h(r) = r$ if $r \in R$, $h(a_i) = u_i/i!$, $h(b_i) = v_i/i!$, and $h(c_i) = (uv)_i/i!$. There exists an isomorphism of A/K into $R\{u, v\}$, where K is the kernel of h and this isomorphism becomes a differential isomorphism by defining $(y_i)_1 = (i+1)y_{i+1}$ for $y_i = a_i$, b_i or c_i and leaving the derivative in R unchanged. Now consider the algebra obtained by replacing a_i, b_i, and c_i by u_i, v_i and $(uv)_i$ respectively. The relation

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$$(uv)_n = \sum_{i=0}^{n} \binom{n}{i} u_i v_{n-i}$$

has become

$$(uv)_n = \sum_{i=0}^{n} u_i v_{n-i} .$$

An M-congruence in the original algebra $R\{u, v\}$ is an ordinary congruence in this isomorphic image of $R\{u, v\}$.

Throughout this paper, the notation and terminology of [1] and [2] are used. All congruences are M-congruences and are taken modulo $[uv]$.

2. **Minimum weight sequences.** Let $R\{u, v\}$ be a *Ritt* algebra in the indeterminates u and v, $\Omega = [uv]$ the *differential ideal* generated by the form $X = uv$, and $P = u_{i_1} \cdots u_{i_m} v_{j_1} \cdots v_{j_n}$ a *power product* of *weight* $w = \sum_1^m i_p + \sum_1^n j_q$, and *signature* (m, n). For each (m_i, n_i), with $1 \leqq m_i \leqq m$ and $1 \leqq n_i \leqq n$, take the minimum weight of all factors of P of signature (m_i, n_i). Subtract $m_i n_i$ from that minimum. The result is one number for each pair (m_i, n_i). The set of these *excess weights* is called the *weight matrix* of P. A theorem of *H.* Levi [1] says that if there is a negative entry in the weight matrix of P, then P is in $[uv]$. To reduce the number of excess weights that need to be computed, the idea of an *ordered* power product and the concept of a *minimum weight sequence* are introduced.

An *ordered* power product Q is of the form: $Q = f_1 f_2 \cdots f_n$, where for each $i, i = 1, 2, \cdots, n$, $f_i = u_{d(i)}$ or $f_i = v_{d(i)}$. For each $m \leqq n$, $Q_m = f_1 \cdots f_m$ is an *initial factor* of Q, of, say, signature (k, l) and weight w. The *excess weight* of Q_m is $e_m = w - kl$. Thus Q has a sequence of excess weights of its initial factors, $\{e_1, \cdots, e_n\}$. The sequence $\{e_1, \cdots, e_n\}$ is called the *initial factor weight sequence* of Q.

Let P be any power product of signature (m, n). Let Q_d be any factor of P of signature (k, l), $k + l = d$ and smallest excess weight, w_d.

DEFINITION 2.1. The sequence $\{w_1, \cdots, w_{m+n}\}$ is the *minimum weight sequence* of P.

A simple method of constructing $\{w_1, \cdots, w_{m+n}\}$ is shown next.

LEMMA 2.2. $Q_d \mid Q_{d+1}$.

Proof. The u (and v) subscripts of Q_t, $t = 1, \cdots, m + n$, form a nondecreasing sequence of natural numbers; call them $r(i)$ and $s(i)$ respectively. Among all pairs (x, y), $x + y = d$, by hypothesis,

$$\sum_1^x r(i) + \sum_1^y s(i) - xy$$

is a minimum for $x = k$ and $y = l$. We claim that the corresponding minimum for $d + 1$ is attained by either $x = k + 1$, $y = l$ or $x = k$, $y = l + 1$. Consider the two expressions:

$$(1) \qquad \sum_1^{k+j+1} r(i) + \sum_1^{l-j} s(i) - (k + j + 1)(l - j)$$

$$(2) \qquad \sum_1^{k+j} r(i) + \sum_1^{l-j+1} s(i) - (k + j)(l - j + 1) \; .$$

Subtract (2) from (1). The difference

$$\Delta = r(k + j + 1) + k + 2j - s(l - j + 1) - l$$

should be positive for (1) to be larger for $j > 0$ than for $j = 0$.

Since w_d is a minimum,

$$r(k + 1) + (k + 1) \geqq s(l) + l \; .$$

Therefore, if $j > 0$, $2j > 1$, $r(k + j + 1) \geqq r(k + 1)$ and $s(l - j) \leqq s(l)$. Thus $\Delta > 0$ and Q_{d+1} must be $Q_d u_r$ or $Q_d v_s$.

LEMMA 2.3. *Q_{d+1} is $Q_d u_r$ or $Q_d v_s$ according as*

$$s - r + l - k > 0$$

or

$$s - r + l - k < 0 \; .$$

(For $s - r + l - k = 0$, either u_r or v_s yields a minimum.)

Proof. Let $P = Q_d u_r$ and the excess weight of P,

$$e_{d+1} = w_d + r - (k + 1)l \; .$$

Replace u_r by v_s and call the new excess weight e^*_{d+1},

$$e^*_{d+1} = w_d + s - (k)(l + 1) \; .$$

The change gives a smaller excess weight if and only if

$$e^*_{d+1} - e_{d+1} < 0 \; ,$$

or

$$s - r + l - k < 0 \; .$$

The following theorems are simple consequences of Lemmas 2.2 and 2.3, and Levi's theorem.

THEOREM 2.4. *For any power product P, there is an ordered power product P^* whose initial factor weight sequence is the minimum weight sequence of P.*

THEOREM 2.5. *If an entry in the minimum weight sequence of P is negative, then $P \equiv 0$ $[uv]$.*

3. Generalized α-terms. If $P = u_{i_1} \cdots u_{i_m} v_{j_1} \cdots v_{j_n}$ and $j_q \geqq m$ for $q = 1, 2, \cdots, n$, then P is called an *α-term.* For example, there is exactly one α-term of signature (m, n) and weight mn, namely $u_0^m v_m^n$. The initial factor weight sequence of $u_0^m v_m^n$ is its minimum weight sequence and consists entirely of zeros. In this section, a study is made of all power products with minimum weight sequences consisting entirely of zeros.

DEFINITION 3.1. A minimum weight sequence consisting entirely of zeros is called a *zero-sequence.*

DEFINITION 3.2. A power product whose minimum weight sequence is a zero-sequence is called a *generalized α-term.*

The power product $P = u_{i_1} \cdots u_{i_m} v_{j_1} \cdots v_{j_n}$ may be symbolically written $P = UV$. The *degree* of P, $d(P)$, is the sum of the degrees of U and V, $d(P) = m + n$. The weight of P, $w(P)$, is the sum of the weights of U and V, $w(P) = w(U) + u(V)$. $w(U)$ is called the *u-weight* of P.

LEMMA 3.3. *Let $P = UV$ be a generalized α-term of degree $d = m + n$. Then P has either u_n or v_m as a factor, but not both u_n and v_m as factors.*

Proof. By Theorem 2.4, let Q be an initial factor of P such that $d(Q) = d(P) - 1$ and the excess weight of Q is zero. By Lemma 2.3, either

$$Qu_r = P \tag{1}$$

or

$$Qv_s = P\,. \tag{2}$$

If (1) holds, then $w(Q) = mn - r$. Also, $w(Q) = (m - 1)n$, hence

$r = n$. *If* (2) holds, by similar reasoning, $m = s$.

Suppose that u_n and v_m are both factors of P. Then let $T = P/u_n v_m$. The excess weight of T is -1 and P can not be a generalized α-term.

THEOREM 3.4. *If P and P^* are generalized α-terms of degree d such that $P = UV$ and $P^* = UV^*$, then $V = V^*$.*

Proof. The proof is by induction on d. For $d = 1$, $V = V^* = 1$ if $U = u_0$, and if $U = 1$, $V = V^* = v_0$. Assume that the theorem holds for power products whose degree is less than d. By Lemma 3.3, two cases may be distinguished:

Case 1. u_n is a factor of U,

Case 2. u_n is not a factor of U.

In Case 1, consider $Q = P/u_n$ and $Q^* = P^*/u_n$. The degree of Q and Q^* is less than d and the induction hypothesis may be applied to Q and Q^* yielding the conclusion that $V = V^*$. In Case 2, v_m must be a factor of V and V^*. The induction hypothesis may then be applied to $R = P/v_m$ and $R^* = P^*/v_m$, yielding the conclusion that $V/v_m = V^*/v_m$; and hence, V and V^* are the same.

THEOREM 3.5. *If A is a generalized α-term of signature (m, n), then*

$$A \equiv (-1)^J u_0^m v_m^n ,$$

where J is the u-weight of A.

Proof. The proof is by induction on J. If $J = 0$, A is an α-term and the theorem is true. By the theorem of [3], assume that A involves u_0 and let

$$A = u_0^a v_a^b u_b^c \cdots .$$

By Lemma 1 of [3], with the roles of u and v interchanged, $i = a$, $j = b$, and $P = u_0^a v_a^{b-1}$,

$$A \equiv -u_0^a v_a^{b-1} u_{b-1} v_{a+1} u_b^{c-1} \cdots .$$

The term on the right is a generalized α-term and has u-weight one less than the u-weight of A. Using the induction hypothesis, the proof may be completed.

Since $u_0^m v_m^n$ is not in $[uv]$ by [1], the following remark summarizes this section.

THEOREM 3.6. *No generalized α-term is in* $[uv]$.

4. A reduction theorem. In ([2], *p.* 429), D. G. Mead has several theorems for $[y^2]$, which, for practical purposes, reduce the degree of the power product under consideration. Similar results may be developed for $[uv]$, with minor complications due to the asymmetrical definition of an α-term. With the help of the symmetry theorems of [4], or [3], proper reduction theorems may be obtained.

For the rest of the paper, adopt the notation of [2]. Let P be any power product of excess weight zero and A the unique α-term of the same weight and signature as P. The M-congruence $P \equiv cA$ is also written $m(P) = c$.

LEMMA 4.1. *Let* $P = u_{i_1} \cdots u_{i_m} v_{j_1} \cdots v_{j_n}$. *Then*

$$(1) \qquad m(P) = m(u_0 u_{i_1} \cdots u_{i_m} v_{j_1+1} v_{j_2+1} \cdots v_{j_n+1})$$

$$(2) \qquad m(P) = (-1)^m m(v_0 u_{i_1+1} u_{i_2+1} \cdots u_{i_m+1} v_{j_1} \cdots v_{j_n}) .$$

Proof. The proof of part 1 is the same as that of Lemma 1, p. 429, [2]. For part 2, the proof of part 1 with the u_i and v_j interchanged may be used, with the symmetry theorem of [4] or [3].

5. The main theorem. By Levi's theorem $(u_i v_j)^t \equiv 0$ for $t = i + j + 1$. For $i = 0$, $u_0^j v_j^j$ is an α-term and not congruent to zero, so that $1 + j$ is the smallest power of $u_0 v_j$ in $[uv]$. We conjecture that, in general, the smallest power of $u_i v_j$ in $[uv]$ is $i + j + 1$. Theorem 5.1 implies that the conjecture is true for $i = 1$.

THEOREM 5.1.

$$(u_1 v_j)^{1+j} \equiv (-1)^{1+j}(1+j)!\ (u_0 v_{1+j})^{1+j} ,$$

for $j \geqq 0$.

Proof. Equivalently we may show that

$$m(u_1 v_j)^{1+j} = (-1)^{1+j}(1+j)! , \qquad \text{for } j \geqq 0 .$$

The proof is by induction on j; since $u_1 v_0 \equiv -u_0 v_1$, $m(u_1 v_0) = -1$ and the theorem is true for $j = 0$. Assume that

$$m(u_1 v_t)^{1+t} = (-1)^{1+t}(1+t)! \qquad \text{for } t < j .$$

By Levi's theorem, $Q = u_0 u_1^j v_j^{j+1} \equiv 0$, and hence, $Q' \equiv 0$. Thus

$$(u_1 v_j)^{1+j} + 2j(u_0 u_1^{j-1} u_2 v_j^{j+1}) + (j+1)^2 u_0 u_1^j v_j^j v_{j+1} \equiv 0 .$$

Applying Lemma 4.1 results in the equation,

$$(1) \qquad m(u_1 v_j)^{1+j} = -2jm(u_1^{j-1} u_2 v_{j-1}^{j+1}) - (j+1)^2 m(u_1^j v_{j-1}^j v_j) .$$

Since $(u_1^j v_{j-1}^{j+1})' \equiv 0$, we also have

$$(2) \qquad m(u_1^{j-1} u_2 v_{j-1}^{j+1}) = -\frac{j(j+1)}{2j} m(u_1^j v_{j-1}^j v_j) .$$

Substituting (2) into (1) yields,

$$m(u_1 v_j)^{1+j} = -(1+j)\, m(u_1^j v_{j-1}^j v_j) .$$

Apply the induction hypothesis to the term on the right, to complete the proof.

6. More general results. Using the same methods, more general results may be obtained.

THEOREM 6.1. *For* $j \geqq 0,\quad k > j-1 \geqq 0,\quad l \geqq j-1 \geqq 0$

$$m(v_j u_1^k v_k^l u_{l+1-j}) = (-1)^{k+l-j+1}\left\{ j! \binom{k}{j}\binom{l}{j} + (j-1)! \binom{k}{j-1}\binom{l}{j-1} \right\}.$$

Proof. The proof is by induction on the triple (j, k, l), and a useful equation is derived first. Differentiate each of the following congruences, for $j \geqq 1$, $k \geqq j$, $l \geqq j-1$:

$$(1) \qquad v_{j-1} u_1^k v_k^l u_{l+1-j} \equiv 0$$

$$(2) \qquad u_0 v_j u_1^k v_{k+1}^l u_{l+1-j} \equiv 0 .$$

The resulting congruences yield, after applying Lemma 4.1, the equations:

$$\begin{aligned}(3) \qquad m(v_{j-1} u_1^{k-1} u_2 v_k^l u_{l+1-j}) = &-\frac{j}{2k} m(v_j u_1^k v_k^l u_{+1-j}) \\ &-\frac{l(k+1)}{2k} m(v_{j-1} u_1^k v_{k+1}^{l-1} u_{l+1-j}) \\ &-\frac{l+2-j}{2k} m(v_{j-1} u_1^k v_k^l u_{l+2-j}) ;\end{aligned}$$

$$(4) \qquad m(v_{j-1} u_1^{k-1} u_2 v_k^l u_{l+1-j}) = -\frac{1}{2k} m(v_j u_1^{k+1} v_{k+1}^l u_{l+1-j})$$

$$- \frac{(j+1)}{2k} m(v_j u_1^k v_k^l u_{l+1-j})$$
$$- \frac{l(k+2)}{2k} m(v_{j-1} u_1^k v_{k+1}^{l-1} u_{l+1-ij})$$
$$- \frac{(l+2-j)}{2k} m(v_{j-1} u_1^k v_k^l u_{l+2-j}) .$$

Combining (3) and (4), and representing $v_j u_1^k v_k^l u_{l+1-j}$ by (j, k, l), the conclusion is:

$$(5) \qquad m(j, k+1, l) = - m(j, k, l) - lm(j-1, k, l-1) .$$

If $m(p, q, r) = (-1)^{q+r-p+1}\left\{p! \binom{q}{p}\binom{r}{p} + (p-1)! \binom{q}{p-1}\binom{r}{p-1}\right\}$, then (5) implies

$$m(p, q+1, r) = (-1)^{(q+1)+r-p+1}\left\{p! \binom{q+1}{p}\binom{r}{p} + (p-1)! \binom{q+1}{p-1}\binom{r}{p-1}\right\} .$$

It remains to show that every triple (j, k, l) with $k \geqq j-1$ can be constructed from triples where the formula is known to hold. Now first consider the cases (α) $j = k$ and (β) $j = k+1$. By rearrangement of the power product $v_j u_1^{j-1} v_{j-1}^l u_{l+1-j}$,

$$(6) \qquad m(j-1, j-1, l-1) = m(j, j-1, l) ,$$

so that if the formula is true for any case (α), then it is true for the next (β) case, for all l.

Start with the case $(0, 0, l)$, $v_0^{l+1} u_{l+1}$. The formula is obviously true for this case (and indeed for $(0, k, l)$) with the convention that

$$(j-1)! \binom{k}{j-1}\binom{l}{j-1} = 0 .$$

Then by (6) the formula must hold also for $(1, 0, l)$. By (5)

$$m(1, 1, l) = -m(1, 0, l) - lm(0, 0, l-1)$$

and

$$m(1, 2, l) = -m(1, 1, l) - lm(0, 1, l-1) .$$

In general, if the formula holds for (j, j, l) (some particular j and all l), then it holds for $(j+1, j, l-1)$, all $l-1$. Using (5), the formula is then true for $(j+1, j+1, l)$. Therefore, the formula holds for all cases α and β; that is, if $k = j$ or $k = j-1$.

Consider next the case $k = j + 1$. For each l by (5),

$$m(1, 3, l) = - m(1, 2, l) - lm(0, 2, l - 1) .$$

The formula holds for $(0, 2, l - 1)$ since it holds for all $(0, k, l)$ and for $(1, 2, l)$ as shown above; and, hence, the formula holds for $(1, 3, l)$. In general, if the formula holds for $(j, j + p, l)$ and for $(j - 1, j + p, l)$ for all l, then by (5)

$$m(j, j + p + 1, l) = - m(j, j + p, l) - lm(j - 1, j + p, l - 1)$$

and the formula holds for $k = j + p + 1$. Thus the formula holds for all $k \geqq j$, and the proof of the theorem is complete.

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