PACIFIC JOURNAL OF MATHEMATICS
Vol. 17, No. 1, 1966

ON TOPOLOGICALLY INDUCED GENERALIZED PROXIMITY RELATIONS II

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In the theory of proximity spaces of Efremovic, (The geometry of proximity, Mat. Sbornic, N.S. 31 (73), (1952), 189-200,) the result:

***A* set *X* with a binary relation "*A* close to *B*" is a proximity space if and only if there exists a compact Hausdorff space *Y* in which *X* can be imbedded so that *A* is close to *B* in *X* if and only if $\bar{A}$ meets $\bar{B}$ in *Y* ($\bar{A}$ denotes the closure of the set *A*) (*Y.M.* Smirnov, on proximity spaces, Mat. Sbornic, N.S. 31 (73), (1952), 543-574.)**

Raises the question: Can we display a set of axioms for a binary relation δ on the power set of a set *X* so that the system (X, δ) satisfies these axioms if and only if there is a topological space *Y* in which *X* can be imbedded so that

(1.1) $\quad A\delta B$ **in** X **if and only if** $\bar{A} \cap \bar{B} \neq \phi$ **in** Y.

In (M.W. Lodato, On topologically induced generalized proximity relations, Proc. Amer. Math. Soc. vol. 15, no. 3, June 1964, pp. 417–422), it is shown that an affirmative answer can be given if *Y* is T_1 and if *X* is *regularly dense* in *Y*. The clusters of S. Leader, On clusters in proximity spaces, Fund. Math. 47 (1959), 205-213, were used in (M.W. Lodato, On topologically induced generalized proximity relations, Proc. Amer. Math. Soc. vol. 15, no. 3, June 1964, pp. 417–422). The present paper generalized this notion and thus relaxes the condition that *X* be regularly dense in *Y*. We actually characterize every system (X, δ) for which there exists a mapping *f* (not necessarily one-to-one) of *X* into a Hausdorff space *Y* such that

(1.2) $\quad A\delta B$ **in** X **if and only if** $\overline{Af} \cap \overline{fB} \neq \phi$ **in** Y.

2. P_s-Spaces. Recall from [3] that a *symmetric generalized proximity space* or *P_s-space* is a system (X, δ) where δ is a binary operation on the power set of X satisfying

(P. 1) $A\delta(B \cup C)$ implies that either $A\delta B$ or $A\delta C$
(P. 2) $A\delta B$ implies that $A \neq \phi$ and $B \neq \phi$
(P. 3) $A \cap B \neq \phi$ implies $A\delta B$
(P. 4) $A\delta B$ and $b\delta C$ for all points b in B imply that $A\delta C$
(P. 5) $A\delta B$ implies $B\delta A$

We read the symbols "$A\delta B$" as "A is close to B"; and we say that "A is remote from B"-in symbols, "$A\phi B$"-if A is not close to B.

Received September 26, 1963.

(2.1) The following facts are evident: (1) If $A\delta B$, $A\subset C$, and $B\subset D$ then $C\delta D$. (2) Define

$$A^{\delta}=\{x\in X: x\delta A\}$$

then in a P_s-space $(A^{\delta})\delta(B^{\delta})$ if and only if $A\delta B$.

3. **Bunches.** Let X be a P_s-space. A *bunch* over X is a class σ of subsets of X satisfying:

(B. 1) $A\delta B$ for all $A, B\in\sigma$

(B. 2) $A\cup B\in\sigma$ implies that $A\in\sigma$ or $B\in\sigma$

(B. 3) $X\in\sigma$

(B. 4) If $A\in\sigma$ and $a\delta B$ for all a in A then $B\in\sigma$.

(3.1) The following facts are easily established:

(1) Every cluster is a bunch.

(2) For x, a point in a P_s-space X, the class σ_x of all subsets A of X such that $x\delta A$ is a bunch over X.

(3) If a point x of X belongs to a bunch σ, then σ is identical to the class σ_x of all subsets A of X such that $x\delta A$.

(4) Any bunch σ from a P_s-space (X,δ) is closed under the operation of supersets: If σ is a bunch from X, $A\in\sigma$ and $A\subseteqq B$, then $B\in\sigma$.

4. Extensions characterized by bunches.

(4.1) THEOREM. *Given a set X and some binary relation δ on the power set of X, the following are equivalent:*

(I) *There exists a T_2 topological space Y and a mapping f of X into Y with $\overline{fx}=Y$ and such that* (1.2) *holds.*

(II) *δ is a P_s-relation satisfying the additional axiom:*

(P. 7) *There exists a family Σ of bunches from X such that*

(i) *$A\delta B$ implies that there exists a $\sigma\in\Sigma$ such that $A, B\in\sigma$, and*

(ii) *if σ and σ' are in Σ and either $A\in\sigma$ or $B\in\sigma'$ for all subsets A and B of X such that $A\cup B=X$, then $\sigma=\sigma'$.*

Proof. Suppose that (I) holds and define δ by (1.2). (P. 1), (P. 2), (P.3), and (P. 5) are trivial consequences of the properties of closure. For (P. 4) suppose that $A\delta B$ and $b\delta C$ for all b in B. Then $\overline{fA}\cap\overline{fB}\neq\phi$ $\overline{fb}\cap\overline{fC}\neq\phi$ for all b in B, which since Y is T_2, implies that $fb\in\overline{fC}$ for all b in B. Thus $fB\subset\overline{fC}$ or $\overline{fB}\subset\overline{fC}$ so that $\overline{fA}\cap\overline{fC}\neq\phi$ showing that $A\delta C$. For (P. 7), define $\sigma_y=\{A\subseteqq X: y\in\overline{fA}\}$ for each point $y\in Y$. Clearly, σ_y is a bunch.

Now let $\Sigma=\{\sigma_y: y\in Y\}$ and we will show that Σ satisfies (i) and (ii).

(i) If $A\delta B$, then $\overline{fA}$ meets $\overline{fB}$ in Y so we can take a point y in $\overline{fA}\cap\overline{fB}$ and σ_y will be a bunch containing both A and B.

(ii) Suppose $\sigma_x \neq \sigma_y$. Then $x \neq y$ in Y so that, using the T_2 property, there exist disjoint open sets U and V, containing x and y respectfully. Thus, $y \notin Y - V = \overline{Y - V}$ and $x \notin Y - U = \overline{Y - U}$ so that $y \notin \overline{fX - V}$ and $x \notin \overline{fX - U}$. Hence, $A = f^{-1}(fX - V) \notin \sigma_y$ and $B = f^{-1}(fX - U) \notin \sigma_x$ and

$$f(A \cup B) = (fX - V) \cup (fX - U) = fX - (V \cap U) = fX$$

so that $A \cup B = X$.

For the converse suppose that (II) holds. Given x in X the class $\sigma_x = \{A \subseteqq X: x\delta A\}$ is a bunch from X, by (3.1), (2.). Thus for any subset A of X, let $\mathscr{A}$ be the set of all bunches σ_a determined by the points a in A and let $\bar{\mathscr{A}}$ be the set of all bunches in Σ which have A as a number. Define the correspondence, $f(x) = \sigma_x$ between X and $\mathscr{X} = fX$ by identifying each x in X with the bunch σ_x determined by it. Let $Y = \Sigma$, the family of bunches satisfying (i) and (ii).

We first show that $fX \subseteqq \Sigma$. Consider any σ_x in fX. Then since by (P. 3) $x\delta x$, by (i) there exists a σ in Y such that $x \in \sigma$. But by (3.1), (3.), $\sigma_x = \sigma$, hence $\sigma_x \in Y$ and $fX \subseteqq Y$.

By (P. 3), $A \in \sigma_a$ for each a in A and so $\mathscr{A} \subset \bar{\mathscr{A}}$.

A subset A of X *absorbs* a subset Φ of Y if and only if A belongs to every bunch in Φ, i.e., if and only if $\bar{\mathscr{A}}$ contains Φ. For any subset Φ of Y we define the closure, $cl\,(\Phi)$, of Φ by

(4.2) $\sigma \in cl(\Phi)$ if and only if every subset E of X which absorbs Φ is in σ.

We next show that

(4.3) $cl(\mathscr{A}) = \bar{\mathscr{A}}$.

For if $\sigma \in cl(\mathscr{A})$ then since A absorbs $\mathscr{A}$, $A \in \sigma$ so that $\sigma \in \bar{\mathscr{A}}$. On the other hand, if $\sigma \in \bar{\mathscr{A}}$ then $A \in \sigma$. Now let P be in every σ_a in $\mathscr{A}$, i.e., $P\delta a$ for every a in A and hence $A \subset P^{\delta}$. Thus, by (B.4), $P \in \sigma$ so that $\sigma \in cl(\mathscr{A})$.

We now show that the Kuratowski closure axioms are satisfied by the closure defined by (4.2).

(K. 1) $\Phi \subset cl(\Phi)$: This is trivial since if E absorbs Φ then $E \in \sigma$ for every $\sigma \in \Phi$.

(K. 2) $cl(\phi) = \phi$: Suppose $\sigma \in cl(\phi)$. Since it is vacuously true that every subset of X absorbs ϕ, we then have that every subset of X is in σ. In particular, ϕ and X are in σ. Thus, $\phi\delta X$, by (B. 1), contradicting (P. 2).

(K. 3) $cl(cl(\Phi)) \subseteqq cl(\Phi)$: Suppose $\sigma \in cl(cl(\Phi))$ and that E absorbs Φ. By (4.2) E absorbing Φ implies that E absorbs $cl(\Phi)$. Hence $E \in \sigma$ showing that $\sigma \in cl(\Phi)$.

(K. 4) $cl(\Phi \cup \Phi') = cl(\Phi) \cup cl(\Phi')$: Suppose that $\sigma \in cl(\Phi \cup \Phi')$ and that A absorbs Φ and A' absorbs Φ'. Then by (3.1), (4.), $A \cup A'$ absorbs $\Phi \cup \Phi'$ so that $A \cup A' \in \sigma$. But by (B. 2) this means that either $A \in \sigma$ or $A' \in \sigma$, i.e., $\sigma \in cl(\Phi)$ or $\sigma \in cl(\Phi')$. On the other hand, $\sigma \in cl(\Phi) \cup cl(\Phi')$ implies that either $\sigma \in cl(\Phi)$ or $\sigma \in cl(\Phi')$. Now if E absorbs $\Phi \cup \Phi'$, then E absorbs Φ and also absorbs Φ'. Hence, $E \in \sigma$ showing that $\sigma \in cl(\Phi \cup \Phi')$ and (K. 4) holds.

Thus, (4.2) defines a topology on Y.

To show that fX is dense in Y, we just note that by (4.3), $cl(\mathscr{X}) = \bar{\mathscr{X}} = Y$.

To show that the topology is T_2 we must show that if σ and σ' are in Y such that $\sigma \neq \sigma'$, then there exist subsets Φ and Φ' of Y such that $\sigma \notin cl(\Phi)$, $\sigma' \notin cl(\Phi')$ and $cl(\Phi) \cup cl(\Phi') = Y$.

So suppose $\sigma \neq \sigma'$, then by (ii) there exist subsets A and B of X such that $A \notin \sigma$, $B \notin \sigma'$ and $A \cup B = X$. Thus, $\mathscr{A}$ and $\mathscr{B}$ are subsets of Y such that $\sigma \notin \bar{\mathscr{A}}$ and $\sigma' \notin \bar{\mathscr{B}}$, (since for instance A absorbs $\mathscr{A}$ but $A \notin \sigma$) and $\bar{\mathscr{A}} \cup \bar{\mathscr{B}} = \overline{\mathscr{A} \cup \mathscr{B}} = \bar{\mathscr{X}} = Y$.

To finish the proof we need only show that (1.2) holds: $A\delta B$ in X if and only if $\bar{\mathscr{A}}$ meets $\bar{\mathscr{B}}$ in Y. If $A\delta B$ there exists, by (i) a $\sigma \in Y$ to which both A and B belong. Thus, by definition of $\bar{\mathscr{A}}$, we have $\sigma \in \bar{\mathscr{A}} \cap \bar{\mathscr{B}}$. On the other hand, if $\sigma \in \bar{\mathscr{A}} \cap \bar{\mathscr{B}}$ then A and B are in σ so that by (B. 1), $A\delta B$.

The proof is now complete.

5. Symmetric P_1-Spaces. A P_s-Spaces (X, δ) in which δ satisfies the additional axiom.

(5.1) $x\delta y$ implies $x = y$

is called a *symmetric P_1-space* (see [4]). The following theorem follows directly from (B. 1) and (5.1).

(5.2) THEOREM. *Every bunch σ from a symmetric P_1-space (X, δ) possesses at most one point.*

(5.3) THEOREM. *Given a set X and a binary relation, δ, on the power set of X, the following are equivalent*:

(I′) *There exists a T_2 topological space Y in which X can be imbedded so that* (1.1) *holds.*

(II′) *δ is a symmetric P_1-relation satisfying* (P. 7).

Proof. The demonstration is very similar to that of theorem (4.1). To see that (I′) implies (5.1), note that $\bar{x} \cap \bar{y} \neq \phi$ implies that $x \cap y \neq \phi$, or $x = y$.

Finally we note that, because of (5.2), the correspondence between X and $\mathscr{X}$ induced by the identification of x with the bunch σ_x determined by it is one-to-one.

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