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SOME NEW RESULTS ON SIMPLE ALGEBRAS

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This paper deals with the problem of proving that a simple algebra (finite dimensional) has an identity element. The main result is contained in the following theorem. Let A be a simple algebra (char. $\neq 2$) in which $(x, x, x) = 0$ and $x^3 \cdot x = x^2 \cdot x^2$. If M is a subset of A such that $(A, M, A) = 0$ and $(M, A, A) \cup (M, A) \cup (A, A, M) \subseteqq M$, then $M = 0$ or there is an identity element in A. This result is then used to prove the three following corollaries (char. $\neq 2$): (1) A simple power associative algebra with all commutators in the nucleus has an identity; (2) A simple power associative algebra with all associators in the middle center has an identity; (3) A simple antiflexible algebra in which $(x, x, x) = 0$ and A^+ is not nil has an identity.

For convenience in terminology, we define an algebra as a finite dimensional vector space on which a multiplication is defined that satisfies both distributive laws. An algebra is nilpotent if there is an integer k such that any product of k elements, no matter how associated, is zero. An element x in an algebra is nilpotent if the subalgebra generated by x is nilpotent. An algebra is nil if it consists entirely of nilpotent elements. A simple algebra is an algebra without proper ideals that is not nil. For char. $\neq 2$, define $x \cdot y = 1/2(xy + yx)$. The algebra A^+ is defined to be the vector space A with miltiplication $x \cdot y$. In addition, define the commutator $(x, y) = xy - yx$ and the associator $(x, y, z) = (xy)z - x(yz)$. Using techniques similar to [3] we will prove the following theorem.

THEOREM 1. *If B is a subspace of an algebra A, there cannot be a nil subset M proper in B with:*
(a) $B = M + MB$
(b) *M a subalgebra*
(c) $(M, M, B) = 0$.

Define $x^1 = x$ and for $k > 1$ define $x^k = x^{k-1}x$. Using this theorem, the following theorem is proved.

THEOREM 2. *Let A be a simple algebra* (char. $\neq 2$) *in which*

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$(x, x, x) = 0$ and $x^3 \cdot x = x^2 \cdot x^2$. If M is a subset of A such that

(a) $(M, A, A) \subseteqq M$

(b) $(M, A) \subseteqq M$

(c) $(A, A, M) \subseteqq M$

(d) $(A, M, A) = 0$

then $M = 0$ or there is an identity element in A.

It should be remarked that, if (d) could be replaced by $(A, M, A) \subseteqq M$, then $M = A$ would satisfy (a), (b), (c) and (d) in any algebra.

Many simple algebras are known to have identities. If the center (the set of x such that $(x, y) = (x, y, z) = (y, z, x) = 0$ for all y and z in A) of a simple algebra is not zero, the algebra has an identity element [4]. In addition, there are many identities which, if satisfied by a simple algebra, force that algebra to have an identity element.

In § 4, Theorem 2 will be used to prove the existence of an identity element in three classes of algebras.

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2. Proof of the first theorem. Let us assume that M is a subset proper in B that satisfies (a), (b) and (c) of Theorem 1. We will show that M is not nil. From (c), it is clear that M is an associative subalgebra even though A is not even assumed to be power associative. The following fact is easily obtained by induction. In an associative algebra such as M, $\prod_{i=1}^{n} a_i = a_1 \cdots a_n$.

LEMMA 2.1. *If $a_1, \cdots, a_n$ are in M and if x is in B then $(\prod_{i=1}^{n} a_i)x = a_1(a_2 \cdots (a_n x) \cdots)$.*

LEMMA 2.2. *There exists an $x \neq 0$ in B and an a in M with $x = ax$.*

Proof. For any x in $B \cap M'$ (set theoretic complement), there is a set of elements $a_1, \cdots, a_n, b$ in M and a set $y_1, \cdots, y_n$ in $B \cap M'$ with $x = b + \sum_{i=1}^{n} a_i y_i$. Throughout this proof we will use a, b with subscripts to indicate members of M and y with subscripts to indicate members of $B \cap M'$. Since M is proper in B, we have $B \cap M' \neq \varnothing$. Let y_0 be in $B \cap M'$.

Clearly, $y = \sum_{i=1}^{n_1} a_{1i} y_{1i} + b_1$. Since y_0 is not in M, some j exists for which $a_{1j} y_{1j}$ is not in M. Define $y_1 = y_{1j}$ and $a_1 = a_{1j}$. Suppose

$a_1, \cdots, a_k, y_1, \cdots, y_k$ have been defined for all $k < m$ with $a_1(a_2 \cdots (a_k y_k) \cdots)$ in $B \cap M'$. Since y_{m-1} is not in M, $y_{m-1} = \sum_{i=1}^{n} a_{mi} y_{mi} + b_m$. There must be some j for which $a_1(a_2 \cdots (a_{m-1}(a_{mj} y_{mj})) \cdots)$ is in $B \cap M'$ because $a_1(a_2 \cdots (a_{m-1} y_{m-1}) \cdots)$ is in $B \cap M'$ and M is a subalgebra. Defining $y_m = y_{mj}$ and $a_m = a_{mj}$, we have inductively defined for all n the set $y_1, \cdots, y_n, a_1, \cdots, a_n$ with $(\prod_{i=1}^{n} a_i) y_n = a_1(a_2 \cdots (a_n y_n) \cdots)$ in $B \cap M'$. For $j < k$, let $m_{jk} = \prod_{i=j}^{k} a_i$. Also, let $r_j = \prod_{i=1}^{j} a_i$ so that when $1 < j < k$, $r_{j-1} m_{jk} = r_k$. From finite dimensionality, there is an s with $m_{1s} y_s, \cdots, m_{ss} y_s$ a linearly dependent set. If $m_{js} y_s$ were in M for $j > 1$, then $r_{j-1}(m_{js} y_s) = r_s y_s$ would be in M. This is not true by the construction of y_s so $m_{js} y_s$ is not in M and hence is not zero for all j. Because the elements $m_{1s} y_s, \cdots, m_{ss} y_s$ are dependent, there exist field elements $\alpha_1, \cdots, \alpha_s$ not all zero with $\sum_{i=1}^{s} \alpha_i m_{is} y_s = 0$. Letting $t = \max\{i \colon a_i \neq 0\}$ will give

$$m_{ts} y_s = -\sum_{i=1}^{t-1} (\alpha_i/\alpha_t)\, m_{is} y_s = -\left[\sum_{i=1}^{t-1} (\alpha_i/\alpha_t) m_{i,t-1}\right] m_{ts} y_s \,.$$

Letting $x = m_{ts} y_s$ and $a = -\sum_{i=1}^{t-1} (\alpha_i/\alpha_t) m_{i,t-1}$, will give $x = ax$ with $x \neq 0$ in B and a in M.

We are now able to prove that M is not nil. By induction $x = ax = a(ax) = a^2 x = \cdots = a^k x$ for all k. Since x is not zero, a is not nilpotent and M is not nil. Since M is an associative algebra, we have the following result.

COROLLARY. *If B is a subspace of an algebra A and if M is a proper subset of B with:*
(a) $M + MB = B$
(b) *M a subalgebra*
(c) $(M, M, B) = 0$
then there is an idempotent in M.

3. Proof of the second theorem. Throughout this section we will assume that A is a simple algebra in which $(x, x, x) = 0$ and $x^3 \cdot x = x^2 \cdot x^2$. In addition, assume char. $\neq 2$. Given a subset of A that satisfies (a), (b), (c) and (d) of Theorem, it is necessary to find a subalgebra of A satisfying the same conditions.

DEFINITION 3.1. The set $\mathcal{N}$ = {subsets of A that satisfy (a), (b), (c) and (d) of Theorem 2}.

LEMMA 3.1. *If M is in $\mathcal{N}$ then either $M = 0$ or $M + MA = A$.*

This lemma is obvious for $M + MA$ is an ideal in A. From now on, assume that $\mathscr{N}$ contains a nonzero member. Each member of $\mathscr{N}$ can easily be extended to a subspace of A, $\mathscr{N}$ is closed under set union, and A is finite dimensional. Consequently, there is a largest member of $\mathscr{N}$ which we will call N.

LEMMA 3.2. *The set N is an associative subalgebra with $N + NA = A$.*

Proof. Clearly N is a nonzero subspace so $N + NA = A$. In any ring, the following identities are satisfied:

$$(1) \qquad (xy, z) + (yz, x) + (zx, y) = (x, y, z) + (y, z, x) + (z, x, y)$$

$$(2) \qquad (wx, y, z) - (w, xy, z) + (w, x, yz) = w(x, y, z) + (w, x, y)z .$$

If x and y are in N, then from (1), we have

$$(3) \qquad (xy, z) = (y, z, x) - (yz, x) - (zx, y) .$$

Hence, (xy, z) is in N. Equation (2) implies the fact that $(w, xy, z) = 0$. If w and x are in N, then $(wx, y, z) = (w, xy, z) + w(x, y, z)$ which is in $N + N^2$. Also, (x, y, xw) is in $N + N^2$. Consequently, $N + N^2$ is in $\mathscr{N}$ so $N + N^2 \subseteqq N$. The fact that N is associative is obvious.

The proof of Lemma 3.2 can be used to construct a subalgebra P satisfying (a), (b), (c) and (d) of Theorem 2 in a quite different way. First, extend M to a subspace of A and call this subspace M. Then for $k \geqq 1$, define $M_k = M_{k-1}^2 + M_{k-1}$. Induction and the proof of Lemma 3.2 imply M_k satisfies (a), (b), (c) and (d) of Theorem 2 for all k. By finite dimensionality, for some k, $M_k = M_{k+1}$. The set $P = M_k$ is a subalgebra satisfying (a), (b), (c) and (d). It is clear that $P + PA = A$.

We know that an associative simple algebra has an identity element. Therefore, if $A = N$, then A has an identity element. From now on, assume $A \neq N$. Theorem 1 implies N is not nil so there is an idempotent in N. An idempotent e is principal if there is no idempotent g with $eg = ge = 0$. Because N is associative, there is an idempotent that is principal in N. We will call this idempotent f. It cannot be assumed that f is principal in A. We will need the following definition.

DEFINITION 3.2. For any subset B of A, $B_{ij}(e) = \{x$ in B such that $ex = ix$ and $xe = jx\}$. Define B_{ij} by $B_{ij} = B_{ij}(f)$.

Because N is an associative algebra with f in N, $N = N_{11} + N_{10} + N_{01} + N_{00}$. In addition, $N_{ij}N_{jk} \subseteqq N_{ik}$ and, for $j \neq k$, $N_{ij}N_{km} = 0$. Up to now, we have not used the fact that $(x, x, x) = 0$. Linearizing this yields this multilinear identity:

$$(x, y, z) + (y, z, x) + (z, x, y) + (z, y, x) + (y, x, z) + (x, z, y) = 0 . \tag{4}$$

If we now let $x = z = f$ and divide by 2, we obtain $(f, y, f) = 0$. Therefore, we have $(f, f, y) = (f, y, f) = (y, f, f) = 0$ which is all that is needed to prove this lemma:

LEMMA 3.3. *Relative to* f, $A = A_{11} + A_{10} + A_{01} + A_{00}$.

LEMMA 3.4. *The subspaces* N_{ij} *and* A_{ij} *obey the following laws*:

$$A_{ij}A_{km} = 0 \text{ if } j \neq k \tag{5}$$

$$N_{ij}A_{jk} \cup A_{ij}N_{jk} \subseteqq N_{ik} \text{ if } i \neq k \tag{6}$$

$$N_{ij}A_{jk} \subseteqq A_{i1} + A_{i0} \tag{7}$$

$$A_{ij}N_{jk} \subseteqq A_{1k} + A_{0k} \tag{8}$$

$$N_{10}A_{01} \subseteqq N_{11} + N_{10} \tag{9}$$

$$N_{01}A_{10} \subseteqq N_{00} + N_{01} \tag{10}$$

$$N_{ii}A_{ii} \subseteqq A_{ii} + N . \tag{11}$$

Proof. To prove (5), let x be in A_{ij} and let y be in A_{km} in $(x, f, y) = 0$. The relations (7) and (8) result from $(f, N_{ij}, A_{jk}) = 0$ and $(A_{ij}, N_{jk}, f) = 0$. In proving (6), we first let $y = f, z = n_{ij}$ (in N_{ij}) and $x = x_{jk}$ (in A_{jk}) in (4) to obtain $(f, x_{jk}, n_{ij}) + (n_{ij}, x_{jk}, f) = 0$. Using (5), we find that $(f, x_{jk}, n_{ij}) = jx_{jk}n_{ij} - f(x_{jk}n_{ij}) = 0$. Hence, $(n_{ij}x_{jk})f = kn_{ij}x_{jk}$. This together with (7) implies $N_{ij}A_{jk} \subseteqq A_{ik}$. In a similar way, (4), (5) and (8) imply $A_{ij}N_{jk} \subseteqq A_{ik}$. We will now prove $N_{ij}A_{jk} \subseteqq N_{ik}$. The proof of $A_{ij}N_{jk} \subseteqq N_{jk}$ is similar. Using z and x as defined above, $xz = 0$ and (z, x) is in N. Therefore $zx = (z, x)$ is in $N \cap A_{ik}$ and we have proved (6). We now take n in N_{10} and a in A_{01}. From (7) and (8) we have $na = b_{11} + b_{10}$ and $an = c_{10} + c_{00}$ with b_{ij}, c_{ij}, in A_{ij}. We also see that b_{11}, c_{11} and $b_{10} - c_{10}$ are in N because (n, a) is in N. Equation (4) implies $(n, a, f) + (f, a, n) = 0$. Upon substitution we get $(b_{11} + b_{10})f - (b_{11} + b_{10}) - f(c_{10} + c_{00}) = 0$ or $-b_{10} - c_{10} = 0$. Therefore $b_{10} = 1/2(b_{10} - c_{10})$ is in N_{10}. This proves (9) and a similar argument proves (10). In [1], a linearization of $(x^2 \cdot x) \cdot x = x^2 \cdot x^2$ was all that was needed to prove that $A_{11} \cdot A_{11} \subseteqq A_{11}$ and $A_{00} \cdot A_{00} \subseteqq A_{00}$. The proof of Lemma 3.4 is completed by observing that this implies $N_{ii}A_{ii} \subseteqq N_{ii} \cdot A_{ii} + (N_{ii}, A_{ii}) \subseteqq A_{ii} + N$.

LEMMA 3.5. *The algebra* $A = N + N_{11}A_{11} + N_{00}A_{00}$.

Proof. As a result of (5) and Lemma 3.2, $A = N + \sum N_{ij}A_{jk}$. From (6), $N_{11}A_{10} + N_{10}A_{00} \subseteqq N_{10}$ and $N_{00}A_{01} + N_{01}A_{11} \subseteqq N_{01}$. The relations in (9) and (10) complete the proof.

LEMMA 3.6. *The set* $B = N_{10}N_{01} + N_{10} + N_{01} + N_{01}N_{10}$ *is an associative ideal in* A.

Proof. The laws $N_{ij}N_{jk} \subseteqq N_{ik}$ and $N_{ij}N_{mk} = 0$ for $j \neq m$ easily establish the fact that B is an associative ideal in N. Using Lemma 3.5, we need only prove that $B(N_{11}A_{11})$, $(N_{11}A_{11})B$, $(N_{00}A_{00})B$ and $B(N_{00}A_{00})$ are in B. We have $(N_{11}A_{11})(N_{10} + N_{01}) \subseteqq (A_{11} + N)(N_{10} + N_{01}) \subseteqq A_{11}N_{10} + B \subseteqq B$. Also, $(N_{10} + N_{01})(N_{11}A_{11}) = [(N_{10} + N_{01})N_{11}]A_{11} \subseteqq N_{01}A_{11} \subseteqq N_{01} \subseteqq B$. Interchanging 0 and 1 will prove that $(N_{00}A_{00})(N_{10} + N_{01})$ and $(N_{01} + N_{01})(N_{00}A_{00})$ are in B. Therefore $(N_{10} + N_{01})A$ and $A(N_{10} + N_{01})$ are in B. Now, $(N_{10}N_{01})A = N_{10}(N_{01}A) \subseteqq N_{10}B \subseteqq B$ and $A(N_{10}N_{01}) = (AN_{10})N_{01} \subseteqq BN_{01} \subseteqq B$. Similarly $(N_{01}N_{10})A$ and $A(N_{01}N_{10})$ are in B so B is an associative ideal in A.

We have assumed that $N \neq A$ which implies $B = 0$. Therefore $A = N_{11} + N_{11}A_{11} + N_{00} + N_{00}A_{00}$. The relations in (7) and (11) together with the fact that $N = N_{11} + N_{00}$ imply $N_{11}A_{11} \subseteqq A_{11}$ and $N_{00}A_{00} \subseteqq A_{00}$.

We are now able to complete the proof of Theorem 2. We have $A = A_{11} + A_{00}$ with $A_{11} = N_{11} + N_{11}A_{11}$ and $A_{00} = N_{00} + N_{00}A_{00}$. The result in Theorem 1 and the fact that N_{00} is nil imply $N_{00} = A_{00}$. Therefore, using (5) and $N_{00}^2 \subseteqq N_{00}$, we see that N_{00} is an ideal of A. Consequently, f is the identity element of A.

4. Applications.

We will mention several cases where Theorem 2 can be used to prove the existence of an identity element in a simple algebra. The following two corollaries are obvious since simple associative and simple commutative power associative algebras have identities. The nucleus of an algebra A is $\{x: xy = yx \text{ for all } y \text{ in } A\}$.

COROLLARY 4.1. *For* char. $\neq 2$, *a simple power associative algebra with all commutators in the nucleus has an identity element.*

COROLLARY 4.2. *For* char. $\neq 2$ *a simple power associative algebra with all associators in the middle center* ($\{x: (x, y) = (y, x, z) = 0$ *for all* y *and* $z\}$) *has an identity element.*

For our last application, let A be a simple algebra that satisfies the following:

$$(x, y, z) = (z, y, x) \quad \text{(the antiflexible law)} \tag{12}$$

$$(x, x, x) = 0 . \tag{13}$$

In [2] it is proved that $x^2x^2 = x^3x = xx^3$ in A. In fact, A^+ is power associative [2]. Furthermore, if we let $Z = \{x: xy = yx \text{ for all } y \text{ in } A\}$, we can show by the arguments of [3] that $(A, Z, A) = 0$ and (Z, A, A), (A, A, Z) are subsets of Z. Because A^+ is power associative, there is an idempotent e in A^+ if A^+ is not nil. Clearly, e is an idempotent in A. The arguments of [2] will show $A = A_{11}(e) + A_{00}(e)$. Therefore, e is in Z and $Z \neq 0$. We have proved this corollary.

COROLLARY 4.3. *If A is a simple antiflexible algebra* (char. $\neq 2$) *that satisfies* $(x, x, x) = 0$ *and if* A^+ *is not a nil algebra, then there is an identity element in* A.

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