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PATHS ON POLYHEDRA. II

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Continuing the author's earlier investigation, this paper studies the behavior of paths on (convex) polyhedra relative to the facets of the polyhedra. In Section 1, the polytopes which are polar to the cyclic polytopes are shown to admit Hamiltonian circuits, and the fact that they do leads to sharp upper bounds for the lengths of simple paths or simple circuits on polyhedra of a given dimension having a given number of facets. Section 2 is devoted to the conjecture, due jointly to Philip Wolfe and the author, that any two vertices of a polytope can be joined by a path which never returns to a facet from which it has earlier departed. This implies a well-known conjecture of Warren Hirsch, asserting that $n-d$ is an upper bound for the diameter of d-dimensional polytopes having n facets. The Wolfe-Klee conjecture is proved here for 3-dimensional polyhedra, and a stronger conjecture (dealing with polyhedral cell-complexes) is established for certain special cases.

Our notation and terminology are as in [10, 11, 12, 13].[1] In particular, a *polyhedron* is a set which is the intersection of finitely many closed halfspaces in a finite-dimensional real linear space, and a *d-polyhedron* is one which is d-dimensional. The *faces* of a polyhedron P are the empty set, P itself, and the intersections of P with the various supporting hyperplanes of P. Two faces are *incident* provided one contains the other. The 0-faces and 1-faces of P are its *vertices* and *edges*, and when P is a d-polyhedron its $(d-1)$-faces and $(d-2)$-faces are called *facets* and *subfacets* respectively. A *proper polyhedron* is one which contains no line, or, equivalently (assuming it is not empty), one which has at least one vertex. A *polytope* is a bounded polyhedron; equivalently, it is a set which is the convex hull of a finite set of points. Two vertices of a polyhedron P are *adjacent* provided they are joined by an edge of P. A *path* on P is a finite sequence $(x_0, x_1, \cdots, x_l)$ of consecutively adjacent vertices, and the integer l is the *length* of the path. The *diameter* of a polyhedron is the smallest number l such that any two vertices of the polyhedron can be joined by a path of length $\leq l$.[2]

The present paper is part of a development of recent years in

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[1] Hirsch's conjecture is reported on p. 168 of [5]. See also p. 160 of [5] and pp. 608-610 of [10]. And see footnote 16 at end of paper.

[2] Diameters of polyhedra are studied in [10,25].

which the importance of linear programming has reawakened interest in the area forming the geometric background of the subject, namely the study of the facial structure of polyhedra.[3] Because of the special importance of the simplex method, the study of paths on polyhedra is especially relevant. Several authors have considered the behavior of such paths relative to the number of vertices of the polyhedron [2, 9, 10, 12, 16]. However, in a practical linear programming problem one has much more direct information about the number of facets of the feasible region than about the number of its vertices, so it seems appropriate to study the behavior of paths relative to the number of facets. Such a study was initiated in [10, 12] and is continued here.

1. Longest simple paths and circuits. A path $(x_0, x_1, \cdots, x_l)$ is called a *simple path* provided no vertex is repeated; it is a *simple circuit* provided $l \geqq 2$ and $x_l = x_0$ but there is no other repetition among the x_i's. A *Hamiltonian path* or *Hamiltonian circuit* on a polyhedron P is a simple path or circuit which runs through all the vertices of P.[4]

Two d-polyhedra P and Q are said to be *combinatorially equivalent* provided there is a biunique correspondence between the faces of P and the faces of Q such that both incidence and dimension are preserved. Two polytopes P and Q will be called *combinatorially dual* provided there is a biunique correspondence between their faces such that incidence is preserved and dimension is complemented, so that the s-faces of one polytope correspond to the $(d-1-s)$-faces of the other.

The *moment curve* M_d is the subset of $\boldsymbol{R}^d$ consisting of all points of the form $(r, r^2, \cdots, r^d)$ for $r \in \boldsymbol{R}$.[5] A *cyclic d-polytope* is one which is combinatorially equivalent to a d-polytope whose vertices are all on M_d [6].

THEOREM 1.1. *If a d-polytope is a cyclic d-polytope or is combinatorially dual to a cyclic d-polytope, then it admits a Hamiltonian circuit.*

Proof. Since the 2-dimensional case is trivial, we assume that $d \geqq 3$. Consider a set V of at least $d+1$ points on M_d, and let $P \cdot = \operatorname{con} V$.[6] It is known that each point of V is a vertex of P, and

[3] For a survey of some of the interconnections, see [**13**].

[4] For results on the existence or nonexistence of Hamiltonian paths or circuits in 3-polytopes, see [**2**, **9**, **20**, **21**, **22**, **23**].

[5] $\boldsymbol{R}$ denotes the real number field.

[6] Equality by definition is indicated by $\cdot =$ or $= \cdot$.

that for $d \geqq 4$ each two vertices of P are joined by an edge of P [4, 6, 17], whence of course P admits a Hamiltonian circuit. It is known [6] that the facets of P are all simplices, each having d vertices, and that a set F of d points of V determines a facet of P if and only if between each two points of $V \sim F$ there is (in the natural ordering on M_d) an even number of points of F.[7] From this it is easy to identify the edges of P when $d = 3$, and to see that P admits a Hamiltonian circuit. That disposes of the cyclic polytopes.

Now we want to produce a Hamiltonian circuit on a polytope Q which is combinatorially dual to a cyclic polytope P, but rather than working directly with the vertices and edges of Q we may consider the facets and subfacets of P. It suffices to show that if P is as in the preceding paragraph, then all of the facets of P can be arranged in a sequence $(F_0, \cdots, F_l)$ such that $F_l = F_0$, there is no other repetition among the F_i's, and $F_{i-1} \cap F_i$ is a subfacet for $1 \leqq i \leqq l$. Since all the facets of P are simplices, the last condition is equivalent to the requirement that there are $d - 1$ vertices common to F_{i-1} and F_i. Recalling Gale's characterization [6] of the facets of P, we see that the problem at hand is purely combinatorial in nature.

For $2 \leqq d < n$, let $V_n \cdot = \{1, \cdots, n\}$ and let $\boldsymbol{F}(d, n)$ denote the class of all d-pointed sets F in V_n such that between each two points of $V_n \sim F$ there is an even number of points of F. The *initial parity* of F is the parity of the set of all points of F which precede the first point of $V_n \sim F$. The set of all members of $\boldsymbol{F}(d, n)$ which have odd [resp. even] initial parity will be denoted by $\boldsymbol{F}_0(d, n)$ [resp. $\boldsymbol{F}_e(d, n)$]. An ordered pair (F, G) of members of $\boldsymbol{F}_0(d, n)$ will be called *admissible* provided $F \cap G$ consists of exactly $d - 1$ points.

LEMMA. *Suppose $2 \leqq d < n$. Let A consist of the first d points of V_n. If d is even let Z consist of the last d points of V_n, and if d is odd let Z consist of the first point and the last $d - 1$ points of V_n. Then the members of $\boldsymbol{F}(d, n)$ which have the same initial parity as A and Z can be arranged without repetition in a sequence $(F_0, \cdots, F_k)$ such that $F_0 = A$, $F_k = Z$, and (F_{i-1}, F_i) is admissible for $1 < i \leqq k$.*

Proof of the lemma. Let the assertion of the lemma be denoted by $L(d, n)$. For $d = 2$, the assertion is obvious; we merely take $F_i \cdot = \{i + 1, i + 2\} \subset V_n$. For $d = 3$, we keep the point 1 fixed and apply the preceding pattern to the remaining two points; that is, $F_i \cdot =$

[7] Gale's proof [6] is given only for the even-dimensional case, but it can be extended without difficulty to the general case.

$\{1, i+2, i+3\}$. The same procedure will carry us from any even value of d to the next odd value, and it remains only to go from an even value to the next even value. Specifically, we suppose that $L(d, n)$ is known for a given even d and for all $n > d$, and we want to establish $L(d+2, m)$ for an arbitrary $m > d+2$. This is accomplished by successive applications of

$$L(d, m-2),\ L(d, m-3), \cdots, L(d, d+1)\ ,$$

running first through the members of $\boldsymbol{F}(d+2, m)$ which contain $\{1, 2\}$, next through those which contain $\{2, 3\}$ but not $\{1\}$, etc. The procedure is illustrated below for the case in which $d + 2 = 6$ and $m = 10$; the reader can easily supply the formal details for the general argument.

	1	2	3	4	5	6	7	8	9	10	
$L(4, 8)$	x	x	x	x	x	x	0	0	0	0	A
	x	x	0	0	0	0	x	x	x	x	
$L(4, 7)$	0	x	x	0	0	0	x	x	x	x	
	0	x	x	x	x	x	x	0	0	0	
$L(4, 6)$	0	0	x	x	x	x	x	x	0	0	
	0	0	x	x	0	0	x	x	x	x	
$L(4, 5)$	0	0	0	x	x	0	x	x	x	x	
	0	0	0	x	x	x	x	x	x	0	
	0	0	0	0	x	x	x	x	x	x	Z

To complete the proof of 1.1, we show that all of the members of $\boldsymbol{F}(d, n)$ can be arranged in a sequence $(F_0, \cdots, F_l)$ such that $F_l = F_0$, there is no other repetition among the F_i's, and the pair (F_{i-1}, F_i) is admissible for $1 \leqq i \leqq l$. Let

$$A\cdot = \{i\colon 1 \leqq i \leqq d\},\quad B\cdot = (i\colon n - d + 1 \leqq i \leqq n\}\ ,$$

$$C\cdot = \{i\colon 1 \leqq i \leqq d) \cup \{n\},\quad \text{and}\quad D\cdot = \{1\} \cup \{i\colon n - d + 1 < i \leqq n\}\ ,$$

all members of $\boldsymbol{F}(d, n)$. Then the sequence $(F_0, \cdots, F_l)$ is formed as follows, using the patterns described in the lemma:

When d is odd, run through $\boldsymbol{F}_0(d, n)$ from A to D; next go directly to B; then run through $\boldsymbol{F}_e(d, n)$ from B to C, and finally return to A;

When d is even, run through $\boldsymbol{F}_e(d, n)$ from A to B; next go directly to D; then run through $\boldsymbol{F}_0(d, n)$ from D to C, and finally return to A. The procedure is illustrated below with $n = 14$ and $d = 5$ or 6.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	
$L(5, 14)$	x	x	x	x	x	0	0	0	0	0	0	0	0	0	A
	x	0	0	0	0	0	0	0	0	0	x	x	x	x	D
$L(5, 14)$	0	0	0	0	0	0	0	0	0	x	x	x	x	x	B
	x	x	x	x	0	0	0	0	0	0	0	0	0	x	C
	x	x	x	x	x	0	0	0	0	0	0	0	0	0	A
$L(6, 14)$	x	x	x	x	x	x	0	0	0	0	0	0	0	0	A
	0	0	0	0	0	0	0	0	x	x	x	x	x	x	B
L(5, 13)	x	0	0	0	0	0	0	0	0	x	x	x	x	x	D
	x	x	x	x	x	0	0	0	0	0	0	0	0	0	C
	x	x	x	x	x	x	0	0	0	0	0	0	0	0	A

Again, the formal description is left to the reader.[8]

Now (following the pattern of [10] and [12]) let us denote the class of all d-polytopes by $\boldsymbol{P}_d$, while the subclasses $\boldsymbol{P}_d^v$ and $\boldsymbol{P}_d^f$ consist respectively of the d-polytopes which are *simple* (each vertex incident to d edges) and those which are *simplicial* (each facet a simplex). For each polyhedron P, let $\lambda(P)$ [resp. $\kappa(P)$] denote the largest number which is realized as the length of a simple path [resp. simple circuit] on P. We are interested in the maxima of $\lambda(P)$ and $\kappa(P)$ as P ranges over various subclasses of $\boldsymbol{P}_d$. Let us define

$$\Lambda_r(d, n) \cdot = \max \{\lambda(P) \colon P \in \boldsymbol{P}_d \text{ and } f_r(P) \leqq n\} \text{ and}$$
$$K_r(d, n) \cdot = \max \{\kappa(P) \colon P \in \boldsymbol{P}_d \text{ and } f_r(P) \leqq n\} ,$$

where $f_r(P)$ denotes the number of r-faces of P. Similarly, we define Λ_r^v and K_r^v (where P_d is replaced by $\boldsymbol{P}_d^v$) as well as Λ_r^f and K_r^f (where $\boldsymbol{P}_d$ is replaced by $\boldsymbol{P}_d^f$). Our attention will be confined to two possible values for r—namely, $r = 0$ and $r = d - 1$—and the results are as follows:[9]

1.2 $K_0(d, n) = K_0^f(d, n) = n = \Lambda_0^f(d, n) + 1 = \Lambda_0(d, n) + 1.$

1.3 $(d-1)\left[\dfrac{n-2}{d-1}\right] + 2 \leqq K_0^v(d, n) \leqq \Lambda_0^v(d, n) + 1 \leqq n ,$

with equality throughout if $d \leqq 3$ *or* $n \equiv 2 \bmod (d-1)$.

[8] It would be interesting to know which of the m-neighborly $(2m)$-polytopes studied by Gale [**6**] and Grünbaum [**8**] are such that their combinatorial duals admit Hamiltonian paths or circuits.

[9] These results are stated only for polytopes, but they can easily be reformulated so as to apply to polyhedra; 4.5 of [**11**] is useful in this connection.

1.4 $$K_{d-1}(d, n) \geqq K^{v}_{d-1}(d, n) \geqq \binom{n - \left[\frac{d+1}{2}\right]}{n-d} + \binom{n - \left[\frac{d+2}{2}\right]}{n-d} \leqq$$
$$\leqq \Lambda^{v}_{d-1}(d, n) + 1 \leqq \Lambda_{d-1}(d, n) + 1 ,$$

with equality throughout if $d \leqq 8$, *or* $n \leqq d + 3$, *or* $n \geqq (d/2)^2 - 1$, *or the* MJSG *conjecture is true.*

1.5 $$\Lambda^{f}_{d-1}(d, n) + 1 \geqq K^{f}_{d-1}(d, n) \geqq \left[\frac{n-2}{d-1}\right] + d ,$$

with equality throughout if $d \leqq 3$ *or* $n \leqq d + 3$.

These estimates for K and Λ may be compared with estimates for the diameter function Δ given in [10, 25] and for the height functions H, S and Z given [12].

Proofs of 1.2–1.5. The truth of 1.2 follows immediately from the fact that the cyclic polytopes admit Hamiltonian circuits.

For 1.3 we define $g(n) \cdot = (d-1)(n-d) + 2$ and consider the d-polytopes Q_n formed as follows: Q_{d+1} is a d-simplex with Hamiltonian circuit $(x_1, x_2, \cdots, x_{g(d+1)}, x_1)$; given a simple d-polytope Q_n with Hamiltonian circuit $(x^n_1, x^n_2, \cdots, x^n_{g(n)}, x^n_1)$, we form Q_{n+1} by truncating Q_n at the vertex $x_{g(n)}$, and thus replacing $x_{g(n)}$ by a new facet which is a $(d-1)$-simplex whose vertices lie on the d edges of Q_n which issue from $x^n_{g(n)}$. The d-polytope Q_{n+1} is simple, has $g(n+1)$ vertices, and admits a Hamiltonian circuit

$$(x^n_1, x^n_2, \cdots, x^n_{g(n)-1}, x^{n+1}_{g(n)}, x^{n+1}_{g(n)+1}, \cdots, x^{n+1}_{g(n+1)}, x^n_1) ,$$

where $x^{n+1}_{g(n)} \in]x^n_{g(n)-1}, x^n_{g(n)}[$ and $x^{n+1}_{g(n+1)} \in]x^n_{g(n)}, x^n_1[$. (With additional care, we can also insure that Q_n admits a Hamiltonian path which is a "maximum gradient" path.) For the details of this construction and for the rest of the proof of 1.3, see the proof of (3) in [12].

To establish all of the inequalities of 1.4, it suffices to show that $K^{v}_{d-1}(d, n) \geqq \varphi(d, n)$, where

$$\varphi(d, n) \cdot = \binom{n - \left[\frac{d+1}{2}\right]}{n-d} + \binom{n - \left[\frac{d+2}{2}\right]}{n-d} .$$

Gale [6] has proved that a cyclic d-polytope with n vertices is simplicial and has exactly $\varphi(d, n)$ facets[7], whence the dual polytopes are simple and have n facets and $\varphi(d, n)$ vertices. Since the dual polytopes admit Hamiltonian circuits by 1.1, the desired inequality follows. (This shows also that equality holds in 1.3 whenever $n = \varphi(d, m)$ for some $m > d$.) For equality in 1.4, it suffices to show that if a

d-polytope has at most n facets, then it has at most $\varphi(d, n)$ vertices. This statement was called the "JSG conjecture" in [11], but should have been called the *MJSG conjecture* because of its first appearance in [17]. It has been established for $n \leqq d + 3$ [7] and $n \geqq (d/2)^2 - 1$ [11], hence in particular for $d \leqq 6$. And see [8] for $d = 7,8$.

For 1.5 it suffices to follow the proof of (7) in [12], noting that Hamiltonian circuits are admitted by the polytopes P_n which are constructed there. (They also admit "maximum gradient" Hamiltonian paths.) Alternatively, we could employ simplicial d-polytopes Q'_n which are dual to the simple polytopes Q_n; Q'_{d+1} is a d-simplex and Q'_{n+1} is formed by adding a pyramidal cap over one of the facets of Q'_n.

For equality in 1.5 when $d > 3$, it would suffice to show that if a simplicial d-polytope has n vertices, then it has at least $(d-1)(n-d)+2$ facets. The equivalent dual statement was made by Brückner [3] for the 4-dimensional case, but his proof was incorrect (as noted in [18]). Grünbaum [8] has settled the case $n \leqq d + 3$.

2. The existence of W_v paths. Consider a path $(x_0, x_1, \cdots, x_l)$ on a polyhedron P, and for $1 \leqq i \leqq l$ let σ_i denote the edge $[x_{i-1}, x_i]$. The path will be called a *W_v path* or a *W_e path* provided the following respective conditions are satisfied:

(W_v) if $i < j < k$ and a facet F of P includes both x_i and x_k, then F includes x_j also;

(W_e) $\sigma_{i-1} \neq \sigma_i (1 \leqq i \leqq l)$; if $i < j < k$ and a facet F of P contains both σ_i and σ_k, then F contains σ_j also.

Every W_v path is simple. A W_e path of length l need not be simple, but it must involve l distinct edges. These and some related types of paths are considered in [14]. Here we are interested mainly in W_v paths, for the conjecture formulated in the introduction may be stated more formally as follows: *Any two vertices of a polytope can be joined by a W_v path.* This will be called the *W_v conjecture.* The connection between the W_v conjecture and Hirsch's conjecture[1] arises from the following remark.

2.1. *If P is a d-polyhedron which has at most n facets and l is the length of a W_v path [resp. W_e path] on P, then $l \leqq n - d$ [resp. $l \leqq n - d + 2$].*

Proof. There are facets G_i such that

$$x_0 \notin G_0 x_l \in G_0 \sim G_1, \cdots, x_{l-1} \in G_{l-2} \sim G_{l-1}, x_l \in G_{l-1} .$$

From condition (W_v) it follows that the G_i's are distinct and x_0 is not included in any of the facets $G_0, \cdots, G_{l-1}$. Since there are at least d facets of P which include x_0, we have $n \geqq d + l$. A similar argument

[14] applies to W_e paths.

For $n > d$, there exist simple d-polytopes which have exactly n facets and which admit W_v paths of length $n - d$. For example, the d-polytopes Q_n of § 1 have this property. Note also that Hirsch's estimate for the diameter of a polyhedron cannot be improved in general, since for $n > d$ there exists a simple d-polyhedron J_n which has n facets and is of diameter $n - d$. To start the construction, let J_{d+1} be a half-cylinder over a $(d-1)$-simplex; that is, J_{d+1} is the linear sum in $\boldsymbol{R}^d$ of a $(d-1)$-simplex and a ray which is not parallel to the hyperplane determined by the simplex. Now suppose that we have constructed a simple d-polyhedron J_n such that any two vertices of J_n can be joined by a path of length $\leqq n - d$ and there are two vertices x_n and z_n of J_n which cannot be joined by any shorter path. Suppose further that x_n is incident to an unbounded edge ρ_n of J_n and to $d-1$ bounded edges whose other endpoints are $y_n^1, \cdots, y_n^{d-1}$. Then J_{n+1} is formed by truncating J_n at the vertex z_n, thus replacing z_n by a facet which is a $(d-1)$-simplex having vertices $z_{n+1} \in \rho_n \sim \{z_n\}$ and $y_{n+1}^i \in]y_n^i, z_n[$. Let $x_{n+1} \cdot = x_n$. Since the only approach to z_{n+1} along the edges of J_{n+1} is through the vertices y_{n+1}^i, it is evident that x_{n+1} and z_{n+1} cannot be joined in J_{n+1} by a path of length $< n + 1 - d$. Thus the induction can be carried through and the d-polyhedra J_n can be constructed as described.[10]

In discussing the W_v conjecture, we employ the notion of a *polyhedral cell-complex* (or simply complex), where this is a finite family $\boldsymbol{K}$ of polyhedra (the *cells* of $\boldsymbol{K}$) in a finite-dimensional real linear space such that each face of a cell of $\boldsymbol{K}$ is itself a cell of $\boldsymbol{K}$, and the intersection of any two cells of $\boldsymbol{K}$ is a face common to both.[11] If P is a polyhedron, the family of all faces of P forms a complex, as does the family of all faces other than P itself; the latter complex will be demoted by $\boldsymbol{B}(P)$ and will be called the *boundary complex* of P. The notions of *vertex*, *edge* and *path* are defined for complexes in the obvious way, and a path $(x_0, x_1, \cdots, x_l)$ in a complex will be called a W_v *path* provided the following condition is satisfied: if $i < j < k$ and a cell of $\boldsymbol{K}$ includes both x_i and x_k, then it includes x_j also. The W_e *paths* in $\boldsymbol{K}$ are similarly defined. These requirements may appear to be stronger than those for W_v paths in polyhedra, since they are not restricted to cells of a particular dimension. However, we note the following fact.

[10] At least for $d \leqq 3$, the unboundedness of the polyhedra in this construction is essential, for when $d \leqq 3$ the maximum diameter of d-polytopes having n facets is $[n(d-1)/d] - d + 2$ [**10**].

[11] This is simply a finite geometric cell-complex in the sense of Alexandroff & Hopf [**1**], except that the cells are not here required to be bounded.

2.2. *If P is a cell of a polyhedral cell-complex $\boldsymbol{K}$, then every W_v path on P is a W_v path in $\boldsymbol{K}$. The same is true of W_e paths.*

Proof. Since the two situations are similar, we consider only the former. Let $(x_0, \cdots, x_l)$ be a W_v path in P, and suppose a cell C of $\boldsymbol{K}$ includes x_i and x_k but not x_j, where $i < j < k$. Then the same is true of $C \cap P$, which is a proper face of P. Since $C \cap P$ is the intersection of all the facets of P which contain $C \cap P$, there must be a facet of $C \cap P$ which includes x_i and x_k but not x_j. This is a contradiction and completes the proof.

The following useful fact was noted by Clyde Kendall and Johns Rulifson.

PROPOSITION (Kendall and Rulifson) 2.3. *Suppose that $\boldsymbol{K}$ is a polyhedral cell-complex in the Euclidean plane, and that the vertex x of $\boldsymbol{K}$ can be joined to the vertex y by a path in $\boldsymbol{K}$. Among all paths from x to y in $\boldsymbol{K}$, let Π be one of minimum Euclidean length. Then Π is a W_v path.*

Proof. The result is an immediate consequence of the fact that if C is an open convex set in the Euclidean plane and u and v are boundary points of C which can be connected by an arc in the complement of C, then the shortest such connecting arc lies in the boundary of C.

With the aid of 2.3, we can *almost* prove the W_v conjecture for 3-polyhedra.

THEOREM 2.4. *Suppose that x and y are vertices of a 3-polyhedron P. If P is unbounded, then x and y can be joined by a W_v path on P. If P is bounded and F is a facet of P, then x and y can be joined by a path on P for which no facet other than F violates the W_v condition.*[12] (See footnote 17 at end of paper.)

Proof. If P is bounded, we merely apply 2.3 to the Schlegel diagram of P in F. This is obtained by choosing a point z outside P but near an inner point of F, so that for each point w of $P \sim F$ the intersection $[w, z] \cap F$ consists of an inner point $\psi(w)$ of F. The projection ψ takes the complex $\{C: C \in \boldsymbol{B}(P) \sim \{F\}\}$ onto a complex $\boldsymbol{K}$ in the plane of F, and then the desired conclusion follows from 2.3.

Now suppose that P is unbounded and assume without loss of

[12]That is, x and y are joined by a path $(x_0, x_1, \cdots, x_l)$ such that if $i < j < k$ and G is a facet of P which includes x_i and x_k, then G includes x_j or $G = F$.

generality that P lies in a 3-dimensional real linear space E. As is easily verified, the interior of P contains a ray ρ which issues from a boundary point of P. We may assume that the boundary point is the origin 0, whence $\rho =]0, \infty[u$ for some $u \in \operatorname{int} P$. Let H be a plane which supports P at 0, whence $E = H + \boldsymbol{R}u$ and each point q of E admits a unique expression in the form $q = q_h + q_r u$ with $q_h \in H$ and $q_r \in \boldsymbol{R}$. Let V denote the (finite) set of all vertices of P and let $m = \max\{v_r : v \in V\} + 1$, whence V is contained in the half-open strip $H + [0, m[u$ and thus the same is true of every bounded face of P. For each point $q \in H + [0, m[u$, let $\psi(q)$ denote the point at which H is intersected by the ray from mu through q. For each proper face C of P, let $C_H \cdot = \psi(C \cap (H + [0, m[u)$. It can be verified that $\{C_H : C \in \boldsymbol{B}(P)\}$ is a cell-complex which is combinatorially equivalent to the boundary complex $\boldsymbol{B}(P)$, and hence the desired conclusion follows from 2.3.

In order to complete the proof of the W_v conjecture for 3-polyhedra, we still must settle (rather than merely "almost settle") the case of 3-polytopes.

THEOREM 2.5. *Any two vertices of a* 3*-polytope P can be joined by a W_v path on P.* (See footnote 17 at end of paper.)

Proof. The proof is by induction on the number n of facets of P, the assertion being obvious when $n = 4$ for then P is a tetrahedron. Suppose that $n > 4$ and that the theorem has been proved for all 3-polytopes having fewer than n facets. Consider an arbitrary 3-polytope Q which has n facets. By a theorem of Steinitz & Rademacher **[19]**[13], the complex $\boldsymbol{B}(Q)$ is (combinatorially equivalent to one which is) obtained by means of a facet-splitting of type 1, 2 or 3 from a complex

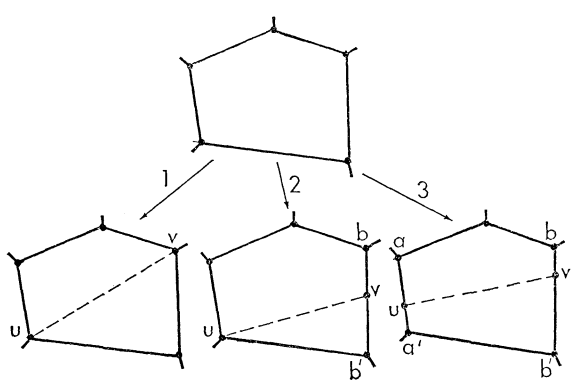

Figure 1

[13] See also Lyusternik **[15]**.

$\boldsymbol{B}(P)$, where P is a 3-polytope having $n-1$ facets. The three types of facet-splitting are depicted in Figure 1 above. Each involves the addition of a new edge $[u, v]$, cutting across one of the facets of P; a facet-splitting of type i involves also the addition of $i-1$ new vertices and the consequent splitting of $i-1$ of the edges of P. We shall follow the notation of Figure 1.

Now consider an arbitrary pair x and y of vertices of Q; we want to show that they can be joined by a W_v path Π on Q. If $\{x, y\} = \{u, v\}$, we let $\Pi = (x, y)$. Now suppose that x and y are both vertices of P, whence (according to the inductive hypothesis) they are joined by a W_v path Π_1 on P. If every edge of Π_1 is also an edge of Q, let $\Pi_2 = \Pi_1$. Alternatively, some edge(s) of Π_1 may be split in the transition from P to Q, and in this case Π_2 is obtained from Π_1 by the corresponding replacement(s) of one edge of P by two edges of Q. (For example, $\cdots, b, v, b', \cdots$ appears in Π_2 if $\cdots, b, b', \cdots$ appears in Π_1.) Then Π_2 is a path on Q, and it can be verified that Π_2 is a W_v path or both u and v appear in Π_2. In the latter case, a W_v path from x to y on Q is formed by simply omitting from Π_2 all vertices which appear between u and v.

For the remaining case, we may assume without loss of generality that $y = v, x \neq u$, and the splitting is of type 2 or type 3. Let Π_1 be a W_v path from x to b on P, and let Π_2 be formed as above. If u appears in Π_2, a W_v path Π_3 from x to y on P is obtained from Π_2 by inserting $v(=y)$ after u and discarding all vertices of Π_2 which appear after u. If u does not appear in Π_2 but v does appear, Π_3 is obtained from Π_2 by discarding all vertices of Π_2 which appear after v. If neither u nor v appears in Π_2, Π_3 is obtained from Π_2 by inserting v after b. In each case, Π_3 is a W_v path from x to y on Q, and this completes the proof.

Finally, we show under very restrictive hypotheses that shortest paths are W_v paths or W_e paths. Here (in contrast to 2.3) shortness is not a metric notion but rather involves the combinatorial notion of length employed in § 1.

PROPOSITION 2.6. *Suppose that p and q are vertices of a polyhedral cell-complex $\boldsymbol{K}$, and that Π is a shortest path from p to q in $\boldsymbol{K}$ (that is, a path involving the smallest possible number of vertices).*

If every cell of $\boldsymbol{K}$ is of diameter $\leqq 1$, then Π is a W_v path.

If every cell of $\boldsymbol{K}$ is of diameter $\leqq 3$, then Π is a W_e path.

If $\boldsymbol{K}$ is the boundary complex of a simple d-polyhedron P and every facet of P is of diameter $\leqq 2$, then Π is a W_v path or p and *q lie together in a facet F of P. (In the latter case, p and q can be joined in F by a path which is both a W_v path and a shortest path in* **K**. *However, Π itself need not be a W_v path.)*

Proof. Let $\Pi = (x_0, x_1, \cdots, x_l)$ and suppose that Π is not a W_v path. Then some cell C of $\boldsymbol{K}$ includes x_i and x_k but neither x_{i+1} nor x_{k-1}, where $0 \leqq i < i+1 \leqq k-1 < k \leqq l$. If $\delta(C) \leqq 1$[14], then $[x_i, x_k] \in \boldsymbol{K}$ and $(x_0, \cdots, x_i, x_k, \cdots, x_l)$ is a path from p to q of length $<l$, contradicting the assumption that Π is a shortest path. This establishes the first assertion of 2.6.

Under the hypotheses of the third assertion, we may assume that C is a facet of P. Since Π is a shortest path and $\delta(C) \leqq 2$, we see that $i + 1 = k - 1$ and there is a vertex y of C, not among the x_j's, such that $[x_i, y]$ and $[y, x_k]$ are both edges of C. Since P is a simple d-polyhedron, each of x_i and x_k is incident to exactly d edges of P and to exactly $d - 1$ edges of C, whence it follows that $i = 0$ or $x_{i-1} \in C$ and also that $k = l$ or $x_{k+1} \in C$. If $x_{i-1} \in C$ or $x_{k+1} \in C$, we may use the fact that $\delta(C) \leqq 2$ to produce a path of length $<l$ from p to q. Since this is impossible, it follows that $i = 0$, $k = l$, and $\{p, q\} \subset C$. The statement in parentheses is then easily verified, and the proof of the third assertion is complete.

For the second assertion of 2.6, let us assume that Π is not a W_e path, whence some cell C of $\boldsymbol{K}$ contains the edges $\sigma_i (= [x_{i-1}, x_i])$ and σ_k of $\boldsymbol{K}$ but neither σ_{i+1} nor σ_{k-1}, where $1 \leqq i < i+1 \leqq k-1 < k \leqq l$. In fact, $i+1 < k-1$, for otherwise $\sigma_{i+1} \subset C$. Thus we have $x_{i-1}, x_k \in C$ with $k \geqq i + 3$, and since $\delta(C) \leqq 3$ it is possible to connect p and q by a path shorter than Π. The contradiction completes the proof.

Figure 2 depicts a complex in which every cell is of diameter $\leqq 2$, and yet no shortest path from p to q is a W_v path. Figure 3

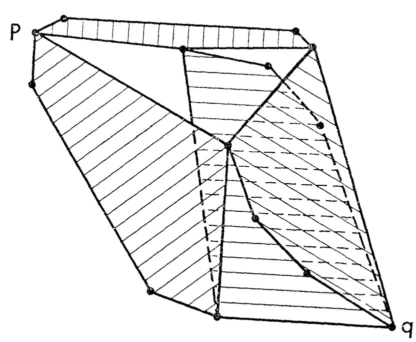

Figure 2

[14] $\delta(C)$ is the diameter of C.

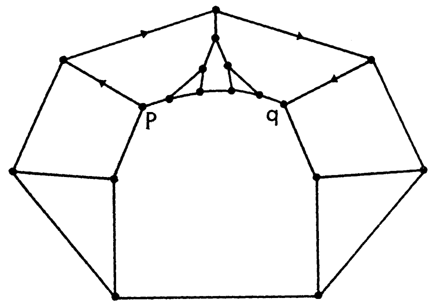

Figure 3

is the Schlegel diagram of a simple 3-polytope in which no shortest path from p to q is a W_v path.[15]

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[15] Here, of course, "shortest path" is in the sense of 2.6. D. Barnette [**24**] has produced a simple 3-polytope having only 14 vertices and at the same time having two vertices p and q such that no shortest path from p to q is a W_v path. He has proved that 14 is the smallest number for which this is possible.

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Footnotes added in Proof.

[16] In [**25**] there is constructed an unbounded 4-polyhedron which has 8 facets and is of diameter 5; necessarily (in view of 2.1) it has two vertices which cannot be joined by any W_v path. Also in [**25**] is a proof that the following two statements are equivalent, though not necessarily on a dimension-for-dimension basis: any two vertices of a simple polytope can be joined by a W_v path; on a d-polytope with n facets, any two vertices can be joined by a path of length $\leqq n$-d.

[17] D. Barnette [**24**] has proved that any two vertices of a 3-polytope can be joined by two independent W_v paths if they do not share an edge and by three independent W_v path if they do not share a facet. D. Walkup has remarked that 2.4 can be used to prove that any two vertices x and y of a 3-polytope P can be joined by a W_v path on P. Choose a vertex u different from x and y and a plane H such that $H \cap P = \{u\}$. Then apply 2.4 to the unbounded polyhedron which is the image of P under a projective transformation carrying H onto the plane at infinity.