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CHAINS OF MODULES WITH COMPLETELY REDUCIBLE QUOTIENTS

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Consider a left module V over a possibly noncommutative ring R. The objective is to investigate finite or infinite sequences of submodules of V of the form $\{0\} = A_0 \subsetneqq A_1 \subsetneqq A_2 \cdots$ or of the form $V = A^0 \supsetneqq A^1 \supsetneqq A^2 \cdots$ where all the quotient modules A_{i+1}/A_i or A^i/A^{i+1} are completely reducible. It is shown that some of the known properties of such series for a module over a ring with minimum condition hold for a more general class of rings, a class which properly includes those satisfying the descending chain condition. The main difficulty which this note has attempted to solve is to generalize these well known theorems from the minimum condition case to a much larger class of rings and modules. The class of rings considered in this note seems to be the natural setting in which to prove these theorems. In spite of the added generality, our proofs are not longer than they would be if the minimum condition were assumed.

All modules considered here will be understood to be left modules. A module V over a ring will be called *simple* provided it contains no proper nonzero submodules and provided also $RV = V \neq \{0\}$. A module is *completely reducible* provided it is a finite or infinite direct algebraic sum of simple modules.

In the next definitions and subsequently, the set inclusion symbol "$\subset$" will always indicate a proper inclusion. The next two definitions are essentially taken from [4, p. 103].

DEFINITION. Suppose V is any left module over an arbitrary ring R. Define $L_0(V)$ to be the zero module. For any positive integer $k = 1, 2, \cdots$ let $L_k(V)$ be defined inductively as the algebraic sum of all submodules Y of V with $L_{k-1}(V) \subset Y$ and with a simple quotient $Y/L_{k-1}(V)$. If $L_\alpha(V)$ has been defined for all $\alpha < \beta$; where β is a limit ordinal, set $L_\beta(V) = \cup\{L_\alpha(V) \mid \alpha < \beta\}$ and define $L_{\beta+1}(V)$ to be the sum of all submodules Y of V with $L_\beta(V) \subset Y$ and with a simple quotient $Y/L_\beta(V)$. The empty sum is taken to be the zero module. The series of submodules $\{0\} = L_0(V) \subsetneqq L_1(V) \subsetneqq L_2(V) \subsetneqq \cdots$ is called the lower Loewy series of V over R.

DEFINITION. For a left module V over a ring R, set $L^0(V) = V$

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and denote by $L^1(V)$ the intersection of all the maximal proper submodules of V. The empty intersection is by convention all of V and the zero module is a maximal submodule of a simple module. For any positive integer $k = 1, 2, \cdots$ the submodule $L^k(V)$ is defined inductively by $L^k(V) = L^1[L^{k-1}(V)]$. If $L(V)$ has been defined for all ordinals $\alpha < \beta$ where β is a limit ordinal, set $L^\beta(V) = \cup \{L^\alpha(V) \mid \alpha < \beta\}$ and define $L^{\beta+1}(V)$ to be $L^{\beta+1}(V) = L^1[L^\beta(V)]$. The series $V = L^0(V) \supseteqq L^1(V) \supseteqq L^2(V) \supseteqq \cdots$ is called the upper Loewy series of V over R.

Clearly, the quotients of the consecutive terms of the lower Loewy series are completely reducible; it will be shown later that this also holds for the upper series for all superscripts. It will also be shown that, in a sense which is more precisely defined in conclusions 3 of the Theorems 2 and 4, the lower and the upper Loewy series are the unique biggest chains of submodules with completely reducible quotients. The indices on the terms of the lower and upper Loewy series will henceforth be positive integers; except in the example at the very end, and even here they will only be ω and $\omega + 1$, where ω is the first infinite ordinal.

The radical of any ring R, denoted by N, will be the usual Jacobson radical, i.e., the intersection of the annihilators of all simple left R-modules U with $RU = U$. In case there do not exist any simple R-modules, $N = R$.

From now on it will be assumed that the module V satisfies the condition that

(1) for any submodule U of V, $RU = U$ · A restriction will be, imposed on the ring by requiring that

(2) R/N is completely reducible,

i.e., either zero or a direct sum of minimal left ideals $\bar{L}$ with $R\bar{L} = \bar{L}$. The ring R considered here is neither assumed to be commutative nor to contain a unity element. Indeed, one of our objectives has been to develop the properties of the two Loewy series so that they would be applicable to rings which cannot in principle contain a unity element, e.g., if R/N is the ring of all linear transformations with finite dimensional ranges on an infinite dimensional vector space. In the latter example, clearly (1) and (2) are satisfied.

2. The lower Loewy series.

The next lemma will be used frequently.

LEMMA 1. *Let R be a completely reducible ring with zero radical. Then any left R-module U with $RU = U$ is completely reducible.*

Proof. The ring R is the algebraic direct sum $R = \bigoplus\{L_i \mid i \in I\}$

where L_i is a minimal left ideal of R with $RL_i = L_i \neq \{0\}$, and I is some indexing set. For any $v \in U$, v is of the form $v = r_1w_1 + \cdots + r_nw_n$ where $r_k \in L_{i(k)}$ and $w_k \in U$ for $i(k) \in I$ and $k = 1, 2, \cdots, n$. Let L be a minimal left ideal of R with $RL = L$. It suffices to show that if for some $v \in U$ we have $Lv \neq 0$, that then Lv is a simple submodule of U. But, if W were a proper submodule of Lv, then $J = \{t \in L \mid tv \in W\}$ would be a left ideal of R properly contained in L.

COROLLARY. *Let R be a ring with radical N such that R/N is completely reducible. Assume that U is any left R-module with $RU = U$. Then U is completely reducible if and only if $NU = \{0\}$.*

Proof. If U is a sum of simple modules, then $NU = \{0\}$ because N annihilates any simple R-module. Conversely, if $NU = \{0\}$, then U viewed as an R/N-module is completely reducible over R/N and hence also over R.

The next theorem is the main result about the lower Loewy series. For any subset B of a ring R, the symbol $B^{\perp}$ denotes the subset of a fixed R-module V which is annihilated by B.

THEOREM 2. *Let the ring R with radical N and the module V be as in* (1) *and* (2). *Then the following holds for all integers $k = 0, 1, 2, \cdots$:*

1. *$L_{k+1}(V)/L_k(V)$ is completely reducible.*
2. *$L_k(V) = N^{k\perp}$.*
3. *Uniqueness: if $\{0\} = A_0 \subseteqq A_1 \subseteqq A_2 \subseteqq \cdots$ is any series of submodules of V with completely reducible quotient modules A_{k+1}/A_k for $k = 0, 1, 2, \cdots$, then $A_k \subseteqq L_k(V)$.*

Proof. Conclusion 1 is clear.

Conclusion 2 is true for $k = 0$. By definition $N^0 = R$. If there exist elements $v \in V$, $v \neq 0$, for which $Rv = 0$, let $U = \{v \in V \mid Rv = 0\}$. By (1), $RU = U$, a contradiction.

Assuming conclusion 2 to be true for $k - 1$, i.e., $L_{k-1}(V) = (N^{k-1})^{\perp}$, we prove it for k. Let U be the completely reducible left R-module $U = L_k(V)/L_{k-1}(V)$. Since $L_k(V)$ is a submodule of V, by assumption (1) we have $RL_k(V) = L_k(V)$ and consequently $RU = U$. Applying Lemma 1 to the module U, we find that $NL_k(V) \subseteqq L_{k-1}(V) = (N^{k-1})^{\perp}$ and hence that $L_k(V) \subseteqq N^{k\perp}$. However, since $NN^{k\perp} \subseteqq (N^{k-1})^{\perp}$, Lemma 1 again guarantees that the module $N^{k\perp}/(N^{k-1})^{\perp}$ is completely reducible, and hence we have the opposite inclusion $N^{k\perp} \subseteqq L_k(V)$.

To establish conclusion 3, assume $A_{k-1} \subseteqq L_{k-1}(V)$; the latter holds for $k = 1$ and 2. The complete reducibility of the module A_k/A_{k-1}

together with Lemma 1 implies that $NA_k \subseteqq A_{k-1}$ and hence that $NA_k \subseteqq L_{k-1}(V)$. Thus $N^k A_k \subseteqq N^{k-1}L_{k-1}(V) = \{0\}$, or $A_k \subseteqq N^{k\perp} = L_k(V)$.

3. The upper Loewy series. The next lemma is false unless the restriction (2) from the introduction is imposed on the ring.

LEMMA 3. *Let R be a ring with radical N such that R/N is completely reducible. Let U be a left R module with $RU = U$. Then for any $v \in U$ with $v \notin NU$, there exists a maximal proper submodule M of U with $NU \subseteqq M$, but $v \notin M$.*

Proof. By Lemma 1, the quotient module U/NU is a direct sum of simple modules. The vector $v + NU$ can be written as a unique finite sum of vectors each of which lies in a simple submodule of U/NU. Let P be a submodule of U with $NU \subseteqq P$, with P/NU simple, and such that $v + NU$ has a nonzero component in P/NU. Then U/NU is the direct sum

$$\frac{U}{NU} = \frac{P}{NU} \oplus \frac{M}{NU}$$

where M is a submodule of U with $NU \subseteqq M$. Clearly, $v \notin M$. Since

$$\frac{U}{M} \cong \frac{U/NU}{M/NU} \cong \frac{P}{NU},$$

it follows that U/M is simple and hence that M is maximal in U.

Next a counterexample is given to show that the last lemma is false if the ring R fails to satisfy (2). Let R be any commutative integral domain with a unit in which every ideal is principal and with a quotient field Q not equal to R. The module Q over R cannot contain a maximal proper submodule M. Otherwise, let $a/b \notin M$ with $a, b \in R$. Then $Q = M + R(1/b)$ and $1/b^2 = \mathrm{m} + n/b$ for some $m \in M$, $n \in R$. Hence $1/b = bm + n$ and $n \notin M$. Thus $R \nsubseteq M$ and for some $c \in R$ with $c \neq 1$ or 0, we have $R \cap M = Rc$. Since $Q = M + R(1/c)$, there exists $m \in M$ and $s \in R$ with $1/c^3 = m + s/c$. Consequently, $1/c = (mc^2 + sc) \in M$, contradicting the fact that $R \nsubseteq M$. If R has a finite number of primes, then the radical N of R is the principal ideal generated by the product of all the primes. In this case $NQ = Q$, and the previous lemma is not applicable. If R has an infinite number of primes then N is zero. Since R cannot contain a single minimal ideal the hypothesis (2) that R/N is a direct sum of minimal left ideals fails.

The following theorem about the upper Loewy series is a perfect analogue of Theorem 2 for the lower series.

THEOREM 4. *Let the ring R with radical N and the module V be* *as in* (1) *and* (2). *Then the following holds for all* $k = 0, 1, 2, \cdots$:

1. $L^k(V)/L^{k+1}(V)$ *is completely reducible.*
2. $L^k(V) = N^k V$.
3. *Uniqueness*: *if* $V = A^0 \supseteqq A^1 \supseteqq A^2 \supseteqq \cdots$ *is any series of submodules of* V *with completely reducible quotient modules* A^k/A^{k+1} *for* $k = 0, 1, 2$, *then* $A^k \supseteqq L^k(V)$.

Proof. By use of Lemma 1, conclusion 1 is an immediate consequence of conclusion 2.

In order to prove conclusion 2, assume by induction that $L^{k-1}(V) = N^{k-1}V$; the latter holds if $k = 1$, since $N^0 = R$ by definition and $RV = V$ by assumption. If $v \in L^{k-1}(V)$ but $v \notin NL^{k-1}(V)$, then by Lemma 3 the element $v \notin L^1[L^{k-1}(V)] = L^k(V)$. Hence $N^k V \supseteqq L^k(V)$. Conversely, if Q is a maximal submodule of $L^{k-1}(V)$, then by Lemma 1, $NL^{k-1}(V) \subseteqq Q$, and hence $N^k V \subseteqq L^k(V)$. Thus $L^k(V)$ equals $N^k V$.

Conclusion 3 is valid for $k = 0$, assume it for $k - 1$, i.e., $L^{k-1}(V) \subseteqq A^{k-1}$. Since the module A^{k-1}/A^k is completely reducible, we have $NA^{k-1} \subseteqq A^k$ and consequently that $L^k(V) = NL^{k-1}(V) \subseteqq NA^{k-1} \subseteqq A^k$.

REMARKS. (i) It is not clear whether the hypothesis in Theorems 2 and 4 that for any submodule U of V, $RU = U$ cannot be weakened to require only that $RV = V$. In order to do this, perhaps some additional hypothesis may have to be placed on R, e.g., $RN = N$.

(ii) The associativity of multiplication in the ring R was never used. For example, all of the previous considerations remain valid for a Lie module over a Lie ring.

(iii) It seems reasonable to conjecture that continuous analogues of the theorems of this paper could be formulated in order to be applicable to the case where the ring R consists of bounded linear transformations on a Hilbert space V. Then one would have to consider chains of subspaces of the Hilbert space V such that the quotient of any two adjacent terms would be a Hilbert space direct sum of closed R-invariant subspaces containing no proper closed R-invariant subspaces. The Jacobson radical would have to be replaced by a possibly bigger ideal—the intersection of all closed regular maximal left ideals of R. (If R were norm-closed and commutative, then any regular maximal ideal would be automatically closed.)

Next some corollaries of Theorems 2 and 4 are derived. In part 3 of the next lemma the ring R is viewed as a left R-module over itself.

COROLLARY 5. *Let the ring* R *and the module* V *be as in* (1) *and* (2) *and let* W *be any submodule of* V. *For conclusion* 3 *assume in addition that for any left ideal* I *of* R, $RI = I$. *Then for all* $k = 0, 1, 2, \cdots$ *the following hold:*

$$L_k(W) = W \cap L_k(V) \tag{1a}$$

(1b) $$L^k(W) \subseteqq W \cap L^k(V)$$

(2a) $$L^k[V/W] = \frac{L^k(V) + W}{W}$$

(2b) $$\frac{L_k(V) + W}{W} \cdot \subseteqq L_k[V/W]$$

(3a) $$L^k(R)V = L^k(V)$$

(3b) $$L_k(R)V \subseteqq L_k(V) .$$

The proof of the last lemma is an immediate consequence of parts 2 of Theorems 2 and 4. In order to show that the inclusion relation in the b parts can be strictly proper, let R be the ring of all $n \times n$ matrices with zeroes above the diagonal, coefficients in any field and with n and k satisfying $0 < k \leqq n - 2$. In this case the radical N consists of all matrices with zeroes on the diagonal; for any integer i in $0 \leqq i \leqq n$, $N^{i\perp} = N^{n-i}$ and $L_i(R) = N^{n-i}$. To obtain strictly proper inclusions in (1b), (2b), and (3b), simply take $V = N$ and $W = N^{n-1}$.

Conclusions (2a) and (2b) of the last corollary can be rephrased by saying that in the category whose objects are all left R-modules V such that $RU = U$ for any submodule U of V and whose maps are module homomorphisms, $L^k(V)$ is a functor but $L_k(V)$ is not.

It is not clear what the most natural hypothesis on the ring R and the module V should be in order for the conclusion of the next corollary to be valid.

COROLLARY 6. *Let R, N, and V be as in* (1) *and* (2). *In addition let V be a faithful R-module. Assume further that for any integer $k = 0, 1, 2, \cdots$ for which $N^k \neq \{0\}$, there exists some $x \in R$ such that $N^k x \neq \{0\}$ but $N^{k+1}x = \{0\}$. Then if N is nilpotent, the lower Loewy series has a finite number of terms, the last term being V. If N is not nilpotent, the lower Loewy series is an infinite ascending chain.*

Proof. It suffices to show that if $N^k \neq \{0\}$, that then $N^{k\perp} \subset (N^{k+1})^\perp$. Take $x \in R$ such that $N^k x \neq \{0\}$ but $N^{k+1}x = \{0\}$. Then $xV \subseteqq (N^{k+1})^\perp$, but by the faithfulness of V, $xV \nsubseteqq N^{k\perp}$; consequently $N^{k\perp} \subset (N^{k+1})^\perp$.

It is interesting to note in connection with the next corollary that there is an example of a ring R whose radical consists entirely of nilpotent elements but with $N^2 = N$, see [2, p. 72] and [7, p. 50].

COROLLARY 7. *Let R and V be as in* (1) *and* (2) *and assume in addition that V is a finitely generated R-module. Then the upper Loewy series of V over R either terminates with zero in a finite number of steps or is an infinite descending chain.*

Proof. Under the assumption that the k-th term $N^k V$ of the upper Loewy series of V is not 0, it has to be shown that $N^{k+1}V \subset N^k V$. It suffices to show, more generally, that for any finitely generated module W over $\boldsymbol{R}$, $NW \neq W$. If $NW = W$, then the fact that N is a left ideal implies that W is also finitely generated over N and not merely over $\boldsymbol{R}$. But now by [5, p. 200, Proposition 2] we have $NW \neq W$, a contradiction.

EXAMPLES. (i) In sharp contrast to the minimum condition case, the following simple examples show that even if the ring $\boldsymbol{R}$ and module V satisfy conditions (1) and (2), the upper and lower Loewy series need not be of the same length. Let $\boldsymbol{R}$ be the ring of all matrices over any field with an infinite number of rows and columns, with only a finite number of nonzero entries, and having zero entries above the main diagonal. The module V is to consist of column vectors with only a finite number of nonzero components. Then N consists of all matrices with zeroes on the main diagonal. Clearly, $\boldsymbol{R}$ and V satisfy (1) and (2). For any integer k, $N^k V$ consists of all column vectors with first k components zero; $N^{k\perp} = \{0\}$ and consequently $\boldsymbol{L}_\omega(V) = \boldsymbol{L}_{\omega+1}(V) = 0$, where ω is the first infinite ordinal. Hence the lower series has each term zero whereas the upper series is an infinite properly descending chain with $L^\omega(V) = \{0\}$. If α is an ordinal which is not a positive integer, then there just does not seem to be in general any way of characterizing the terms $\boldsymbol{L}_\alpha(V)$ and $\boldsymbol{L}_\alpha(V)$ of the Loewy series in terms of the radical. In the above example the expected analogue of conclusion 2 in Theorem 2, i.e., $\boldsymbol{L}_\omega(V) = (\bigcap_1^\infty N^k)^\perp$, is false.

(ii) An example satisfying (1) and (2) where $\boldsymbol{L}_\omega(V) \subset \boldsymbol{L}_{\omega+1}(V) = V$ can be constructed by taking $\boldsymbol{R}$ as the ring of all matrices with an infinite number of rows and columns, with zeroes below the main diagonal, with only a finite number of nonzero entries off the main diagonal, and having all the entries on the diagonal eventually constant. The module V is to consist of all column vectors with an infinite number of components but with all the components from some point on eventually constant. The radical N is the set of all matrices with zeroes on the diagonal; $N^{k\perp}$ consists of all vectors having all components zero except possibly the first k. Hence $\boldsymbol{L}_\omega(V)$ is the subspace of all vectors having only a finite number of nonzero components. Since $V/\boldsymbol{L}_\omega(V)$ is isomorphic to the coefficient field, $\boldsymbol{L}_{\omega+1}(V) = V$. Although the lower Loewy series is a proper infinite ascending chain, the upper series contains only two distinct terms.

REFERENCES

1. E. Artin, C. J. Nesbitt, and R. M. Thrall, *Rings with minimum condition*, University

of Michigan Press Ann Arbor, 1955.
2. N. Bourbaki, *Modules et anneaux semi-simples*, Algebre, Chapter 8, Hermann, Paris, 1958.
3. R. Brauer, *On sets of matrices with coefficients in a division ring*, Trans. Amer, Math. Soc. **49** (1941), 502-548.
4. R. Brauer and E. Weiss, *Non-commutative rings*, Harvard mimeographed notes (1962).
5. N. Jacobson, *Structure of rings*, Amer. Math. Soc. Colloq. Pub. **37** (1956).
6. C. E. Rickart, *Banach algebras*, Van Nostrand, New York, 1960.
7. A. D. Sands, *Primitive rings of infinite matrices*, Proc. Edin, Math. Soc. (2) **14** (1964), 47-53.

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