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A PROBLEM COMPLEMENTARY TO A PROBLEM OF ERDÖS

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Let $f(x)$, $g(x)$, and $h(x)$ be rational integer coefficient polynomials of positive degree and with positive leading coefficients and satisfying

$$f(x) = g(x) + h(x) \,. \tag{1.1}$$

$k(x)$ also being such a polynomial of degree $\geqq 0$, let

$$Q(x) = (f(x))! / ((g(x) + k(x))!\ (h(x))!) \,. \tag{1.2}$$

Question 1: Is $Q(x)$ integral for an infinity of integers x, at least when $k(x)$ is of degree zero, say $k(x) = k(\geqq 1)$?

Question 2: Is $Q(x)$ nonintegral for all sufficiently large integers x, at least when the degree of $k(x)$ is $\geqq 1$? No general answer is known to both these questions. In this paper, we consider the question of existence of an infinity of integers x for which $Q(x)$ is not an integer: in the context of question 1, we obtain certain conditions on the coefficients of $g(x)$ and $h(x)$ and k to ensure the existence of an infinity of integers x for which $Q(x)$ is not an integer, and in the context of question 2, we prove $Q(x)$ is nonintegral infinitely often.

The method rests upon a generalization of the usual representation of an integer a in the scale of a prime p so as to include negative coefficients also and the consequent generalization of the well known result of Legendre concerning the exponent of the highest power of the prime p that divides $a!$.

As regards to question 1, which is a generalization of a problem of Erdös (Research problem, American Mathematical Monthly, May 1947) who takes $g(x) = h(x) = x$, we know, however, by (i) of Theorem I of [1] that some multiple of $Q(x)$, i.e., $Q(x)L(x)$ is an integer infinitely often where $L(x)$ is the integer coefficient G.C.D. (in fact, the monic G.C.D. over the rationals) with least positive leading coefficient of the polynomials

$$\prod_{i=1}^{k} (f(x) + i),\ \prod_{i=1}^{k} (g(x) + i),\ \text{and}\ \prod_{i=1}^{k} (h(x) - i + 1) \,.$$

In the case of Erdös problem $(g(x) = h(x) = x)$, $L(x) = 1$, and it is easily seen that $Q(x)$ is an integer for all integers $x \geqq 1$ in case $k = 1$, while $Q(x)$ is not an integer for all integers $x = 1 + 2^j$ in case

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$k = 3$. Also, it is easy to give examples of similar situations with degrees of $g(x)$ and $h(x)$ greater than 1 and with all coefficients of $g(x)$ and $h(x)$ positive. Our generalization mentioned above enables to construct examples of similar situations in which some of the coefficients of $g(x)$ and $h(x)$ may be negative.

For convenience, we shall write, for any positive integers a, b, and c, $h(a, c)$ to stand for the exponent of the highest power of c that divides a and $D(a/b, c)$ for $h(a, c) - h(b, c)$.

THEOREM I. *If $k(x)$ is of positive degree*

(i) $\lim_{t\to\infty} D(Q(p^t), p) = -\infty$ *for each prime p;*

(ii) $\lim_{p\to\infty} D(Q(p^2), p) = -\infty$

(iii) *If $k(x)$ is of degree at least 2,* $\lim_{p\to\infty} D(Q(p), p) = -\infty$.

Theorem I obviously implies that $Q(x)$ is not an integer when x is sufficiently large power of a prime or the square of a sufficiently large prime and if $k(x)$ is of degree $\geqq 2$, when x is any sufficiently large prime.

THEOREM II. (a) *If $k(x)$ is of degree zero, say $k(x) = k$,*

$g(x) = a_0 + a_1x + \cdots + \cdots,$

$h(x) = b_0 + b_1x + \cdots + \cdots,$ *and*

$f(x) = c_0 + c_1x + \cdots + \cdots$

so that for each i, $c_i = a_i + b_i$, then for sufficiently large primes p,

$$D(Q(p), p) \geqq 0 \text{ if} \tag{1.3}$$

$$\text{either } a_0 \geqq 0 \text{ or } a_0 < 0 \text{ and } a_0 + k < 0 \text{ and} \tag{1.4}$$

$$D(Q(p), p) \geqq -r \text{ if} \tag{1.5}$$

$$a_0 < 0, a_1 = a_2 = \cdots = a_{r-1} = 0 \neq a_r \text{ and } a_0 + k > 0 . \tag{1.6}$$

(b) *The inequality in* (1.3) *becomes an equality if together with* (1.4), *the following condition*

(1.7) *Not both a_i and b_i are negative and $c_i < 0$ for $i > 0$ implies $a_ib_i \neq 0$.*
holds. The inequality in (1.5) *becomes an equality if* (1.6) *and* (1.7) *hold.*

THEOREM III. (a) *If k and n are integers, $k \geqq 1$, $n > 1$ there exists an infinity of integers x such that*

$$(nx)!/\{(x + k)!\}^n \tag{1.8}$$

is not an integer.

(b) *If* a_1, a_2 *and* c_1 *are positive integers and if there is a prime* p *such that*

$$a_1 + a_2 < p \leqq a_1 + c_1 , \tag{1.9}$$

there exists an infinity of integers x *such that*

$$((a_1+a_2)\, x)!/((a_1x+c_1)!(a_2x)!) \tag{1.10}$$

is not an integer.

REMARK. We do not know whether (1.8) is an integer infinitely often in case $k > 1$; however, we know that it is in case $k = 1$ (see Mordell's paper listed under references in [1]). Also (1.10) is integer infinitely often (see Theorem IV of [1]).

§2: DEFINITION 1. Let a be a positive integer and p a prime. An expression

(2.1) $a_0 + a_1p + a_2p^2 + \cdots + a_np^n$, where

(2.1a) (i) $a = a_0 + a_1p + a_2p^2 + \cdots + a_np^n$, and

(ii) $a_n > 0$, $|a_i| < p$ for $0 \leqq i \leqq n$

is called a representation of order n of a in the scale of p; the representation is called proper if $a_i \geqq 0$ for each i and improper otherwise.

The proper representation (which is unique) is the usual representation of a in the scale of p. It is easily seen that if n_0 is the order of the proper representation, there is no representation of order $< n_0$ while to each $n > n_0$, there are representations of order n.

DEFINITION 2. If R is a representation of a in the scale p given by (2.1), we denote

(i) by $S_R(a, p)$ the integer $\sum_{i=0}^{n} a_i$, and

(ii) by $I_R(a, p)$ the number of negative terms plus the number of zeros following immediately a negative term in the sequence of integers

$$a_0, a_1, \cdots a_n , \tag{2.2}$$

which may be called the digits of a in this representation R of a in the scale of p.

EXAMPLE. $15{,}524 = -1 + 0.3 + 0.3^2 + 2.3^3 - 3^4 + 3^5 + 0.3^6 - 2.3^7 + 0.3^8 + 3^9.$

In this representation R of 15,524 in the scale of 3, $S_R(15{,}524{,}3) = 0$ and $I_R(15{,}524{,}3) = 6$

LEMMA 1. *If* R *is the representation of* a *in the scale of* p *given by* (2.1), *then*

(i) *for each* i *in* $0 \leqq i \leqq n$

$$T_i = a_n p^{n-i} + a_{n-1} p^{n-i-1} + \cdots + a_i > 0 . \tag{2.3}$$

(ii) *If in the sequence of integers* (2.2), *there are* N *blocks* $B_1, B_2, \cdots, B_N$ *of negative terms each not immediately followed by a zero and there are* M *blocks of negative terms* $C_1, C_2, \cdots C_M$, *the block* C_i *being immediately followed by a block* D_i *of zeros and if* r_i *is the number of terms in* B_i *and* s_i *and* t_i *respectively are the number of terms in* C_i *and* D_i, *then*

$$h(a!, p) = (((a - S_R(a, p))/(p - 1)) - \left\{\sum_{i=1}^{N} r_i + \sum_{i=1}^{M} (s_i + t_i)\right\} \tag{2.4}$$

REMARKS. (i) The number in the curly brackets above is $I_R(a, p)$.
(ii) If $N = 0$ and $M = 0$, so that the representation is proper, Lemma 1 reduces to the well known result due to Legendre.

Proof (i) We have $a = pT_1 + a_0 > 0$; we observe that $T_1 \not< 0$; for, otherwise, it would follow that a_0 is greater than a positive multiple of p, contradicting (2.1a).

Further $T_1 \neq 0$; for, if it were zero, then from $T_1 = pT_2 + a_1$, it would follow that a_1 is divisible by p and so again by (2.1a) that $a_1 = 0$ and consequently $T_2 = 0$. Thus proceeding, we arrive at the contradiction $a_n = 0$.

Starting with T_1, we get $T_2 > 0$ and so on.

(ii) We have from (2.1a) and (2.3)

$[a/p] = T_1 + \theta_0$ where $\theta_0 = [a_0/p]$, so that

$$\theta_0 = 0 \quad \text{if } a_0 \geqq 0$$
$$= -1 \quad \text{if } a_0 < 0 .$$

$[a/p^2] = [[a/p]/p] = T_2 + \theta_1$ where $\theta_1 = [(a_1 + \theta_0)/p]$ so that

$$\theta_1 = 0 \quad \text{if either } a_1 \geqq 0,\ \theta_0 = 0 \quad \text{or } a_1 > 0,\ \theta_0 = -1 ;$$
$$= -1 \text{ if either } a_1 \leqq 0,\ \theta_0 = -1 \text{ or } a_1 < 0,\ \theta_0 = 0 .$$

In general, if $1 \leqq r \leqq n + 1$,

$$[a/p^r] = T_r + \theta_{r-1}, \text{ where } \theta_{r-1} = [(a_{r-1} + \theta_{r-2})/p] \text{ so that}$$

$$\theta_{r-1} = 0 \quad \text{if either } a_{r-1} \geqq 0,\ \theta_{r-2} = 0 \quad \text{or } a_{r-1} > 0,\ \theta_{r-2} = -1 ;$$
$$= -1 \text{ if either } a_{r-1} \leqq 0,\ \theta_{r-2} = -1 \text{ or } a_{r-1} < 0,\ \theta_{r-2} = 0 .$$

It is clear, now, that if a_i is the first negative term and a_j is the first positive term that occurs immediately after a_i in the sequence (2.2), then $\theta_i = \theta_{i+1} = \cdots = \theta_{j-1} = -1, \theta_j = 0$, even though there are some l's such that $i < l < j$ and $a_l = 0$. The lemma is clear since

$$h(a!, p) = \sum_{r=1}^{\infty} [a/p^r] .$$

NOTE. From the proof, it is clear that, if in (2.2) two blocks of negative terms include between them a block of zeros, the three blocks taken together can be regarded as a negative block.

As an immediate consequence of the lemma, we have the following:

COROLLARY. *If R and R' are any two representations of a in the scale of p,*

$$S_R(a, p) - S_{R'}(a, p) = (p - 1)\{I_{R'}(a, p) - I_R(a, p)\} .$$

DEFINITION 3. For any polynomial $\varphi(x)$ over the domain of integers given by

$$\varphi(x) = e_0 + e_1 x + e_2 x^2 + \cdots + e_n x^n ,$$

$$S_\varphi(p) = \sum_{\substack{i=0 \\ e_i \neq 0}}^{n} S_{R_0}(|e_i|, p)\,\mathrm{sgn}(e_i) \tag{2.5}$$

where R_0 denotes proper representation; and

$$S(\varphi) = \sum_{i=0}^{n} e_i . \tag{2.6}$$

LEMMA 2. *Let $\varphi(x) = e_0 + e_1 x + e_2 x^2 + \cdots + e_n x^n$, $e_n > 0$, be an integer coefficient polynomial and p a prime, also if $e_i \neq 0$ let λ_i, μ_i be the exponents of the smallest and highest powers of p that occur in the proper representation of $|e_i|$ in the scale of p; let $e_{i_1}, e_{i_2}, \cdots, e_{i_m}$ be the negative terms each not immediately followed by a zero and $e_{j_1}, e_{j_2}, \cdots e_{j_l}$ be the negative terms each immediately followed by a zero, say e_{j_r} is followed by a block of U_r zeros in the sequence $e_0, e_1, \cdots e_n$; further, let t satisfy*

(2.7) (i) $t > \underset{\substack{0 \leq i \leq n \\ e_i \neq 0}}{\mathrm{Max}}\ \mu_i$ and

(ii) $\varphi(p^t) > 0$; *then*

$$\begin{aligned} h(\varphi(p^t)!, p) = {} & ((\varphi(p^t) - S_\varphi(p))/(p - 1)) \\ & - \{(\sum_{r=1}^{l} U_r) + l + m\}t - (\sum_{r=1}^{m} \lambda_{i_r+1} - \lambda_{i_r}) \\ & - \sum_{r=1}^{l} (\lambda_{j_r+U_r+1} - \lambda_{j_r}) . \end{aligned}$$

Proof. The lemma follows, if we express each $|e_i| \neq 0$ in the proper representation of p and make use of Lemma 1, the note at the end of its proof and (2.5).

§ 3: *Proof of Theorem I.* (i) Choose t so large that conditions (i) and (ii) of (2.7) are satisfied for $f(x)$, $g(x) + k(x)$ and $h(x)$. By Lemma 2,

$$h(f(p^t)!, p) = ((f(p^t) - S_f(p))/(p - 1)) + A_1 t + B_1 \tag{3.1}$$

where A_1 and B_1 are numbers independent of t. Similarly,

$$\begin{aligned} &h((g(p^t) + k(p^t))!, p) \\ &= ((g(p^t) + k(p^t) - S_{g+k}(p))/(p - 1)) + A_2 t + B_2 \end{aligned} \tag{3.2}$$

and

$$h(h(p^t)!, p) = ((h(p^t) - S_h(p))/(p - 1)) + A_3 t + B_3 , \tag{3.3}$$

where A_2, B_2, A_3 and B_3 are independent of t. From (3.1), (3.2) and (3.3), it follows that

$$\begin{aligned} D(Q(p^t), p)/t &= (- k(p^t)/(p - 1)t) \\ &+ (\{S_{g+k}(p) + S_h(p) - S_f(p)\}/(p - 1)t) + (A_1 - A_2 - A_3) \\ &+ (B_1 - B_2 - B_3)/t . \end{aligned} \tag{3.4}$$

Taking limits on both sides of (3.4) as $t \to \infty$, and observing that the expression in curly brackets on $R. H. S.$ of (3.4) is independent of t, we get (i).

(ii) Choose p large enough to ensure the substitution of p for x in $f(x)$, $g(x) + k(x)$ and $h(x)$ gives the representation of the numbers $f(p)$, $g(p) + k(p)$ and $h(p)$ in the scale of p. (ii) follows by an application of Lemma 1 and proceeding to the limit as $p \to \infty$.

(iii) The proof is similar to that of (ii).

Proof of Theorem II. (a) Choose p large enough as in the proof of (ii) of Theorem I. In this representation, say R_p, $a_0 + a_1 p + \cdots + \cdots$ of $g(p)$ in the scale of p, obviously $S_{R_p}(g(p), p) = S(g)$. Also $I_{R_p}(g(p), p) =$ the number of negative terms plus the number of zeros immediately following a negative term in $a_0, a_1, \cdots$; let us denote this number by $I(g)$, and similarly for others.

First, we prove that

$$I(g) + I(h) - I(f) \geqq 0 . \tag{3.5}$$

To prove (3.5), let us observe that

$$c_i < 0,\ a_i b_i = 0,\ a_i \neq 0 \text{ implies } a_i < 0$$
$$c_i < 0,\ a_i b_i = 0,\ b_i \neq 0 \text{ implies } b_i < 0$$

$c_i < 0$, $a_i b_i \neq 0$ implies one of a_i and b_i is negative; so that the contribution to $I(f)$ by a negative c_i is balanced by the contribution of a negative a_i or b_i to $I(g) + I(h)$. Further, let $c_i = 0$, $c_j < 0$, $c_{j+1} = c_{j+2} = \cdots = c_i$, if $a_i b_i \neq 0$, one of a_i and b_i is negative, if $a_i = 0 = b_i$, let λ be the largest integer such that $\lambda < i$ and one of a_λ, b_λ is not zero; clearly $\lambda \geqq j$ and one of a_λ, b_λ is negative. So in any case, the contribution of c_i to $I(f)$ is balanced and (3.5) is clear. Next, we observe that

$$I(g + k) = I(g) \text{ if and only if (1.4) holds}\,, \tag{3.6}$$

and

$$I(g + k) = I(g) - r, \text{ if and only if (1.6) holds}\,. \tag{3.7}$$

Further, by Lemma 1,

$$D(Q(p), p) = I(g + k) + I(h) - I(f)\,. \tag{3.8}$$

Now (1.3) follows from (3.8), (3.6) and (3.5) and (1.5) follows from (3.8), (3.7) and (3.5).

It is easily verified that (1.7) implies the equality sign in (3.5) and the proof is complete.

We now consider an example: Taking $g(x) = 1 - x^r + x^n$, $h(x) = -2 + x^r + x^n$ and $k =$ any odd integer > 1, it can be shown by an application of Lemma 1, that

$$(2x^n - 1)!/((x^n - x^r + 1 + k)!(x^n + x^r - 2)!)$$

is not an integer for $x = 2^t$ where t is sufficiently large. In particular, taking $n = 2$, $r = 1$, it is easily verified that $L(x) = 1$ and so it follows that

$$(2x^2 - 1)!/((x^2 - x + 1 + k)!(x^2 + x - 2)!)$$

is an integer infinitely often and a non integer infinitely often.

Proof of Theorem III (a) It is easily verified by taking proper representations, that, in case $k \geqq 2$

$$D((np^t)!/\{(p^t + k)!\}^n, p) < 0 \text{ where}$$

$p \mid k$ and t is sufficiently large and in case $k = 1$, $D(\{n(-1 + 2^t)\}!/\{(-1 + 2^t + 1)!\}^n, 2) < 0$, where t is sufficiently large. Hence (i).

(ii) Again, by taking proper representations in the scale of p where p satisfies (1.9), it is easy to verify that for $x = 1 + p + p^2 + \cdots + p^t$ (t sufficiently large) that

$$D(((a_1 + a_2)x)!/(a_1 x + c_1)!(a_2 x)!, p) < 0\,.$$

REFERENCE

1. J. Chidambaraswamy, *Divisibility properties of certain factorials*, Pacific J. Math. **17** (1966), 215-226.

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