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HOMOMORPHISMS AND SUBDIRECT DECOMPOSITIONS OF SEMIGROUPS

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Dedicated to Professor Alexander Doniphan Wallace on the occasion of his sixtieth birthday

Subdirect decompositions of rings seem to be an important tool in the theory of rings promoting the development of this theory. It is a very natural thing to study subdirect products of semigroups but to the author's knowledge the only paper on the topic is that of G. Thierrin [22] where certain properties of subdirectly irreducible semigroups are considered.

Subdirect decompositions of semigroups are closely connected with homomorphisms of these semigroups, so we describe in the first section the structure of an arbitrary congruence on a semigroup. The second section is devoted to certain special subsets and elements of a semigroup. Main notions of the section are those of disjunctive element (i.e., an element that does not form a congruence class modulo any nontrivial congruence) and of core of a semigroup (i.e., a least nonnull ideal). Subdirectly irreducible semigroups are considered in the third, fourth and fifth sections. We consider certain general properties of such semigroups and find characterizations of special classes of such semigroups (e.g. nilpotent, idempotent, commutative). Section 6 treats homomorphically simple (h-simple) semigroups, i.e., semigroups having no nontrivial congruences. Section 7 is devoted to consideration of certain semigroups having special subdirect decompositions. By analogy with f-regular rings [3] we introduce f-regular semigroups. There are considered also completely reductive semigroups, i.e., semigroups having no nononreductive homomorphic images.

Several results of this paper have been published without proofs in our note [18]. Certain results of [18] had been previously found in [22] but we did not know this when [18] was published. All concepts of the theory of semigroups that are not defined here are defined in [6, 12]. We use the symbols Λ, $\rightarrow$, $\leftrightarrow$, $\bigwedge$ respectively for conjunction, implication, (logical) equivalence, universal quantifier and follow the ordinary agreement as to the use of brackets in statements. If ε is an equivalence relation, then $\varepsilon\langle g\rangle$ is the ε-class containing g and $g_1 \equiv g_2(\varepsilon)$ or $g_1 \equiv g_2$ means that g_1 and g_2 are in the relation ε. If G is a semigroup then G^1 denotes G with adjoined identity (unless G already has an identity), G° denotes G with adjoined zero (unless G already has a zero). Variables g and h (with or without indices) take values in the set of all elements

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of G, variables x and y take values in the set of all elements of G^1 (i.e., x and y may be void symbols ([12], p. 7)). A one-element set is often denoted in the same way as its element. As a rule, one-element semi-groups are excluded from consideration. Δ_G is the identity relation on the set G.

Congruences on semigroups. It is known that the consideration of homomorphisms may be limited to the consideration of congruences on semigroups (it is necessary to emphasize that such a limitation can lead to an essential loss of information—e.g., when one considers automorphisms).

Let $\Theta(G)$ denote the complete lattice of all congruences on a semigroup G. A minorant basis of $\Theta(G)$ is any subset B of $\Theta(G)$ such that each element of $\Theta(G)$ is the greatest lower bound (i.e., the intersection) of some subset of B.

If H is a subset of a semigroup G then the equivalence $\mathscr{C}_H$ defined as follows:

$$g_1 \equiv g_2(\mathscr{C}_H) \leftrightarrow (\bigwedge x, y)[xg_1y \in H \leftrightarrow xg_2y \in H] \tag{1.1}$$

is a congruence [17, 21] - the principal congruence determined by H. R. Pierce [14] and R. Croisot [7] define $\mathscr{C}_H$ in another way (they do not allow x and y in (1.1) to be void). Clearly $\mathscr{C}_H = \mathscr{C}_{H'}$, where H' is the complementation of H in G. E.g., $\mathscr{C}_\phi = \mathscr{C}_G = G \times G$.

Let ε be an equivalence on a semigroup G. The greatest congruence included in ε is called the *stable opening* of ε (cf. [16]). We omit the straightforward proof of the following

LEMMA 1.1. *Every equivalence ε on a semigroup G possesses a stable opening $\bar{\varepsilon}$ and $\bar{\varepsilon} = \bigcap(\mathscr{C}_{\varepsilon\langle g\rangle})_{g\in G}$.*

Every congruence coincides with its stable opening, so every congruence ε coincides with intersection of the family of principal congruences generated by ε-classes.

THEOREM 1.2. *The set of all principal congruences on a semigroup forms a minorant basis of the lattice of all congruences.*

This theorem is no longer true if principal congruences are understood in the sense of R. Croisot [7].

Let G be a subsemigroup of a ring R. Then every congruence of the ring R induces a congruence on G. It is easy to prove that if R is the semigroup ring of G over the ring of integers, then every congruence of G is induced by some congruence of R.

2. Disjunctive element and cores of semigroups. A subset H of a semigroup G is called *indivisible by an equivalence* ε (by a subset F) if H is contained in some ε-class ($\mathscr{C}_F$-class). H is called *saturated for* ε, if H is the union of a family of ε-classes. M. Teissier [21] has proved that $\mathscr{C}_H$ is the greatest congruence for which H is saturated.

Let us associate with every subset H the subset $r(H)$ defined by the formula

$$g \in r(H) \leftrightarrow (\bigwedge x, y)[xgy \notin H] \tag{2.1}$$

$r(H)$ is called the (bilateral) *residue* of H in G. It is an ideal of G and, if nonempty, an $\mathscr{C}_H$-class. If x and y in (2.1) are not permitted to be void, one comes to the definition of residue in the sense of R. Croisot [7]. In our previous papers we designated $\mathscr{C}_H$ by $\underset{H}{\varepsilon}$ and $r(H)$ by W_H. It follows from (2.1) that $r(H) \subset H'$, i.e., $r(H) \cap H = \phi$.

H is called *neat* if $r(H) = \phi$. If G contains a zero 0 and $0 \notin H$ then H cannot be neat because $0 \in r(H)$. H is called 0-*neat* if $r(H)$ contains at most one element (i.e., $r(H)$ coincides with ϕ or with $\{0\}$). Both notions are identical for a semigroup without zero.

A subset H is called *disjunctive* if the only subsets indivisible by $\mathscr{C}_H$ are empty and one-element. This means that

$$\mathscr{C}_H = \Delta_G \,. \tag{2.2}$$

Disjunctive subsets were considered by E. J. Tully, Jr. [23], M. P. Schützenberger [20] and (in a slightly different sense) by R. Pierce [14].

An element g is called 0-*neat* (*disjunctive*) if $\{g\}$ is 0-neat (disjunctive). In [18] disjunctive elements were called separative.

The least nonempty ideal of a semigroup G (if it exists) is called the *kernel* of G. The kernel of a semigroup with zero is trivial. We call an ideal *nonnull* if it contains at least two elements. The least nonnull ideal (if it exists) is called the *core* of G. The core and the kernel of a semigroup without zero coincide.

THEOREM 2.1. *Let G be a semigroup with disjunctive zero. Then an element of G is disjunctive if and only if it is* 0-*neat.*

Proof. If an element k is disjunctive then by (2.2) $r(k)$ contains at most one element. Hence k is 0-neat. Now let k be 0-neat. If $k = 0$ then k is disjunctive, so let $k \neq 0$. For every $g_1, g_2 \in G$ there exist x, y such that exactly one of elements xg_1y, xg_2y is equal to 0 (this follows from disjunctivity of 0). Let $xg_1y \neq 0$. Then there exist x_1, y_1 such that $x_1xg_1yy_1 = k$. But $xg_2y = 0$, so $x_1xg_2yy_1 = 0 \neq k$, and $g_1 \not\equiv g_2(\mathscr{C}_k)$, i.e., k is disjunctive.

Let K be the core of G. Since K^2 is an ideal, one has two alternatives: $K^2 = K$ or $K^2 = 0$. In the first case we call K *globally idempotent*, in the second case K is called *nilpotent*.

The set of all right annihilating elements of a semigroup G (i.e., such elements g that $Gg = 0$) is called the *right annihilator* of G. The *left annihilator* is defined dually. The set of all (left and right) annihilating elements is called the *annihilator* of G. The right annihilator is called *trivial* if it contains at most one element.

A core K is called *primitive* if it contains exactly one nonzero element.

THEOREM 2.2. *If a semigroup G having a core K contains a nonzero element g such that $GgG = 0$, then K is included in the right or in the left annihilator of G. If K is included in both, then K is primitive and coincides with the annihilator. In every case the core K is nilpotent.*

Proof. Let $M(G)$ be the set of all $g \in G$ such that $GgG = 0$. $M(G)$ is a nonnull ideal, so $K \subset M(G)$ and $GKG = 0$. Therefore GK and KG are ideals. If GK is nonnull, then $K \subset GK$, so $GK = K$ and $KG = 0$, i.e., K is contained in the left annihilator. Analogously, if KG is nonnull, then K is contained in the right annihilator. If $GK = KG = 0$ then K is contained in the annihilator. Every subset of the annihilator is an ideal, hence the annihilator cannot contain more than two elements. So K coincides with the annihilator and is primitive. Clearly, in every case $K^2 = 0$.

It is easy to prove that *if the core K is globally idempotent then $KgK = K$ for every nonzero $g \in G$.*

3. General properties of subdirectly irreducible semigroups. Let $(G_i)_{i \in I}$ be a nonempty family of semigroups. Its direct product is a semigroup $\times(G_i)_{i \in I}$ whose set of elements is the Cartesian product of the family of sets of elements G_i, the operation is defined componentwise.

A subsemigroup G of the semigroup $\times(G_i)$ is called a *subdirect product* of the family $(G_i)_{i \in I}$ of semigroups if $pr_i(G) = G_i$ for all $i \in I$ (here pr_i denotes the natural projection of $\times(G_i)$ on G_i). Clearly, all pr_i are homomorphic mappings of G.

We say that a semigroup S is *decomposable as a subdirect product* of a family $(G_i)_{i \in I}$ of semigroups if S is isomorphic to the subdirect product of $(G_i)_i$. This isomorphism is called a *subdirect decomposition* of S.

If the subdirect decomposition followed by some one of the projections pr_i is an isomorphism of S with G_i, then this decomposition is called *trivial*. A semigroup is called *subdirectly irreducible* if each of its subdirect decompositions is trivial. In the same way direct decompositions and directly irreducible semigroups are defined.

Two following theorems are true ([2], Th. 10, p. 92; [1], p. 765):

THEOREM 3.1. *Every semigroup is decomposable as a subdirect product of a family of subdirectly irreducible semigroups.*

THEOREM 3.2. *A semigroup G is subdirectly irreducible if and only if it possesses the least nonidentical congruence.*

If a family (ε_i) of congruences on G has Δ_G as its intersection, then G is decomposable as a subdirect product of a family (G/ε_i) of semigroups ([2], Th. 9, p. 92).

THEOREM 3.3. *Every subdirectly irreducible semigroup has at least two different disjunctive elements.*

Proof. Consider the congruence $\bigcap(\mathscr{C}_g)_{g\in G}$ on a subdirectly irreducible semigroup G. Each of $\{g\}$ is a $\mathscr{C}_g$-class, so our congruence is equal to Δ_G. Since G is subdirectly irreducible, $\mathscr{C}_{g_1} = \Delta_G$ for some g_1 (by Theorem 3.2). Now consider the congruence $\bigcap(\mathscr{C}_g)_{g\neq g_1}$. Every $g \neq g_1$ forms its congruence class, so our congruence is equal to Δ_G and there exists $g_2 \neq g_1$ such that $\mathscr{C}_{g_2} = \Delta_G$. g_1 and g_2 are two different disjunctive elements, by (2.2).

Disjunctive elements are 0-neat, every nonzero 0-neat element belongs to every nonnull ideal, i.e., belongs to a core.

COROLLARY 3.3.1. [22]. *Every subdirectly irreducible semigroup has a core.*

All disjunctive elements of a semigroup belong to its core. But a semigroup having a core need not be subdirectly irreducible (as it is for rings).

THEOREM 3.4. *Let $\mathscr{H}$ be a family of subsets of a subdirectly irreducible semigroup and let the intersection $\bigcap H$ or the union $\bigcup H$ of this family be disjunctive. Then at least one subset in the family $\mathscr{H}$ is disjunctive.*

Proof. It is easy to prove (cf. [17], p. 387) that

$$\bigcap (\mathscr{C}_{H_i})_{i\in I} \subset \mathscr{C}_{\bigcap (H_i)_{i\in I}}\,, \qquad \bigcap (\mathscr{C}_{H_i})_{i\in I} \subset \mathscr{C}_{\bigcup (H_i)_{i\in I}}\,. \tag{3.1}$$

If $\mathscr{H} = (H_i)$ and $\bigcap \mathscr{H}$ or $\bigcup H$ are disjunctive, then $\bigcap (\mathscr{C}_{H_i}) = \Delta_G$. Hence there exists $i \in I$ such that $\mathscr{C}_{H_i} = \Delta_G$, i.e., H_i is disjunctive.

COROLLARY 3.4.1. *Let G be a subdirectly irreducible semigroup, K its core, ε_0 the least nonidentical congruence on G, k a disjunctive element of G, $H \subset G$. Every disjunctive subset of G contains at least one disjunctive element. H is saturated for ε_0 if and only if it is not disjunctive. H is disjunctive if and only if $H \cap K$ is such. If $g \in G$ and $\{k, g\}$ is not disjunctive, then every subset of G containing k but not g is disjunctive; excluding at most one, every two-element subset containing k is disjunctive.*

This follows from Theorem 3.4 and evident equality $H = (H \cap K) \cup (H \backslash K)$. If $\{k, g\}$ is not disjunctive then every subset containing k but not g intersects with $\{k, g\}$ by $\{k\}$.

Let g, h be elements of a semigroup G. If $gh = h$ then g is called a *left unit* for h. If $gh = hg = h$, then g is called a *unit* for h. An element e is called *central* if $eg = ge$ for every $g \in G$.

THEOREM 3.5. *Let G be a subdirectly irreducible semigroup and e a central element of G. If e is a unit for some nonzero element then e is the identity of G.*

Proof. Let e be a unit for $g \neq 0$, K be the core of G and $k \in K$. There exist x, y such that $xgy = k$ (since k is 0-neat), so $ke = k$, since e is central. So for every $k \in K$ $ke = k$ and for an arbitrary positive $nke^n = k$. Consider a binary relation $\varepsilon_{(e)}$ defined by the formula: $g_1 \equiv g_2(\varepsilon_{(e)}) \leftrightarrow g_1 e^m = g_2 e^n$ for some positive integers m and n. Clearly, $\varepsilon_{(e)}$ is a congruence and it induces on K the identical congruence. Let $\varepsilon^K = K \times K \cup \Delta_G$ be the congruence generated by K. Then $\varepsilon^K \cap \varepsilon_{(e)} = \Delta_G$. Since G is subdirectly irreducible and $\varepsilon^K \neq \Delta_G$, $\varepsilon_{(e)} = \Delta_G$. For every $g \in G$ $g \equiv ge(\varepsilon_{(e)})$, so $g = ge$.

Every central idempotent is a unit for itself. So we have:

COROLLARY 3.5.1. [22]. *A subdirectly irreducible semigroup does not contain central idempotents different from zero and identity.*

THEOREM 3.6. *Semigroups G and G^1 (G and G^0) are simultaneously subdirectly irreducible or reducible.*

Proof. If G has an identity, then $G = G^1$ and there is nothing to prove. Let G be a semigroup without identity. If ε is a congruence on G, then $\varepsilon^1 = \varepsilon \cup \{(1, 1)\}$ is a congruence on G^1. If ε_0 is the least nonidentical congruence on G^1, then it induces the least nonidentical congruence on G. It is easy to prove that if ε_0 is the least nonidentical congruence on G then ε_0^1 is the least nonidentical congruence on G^1. The proof for G and G^0 is analogous.

THEOREM 3.7. *A semigroup G with disjunctive zero is subdirectly irreducible if and only if it satisfies one of the following conditions* (*which are equivalent*):

(1) *G contains at least two different* 0*-neat elements.*

(2) *G contains at least two different disjunctive elements.*

(3) *G has a core.*

Proof. By Theorem 2.1, conditions (1)–(3) are equivalent. They are necessary for subdirect irreducibility (Theorem 3.3). Now let G satisfy (1)–(3), k be a 0-neat element different from 0. If ε is a nonidentical congruence, then $\{0\}$ cannot be an ε-class (otherwise $\varepsilon \subset \mathscr{C}_0 = \Delta_G$). So $0 \equiv g(\varepsilon)$ for some nonzero g. Since k is 0-neat there exist x and y such that $xgy = k$, so $0 = x0y \equiv xgy = k$. This is true for every $k \in K$, so $\varepsilon^K \subset \varepsilon$ and the congruence ε^K generated by K is the least nonidentical congruence on G. G is subdirectly irreducible, by Theorem 3.2.

COROLLARY 3.7.1. *A semigroup with primitive core is subdirectly irreducible if and only if its zero is disjunctive.*

A subset H of a semigroup G is called a *left reductor* if it has the property:

If $g_1, g_2 \in G$ and $hg_1 = hg_2$ *for every* $h \in H$, *then* $g_1 = g_2$.

H is a *right reductor* if it satisfies the dual property.

We define the congruences ε_r and ε_l by the formulas

$$g_1 \equiv g_2(\varepsilon_r) \leftrightarrow (\bigwedge g)[g_1 g = g_2 g] \tag{3.2}$$

$$g_1 \equiv g_2(\varepsilon_l) \leftrightarrow (\bigwedge g)[g g_1 = g g_2]\ . \tag{3.3}$$

G is called *right* (*left*) *reductive* if $\varepsilon_r = \Delta_G$ ($\varepsilon_l = \Delta_G$).

Let G be a subdirect product of a family (G_i) of right reductive semigroups and $g_1 g = g_2 g$ for every $g \in G$. Then $pr_i(g_1)pr_i(g) = pr_i(g_2)pr_i(g)$. The elements $pr_i(g)$ run over the whole set G_i, so $pr_i(g_1) = pr_i(g_2)$, i.e., $g_1 = g_2$. We have proved:

THEOREM 3.8. *A subdirect product of a family of right* (*left*) *reductive semigroups is right* (*left*) *reductive.*

Let G be subdirectly irreducible, K its core. Denote the congruence $\varepsilon_r \cap \varepsilon_1 \cap \varepsilon^K$ by ε. Let $g_1 \equiv g_2(\varepsilon)$. If g_1 or g_2 does not belong to K, then $g_1 = g_2$. Let $g_1, g_2 \in K$. Then

$$(3.4) \qquad (\bigwedge g)[g_1 g = g_2 g \wedge g g_1 = g g_2] .$$

Let k be disjunctive and different from g_1 and g_2. If $x g_1 y = k$, then x or y is not void, so $x g_1 y = x g_2 y = k$. Therefore $g_1 \equiv g_2(\mathscr{C}_k)$, i.e., $g_1 = g_2$. Two alternatives are possible: 1) $\varepsilon = \Delta_G$. Therefore ε_r or ε_1 is identical, i.e., G is right or left reductive. 2) $\varepsilon \neq \Delta_G$. Then there exist $g_1 \neq g_2$ such that $g_1 \equiv g_2(\varepsilon)$. G has no disjunctive elements different from g_1, g_2. So g_1 and g_2 are disjunctive (by Theorem 3.3). We have proved:

THEOREM 3.9. *If a subdirectly irreducible semigroup is neither right nor left reductive, then it contains exactly two disjunctive elements g_1 and g_2 and these elements satisfy* (3.4).

4. Special classes of subdirectly irreducible semigroups. A *homogroup* is a semigroup which contains a kernel that is a group [5, 6]. A semigroup is a homogroup if and only if the intersection of all right, left and two-sided ideals of the semigroup is not empty. Every semigroup with zero is a homogroup. If K is the kernel of a homogroup G then the identity of the group K is a central idempotent of G([12], p. 252). If G is subdirectly irreducible, then this central idempotent is a zero or an identity of G, by Corollary 3.5.1. In the second case $G = K$. So we have:

THEOREM 4.1. *Every subdirectly irreducible homogroup without zero is a group.*

It follows that a subdirectly irreducible semigroup which is not a group does not contain nonzero zeroid elements in the sense of [5]. It follows also that the core of this semigroup is not a group.

A semigroup is called a *nilsemigroup* if some power of every element is equal to zero (the power may be different for different elements).

LEMMA 4.2. *If a nilsemigroup has a core, this core is primitive and coincides with the annihilator of the semigroup.*

Proof. Let G be a nilsemigroup and K its core, k_1, k_2 be two nonzero elements of K. Since k_1 and k_2 are 0-neat, there exist x, y, x_1, y_1 such that $x k_1 y = k_2$ and $x_1 k_2 y_1 = k_1$. So $(x_1 x)^n k_1 (y y_1)^n = k_1$ for every n, i.e., $(x_1x)^n \neq 0$ and $(yy_1)^n \neq 0$. Hence, x, y, x_1, y_1 are void and $k_1 = k_2$, i.e., K contains a single nonzero element, say, k. If $GK = GK \cup \{0\}$ is not equal to 0, then $GK = K$ and $gk = k$ for some $g \in G$. So $g^n k = k$ for every n. But $g^n = 0$ for some n. So $GK = 0$. Analogously, $KG = 0$. Hence K is the annihilator, by Theorem 2.2.

By this lemma and Corollary 3.7.1,

THEOREM 4.3. *A nilsemigroup is subdirectly irreducible if and only if it contains a disjunctive zero and has the core.*

LEMMA 4.4. *If a semigroup with a nontrivial annihilator contains a disjunctive element, then this semigroup has a disjunctive zero.*

Proof. If 0 is disjunctive, there is nothing to prove. Let k be a nonzero disjunctive element. k is 0-neat, so G has the core. The core necessarily coincides with the annihilator, so k is annihilating. Let g_1 and g_2 be two different elements. Then there exist x and y such that exactly one of elements xg_1y, xg_2y is equal to k. Let $xg_1y = k$. If $xg_2y \neq 0$, then for some x_1, y_1 $x_1xg_2yy_1 = k$, since $k =$ 0-neat. So x_1 and y_1 are not both void. The element k is annihilating, therefore $x_1xg_1yy_1 = x_1ky_1 = 0$. So 0 is disjunctive.

THEOREM 4.5. *A nilsemigroup is subdirectly irreducible if and only if it contains a nonzero disjunctive element.*

Proof. Such a semigroup has a core. The semigroup is subdirectly irreducible, by Lemmas 4.2, 4.4 and Theorem 4.3.

Homomorphic images of nilsemigroups are nilsemigroups, so every nilsemigroup is decomposable as a subdirect product of a family of subdirectly irreducible nilsemigroups.

A semigroup G is called *nilpotent* if $G^n = 0$ for some positive n. A subdirect product of nilpotent semigroups need not be nilpotent, but it is easy to prove that *a semigroup G is isomorphic to a subdirect product of a family of nilpotent semigroups if and only if the ideal $\bigcap G^n$* (for all positive n) *is null.*

THEOREM 4.6. *A nilpotent semigroup is subdirectly irreducible if and only if it contains a disjunctive element.*

Proof. By Theorem 4.5, it is sufficient to prove that a nilpotent semigroup with disjunctive zero is subdirectly irreducible.

If $G_n = 0$ and $G^{n-1} \neq 0$, then G^{n-1} is included in the annihilator of G. Let k_1, k_2 be nonzero annihilating elements. $xk_1y = 0$ means that x or y is not void. Therefore, $k_1 \equiv k_2(\mathscr{C}_0)$, i.e., $k_1 = k_2$. So G^{n-1} is the annihilator containing exactly two elements. Clearly, G^{n-1} is a core, hence G is subdirectly irreducible, by Theorem 4.3.

COROLLARY 4.6.1. *A nilpotent semigroup is subdirectly irreducible if and only if it has a disjunctive zero.*

THEOREM 4.7. *Let G be a subdirectly irreducible idempotent semigroup, K be a core of G. If G is a semigroup without zero, then one of the following two properties hold:*

(1) *K is the set of all right zeros of G and a left reductor.*

(2) *K is the set of all left zeros of G and a right reductor.*

If G has a zero, then the complementation of the zero is a subsemigroup satisfying (1) or (2).

Proof. Let G have a zero. Then K is 0-simple, hence K is a completely 0-simple semigroup ([6], Corollary 2.56). Completely 0-simple idempotent semigroups are rectangular bands with adjoined zeros (cf. [6], Exercise 2.7.9). Let $g_1g_2 = 0$, $g_1 \neq 0$, $g_2 \neq 0$. Then $Kg_1K = K$ (see the last sentence of § 2). Therefore $Kg_1 \neq 0$ and $g_2K \neq 0$, i.e., there exist $k_1, k_2 \in K$ such that $k_1g_1 \neq 0$ and $g_2k_2 \neq 0$. So $k_1g_1 \cdot g_2k_2 = 0$. This equality contradicts to the fact that K is a rectangular band with adjoined zero. Hence, $g_1 = 0$ or $g_2 = 0$, i.e., the complementation of 0 is a subsemigroup. This subsemigroup is subdirectly irreducible, by Theorem 3.6. $K\backslash\{0\}$ is the core of this subsemigroup, so it does not contain zero.

Now let G be a subdirectly irreducible idempotent semigroup without zero. We have just seen that K is a rectangular band. Define two equivalences ε_1 and ε_2 on G. $g_1 \equiv g_2(\varepsilon_1)$ means that $g_1 = g_2$ or that $g_1, g_2 \in K$ and $g_1g_2 = g_2$ (the last equality implies $g_2g_1 = g_1$). Let $g_1 \equiv g_2(\varepsilon_1)$. If $g_1 = g_2$, then $gg_1 \equiv gg_2(\varepsilon_1)$ and $g_1g \equiv g_2g(\varepsilon_1)$. If $g_1 \neq g_2$ then $g_1, g_2 \in K$ and $g_1g_2 = g_2$. Therefore $gg_1gg_2 = gg_1gg_1gg_2 = gg_1g_2 = gg_2$ and $gg_1 \equiv gg_2(\varepsilon_1)$. $g_1gg_1 = g_1 \cdot g_1gg_1 \cdot g_1 = g_1$, since K is a rectangular band and $g_1gg_1 \in K$. Hence, $g_1gg_2g = g_1gg_1g_2g = g_1g_2g = g_2g$ and $g_1g \equiv g_2g(\varepsilon_1)$. So ε_1 is a congruence. ε_2 is defined in a dual way ($g_1g_2 = g_1$, if $g_1, g_2 \in K$). Clearly, $\varepsilon_1 \cap \varepsilon_2 = \Delta_G$. This means that ε_1 or ε_2 is identical, i.e., K is a right zero semigroup or a left zero semigroup. Let K be a right zero semigroup, i.e., $k_1k_2 = k_2$ for $k_1, k_2 \in K$. Then $gk = gk \cdot k = k$ for every $k \in K$ and $g \in G$, i.e., K is a set of right zeros of G. If g is a right zero, then $g = Kg \subset K$, i.e., K is the set of all right zeros. Define $g_1 \equiv g_2(\varepsilon)$ if and only if $kg_1 = kg_2$ for all $k \in K$. ε is a congruence, since K is an ideal. Clearly, $\varepsilon \cap \varepsilon^K = \Delta_G$, so $\varepsilon = \Delta_G$.

This means that K is a left reductor.

Semigroups satisfying (2) were considered by E. S. Ljapin [11].

5. Commutative subdirectly irreducible semigroups. Commutative subdirectly irreducible rings have been described in [13]. We shall consider now commutative subdirectly irreducible semigroups. We distinguish three kinds of such semigroups.

Semigroups of the first kind are subdirectly irreducible abelian groups (with or without adjoined zero).

THEOREM 5.1. *An abelian group is subdirectly irreducible if and only if it is a subgroup of p^∞-group* (i.e., *if it is a p^∞-group or a cyclic group of order p^n, where p is a prime*).

Proof. A subdirectly irreducible abelian group G has a least nonunit subgroup A, by Theorem 3.2. Since A does not contain any proper nonunit subgroup, it is a cyclic group of a prime order p. If $g \neq 1$ and $[g]$ is the cyclic subgroup of G generated by g, then $A \subset [g]$, so g is an element of a finite order pm. Therefore the group $[g]$ has a subgroup of order m. In the same manner we prove that $m = 1$ or m is a multiple of p. So g is an element of order p^n, i.e., G is a p-group. It is well-known ([10], § 25, p. 164) that every directly irreducible p-group is a p^∞-group or cyclic. So G coincides with one of the groups listed in Theorem 5.1. Clearly, these groups possess least nonunit subgroups and are subdirectly irreducible.

Thus, the concepts of direct and subdirect irreducibility are identical for periodic Abelian groups. However, these concepts differ in the general case. E.g., a directly irreducible Abelian torsion-free group is not subdirectly irreducible.

Semigroups of the first kind are exactly subdirectly irreducible commutative semigroups with globally idempotent cores. The complementation of zero in a commutative semigroup with globally idempotent core is a subsemigroup (otherwise the core is nilpotent). And a subdirectly irreducible commutative semigroup without zero and with globally idempotent core is a group, by Theorem 4.1.

Semigroups of the second kind are subdirectly irreducible commutative semigroups having nontrivial annihilator. By Theorem 2.2, the core of such a semigroup is primitive and coincides with the annihilator. By Lemma 4.4,

THEOREM 5.2. *A commutative semigroup with a nontrivial anni-*

hilator is subdirectly irreducible if and only if it contains a nonzero disjunctive element.

COROLLARY 5.2.1. *A commutative semigroup is a semigroup of the second kind if and only if contains a nonzero annihilating disjunctive element.*

COROLLARY 5.2.2. *Subdirectly irreducible commutative nilsemigroups are semigroups of the second kind.*

Subdirectly irreducible commutative semigroups different from the semigroups of the first two kinds are *semigroups of the third kind.* Hence, semigroups of the third kind are subdirectly irreducible commutative semigroups with a nilpotent core and trivial annihilator.

A divisor of zero is called *nontrivial* if it is different from zero. The set of all nondivisors of zero of a commutative semigroup is either empty or forms a subsemigroup.

THEOREM 5.3. *A commutative semigroup is a semigroup of the third kind if and only if it contains an identity, a nontrivial divisor of zero and a nonzero disjunctive element, and the set of all nondivisors of zero forms a subdirectly irreducible group.*

Proof. Let G be a semigroup of the third kind with a core K, F be the set of all elements annihilating the core, i.e., $f \in F \leftrightarrow Kf = 0$. Since $K^2 = 0$, $K \subset F$. Clearly, F is an ideal. G is not of the second kind, so K is not the annihilator. Therefore $G \neq F$. Let A be the complementation of F in G, K_0 be the set $K \backslash \{0\}$, F_0 be the set $F \backslash K$. So $\{\{0\}, K_0, F_0, A\}$ is a partition of G (F_0 may be empty). If $a \in A$, then $aK \neq 0$, i.e., $aK = K$, because aK is an ideal of G. The set of all g such that $ag = 0$ forms an ideal. K is not included in this ideal, therefore this ideal is null, i.e., a is not a divisor of zero. Since elements of F are divisors of zero, A is the set of all nondivisors of zero. G has the trivial annihilator, so for every $k \in K_0$ $K = GK = Ak \cup \{0\}$, i.e., there exists such an element $e \in A$ that $ek = k$. By Theorem 3.5, e is an identity of G. Let $a \in A$ and $k \in K_0$. Then $Aak = K_0$, so there exists $a_1 \in A$ such that $a_1ak = k$. By Theorem 3.5, $a_1a = e$, so a_1 is the inverse of a and A is a subgroup of G. Let $a_1k = a_2k$ for $k \in K_0$, $a_1, a_2 \in A$. Then $k = a_1^{-1}a_2k$, $a_1^{-1}a_2 = e$ and $a_1 = a_2$. $Ak = K_0$, therefore the sets A and K_0 have the same cardinality.

Let ε be a congruence on a group A generated by subsemigroup $\mathfrak{a}$. Define a binary relation $\varepsilon_{\mathfrak{a}}$ by the formula: $g_1 \equiv g_2(\varepsilon_{\mathfrak{a}}) \leftrightarrow g_1 \in g_2\mathfrak{a}$. It is easy to verify that $\varepsilon_{\mathfrak{a}}$ is a congruence on G that induces on A the congruence ε. Let (ε_i) be a family of congruences on A with identical intersection and $(\varepsilon_{\mathfrak{a}_1})$ be the family of corresponding congruences on G. If $k_1, k_2 \in K_0$, then $k_1 = ak_2$ for some $a \in A$. Therefore $k_1 \equiv k_2(\varepsilon_{\mathfrak{a}_1})$ means that $k_1 \in k_2\mathfrak{a}_i$ or that there exists $a_i \in \mathfrak{a}_i$ such that $k_1 = k_2 a_i$, or that $k_2 a = k_2 a_i$, or $a = a_i$, or $a \in \mathfrak{a}_i$. So $k_i \equiv k_2(\bigcap(\varepsilon_{\mathfrak{a}_i})) \leftrightarrow a \in \bigcap(\mathfrak{a}_i) \leftrightarrow a = 1 \leftrightarrow k_1 = k_2$. Therefore $\varepsilon^K \cap (\bigcap(\varepsilon_{\mathfrak{a}_i})) = \Delta_G$ and, since G is subdirectly irreducible, there exists i such that $\varepsilon_{\mathfrak{a}_i} = \Delta_G$ and $\varepsilon_i = \Delta_A$. Hence, A is subdirectly irreducible.

Now let G be a commutative semigroup satisfying the conditions of our theorem. G has a trivial annihilator and contains a nontrivial divisor of zero. Hence, G is not a semigroup of the first or the second kind. It is sufficient to prove the subdirect irreducibility of G.

Let A be the set of all nondivisors of zero, k_0 be a nonzero disjunctive element and K the core of G (K exists since G has a nonzero disjunctive element). If g is a nontrivial divisor of zero, then $gg_1 = 0$ for some $g_1 \neq 0$. For every $k \in K$ there exist x and y such that $xg_1y = k$, so $gk = xgg_1y = 0$ and k is an annihilating element for the set $F = G \backslash A$, i.e., $FK = 0$. If A is one-element, then the set of all divisors of zero, i.e., the complementation of A, is a semigroup satisfying all conditions of Theorem 5.2, i.e., $G = H^1$ where H is a semigroup of the second kind. G is subdirectly irreducible, by Theorem 3.6.

Let A have more than one element. Since A is subdirectly irreducible, it has a least nonunit subsemigroup $\mathfrak{a}$. Let ε be a nonidentical congruence on G. k_0 does not form an ε-class, so there exists $g \neq k_0$ such that $k_0 \equiv g(\varepsilon)$. If $g \notin K$, then for some x, y $xgy = k_0$, whence $xk_0y \equiv k_0(\varepsilon)$. x and y are not both void. If $xy \in A$, then $g = (xy)^{-1}k_0 \in K$. So $xy \in A$ and $xk_0y = 0$. Therefore $k_0 \equiv 0(\varepsilon)$ and $ak_0 \equiv 0(\varepsilon)$ for every $a \in \mathfrak{a}$, i.e.,

$$\mathfrak{a}k_0 \times \mathfrak{a}k_0 \subset \varepsilon \,. \tag{5.1}$$

If $g \in K_0$, then since $K = Gk_0 = Ak_0 \cup \{0\}$, $g = a_0k_0$ for some $a_0 \in A$. In this case the set of all $a \in A$ such that $ak_0 \equiv k_0(\varepsilon)$ forms a nonunit subgroup of A (this subgroup contains $a_0 \neq 1$). So $\mathfrak{a}$ is included in this subgroup and (5.1) is valid. Hence, (5.1) is always valid. Let ε_0 be the intersection of all nonidentical congruences on G. By formula (5.1), $\mathfrak{a}k_0$ is not divisible by ε_0, so ε_0 is not identical and G is subdirectly irreducible, by Theorem 3.2.

COROLLARY 5.3.1. *A semigroup G is of the second kind if and only if G^1 is a semigroup of the third kind in which all elements different from identity are divisors of zero.*

The "if" part has just been proved and the "only if" part follows from Theorems 3.6 and 5.2.

COROLLARY 5.3.2. *Every semigroup of the third kind has the following structure: it is the union of four mutually disjoint sets* $\{0\}$, K_0, F_0, A, *where* F_0 *may be empty,* K_0 *and* A *are not empty.* 0 *is a zero,* A *is the set of all nondivisors of zero;* A *forms a subdirectly irreducible Abelian group;* $K = K_0 \cup \{0\}$ *is the nilpotent core and* $KF = 0$, *where* $F = F_0 \backslash K$. *Sets* A *and* K_0 *are of the same cardinality and* K_0 *is the set of all disjunctive elements of a semigroup (if* A *is one-element, then* 0 *is also disjunctive).* $AF_0 = F_0$ *and* F *is an ideal of the semigroup. For every* $a \in A$ *and* $k \in K_0 a K_0 = K_0$, $Ak = K_0$.

The greater portion of these propositions has been proved above. It follows from the formula $A_k = K_0$ that K_0 is the set of all disjunctive elements.

Note that semigroups of the third kind are a particular case of Rauter's "Übergruppen" [15].

A semigroup is called *periodic* if each of its elements generates a finite subsemigroup. Using the terminology of [4], we may say that semigroups of the third kind are extensions by semigroups of the first kind of semigroups all whose elements are divisors of zero. Periodic semigroups of the third kind are extensions by Abelian groups with zero of nilsemigroups. Finite commutative nilsemigroups are nilpotent, so subdirectly irreducible finite commutative semigroups are cyclic groups (possibly, with zero), finite nilpotent semigroups and extensions of nilpotent semigroups by cyclic groups with zero.

Though all subdirectly irreducible Abelian groups are periodic, this is not true for semigroups. We possess examples of nonperiodic semigroups of the second and of the third kinds.

6. Homomorphically simple semigroups.

A semigroup is called *homomorphically simple* (or, *h-simple*) if it has only two congruences: identical and universal. Clearly, such semigroups are subdirectly irreducible. They have no proper nonnull ideals.

A subset B of a set A is *proper*, if $B \neq \varnothing$ and $B \neq A$.

THEOREM 6.1. *A semigroup is h-simple if and only if each of its proper subsets is disjunctive.*

Proof. Let G be h-simple, H be a proper subset of G. Clearly, $\mathscr{C}_H$ is not universal. So $\mathscr{C}_H = \Delta_G$, i.e., H is disjunctive. If every proper subset of a semigroup is disjunctive, then the semigroup is h-simple, by Theorem 1.2.

THEOREM 6.2. *A semigroup with zero is h-simple if and only* *if each of its elements is disjunctive.*

Proof. Every element of an h-simple semigroup is disjunctive, by Theorem 6.1. If all elements of a semigroup with zero are disjunctive and ε is a nonidentical congruence, then $\varepsilon\langle 0\rangle$ is a nonnull ideal. Every disjunctive element belongs to the core, so our semigroup has no proper nonnull ideals, i.e., $\varepsilon\langle 0\rangle$ coincides with the whole semigroup and ε is universal.

It follows from Theorems 2.1 and 6.2:

COROLLARY 6.2.1. *A semigroup with zero is h-simple if and only if it is* 0*-simple and contains a disjunctive zero.*

This condition is very similar to a somewhat more strong condition of L. M. Gluskin [8].

THEOREM 6.3. *If a semigroup with disjunctive zero has a globally idempotent core, this core is an h-simple semigroup.*

Proof. Let k_1, k_2 be distinct elements of a core K. Then there exist x and y such that exactly one of elements xk_1y, xk_2y (say, xk_1y) is equal to 0. $Kxk_2yK = K$ (cf. with the last sentence of § 2). Therefore $k_3xk_2yk_4 \neq 0$ and $k_3xk_1yk_4 = 0$ for some $k_3, k_4 \in K$. Since $k_3x, yk_4 \in K$, 0 is a disjunctive element of a semigroup K. K has no proper nonnull ideals (otherwise K is nilpotent). Hence, K is h-simple, by Corollary 6.2.1.

THEOREM 6.4. *An h-simple noncommutative semigroup has no central elements different from zero and identity.*

Proof. Let e be a central element of a noncommutative h-simple semigroup G. Consider the congruence $\varepsilon_{(e)}$ constructed in the proof of Theorem 3.5. If $\varepsilon_{(e)} = \Delta_G$, then for every $g \in G$ $g = ge$, since $g \equiv ge(\varepsilon_{(e)})$. Hence, e is an identity of G.

Now let $\varepsilon_{(e)}$ be universal. Consider the congruence ε_e: $g_1 \equiv g_2(\varepsilon_e) \leftrightarrow g_1e = g_2e$. If ε_e is universal, then $g_1e = g_2e$ for all $g_1, g_2 \in G$. In particular, $ge = e^2$, i.e., the principal ideal generated by e is $\{e, e^2\}$. G is not commutative, so this ideal is null, i.e., e is a zero of G. If $\varepsilon_e = \Delta_G$, then $g_1e = g_2e \rightarrow g_1 = g_2$. For every g_1, g_2 there exist m and n such that $g_1e^m = g_2e^n$. Let $m \geqq n$. Then $g_1e^{m-n} = g_2$ or $g_1 = g_2$. In both cases $g_1g_2 = g_2g_1$. But this is impossible (G is not commutative).

H-simple semigroups are examples of semigroups where every congruence is principal. Other examples of such semigroups are groups.

Let H be a subset of a semigroup G saturated for a congruence ε.

Then $\varepsilon = C_H$ if and only if the factor-set H/ε is a disjunctive set of a factor-semigroup G/ε. When H is an ε-class, this was proved in [21, 23]. We omit the analogous proof for the general case. As a consequence:

PROPOSITION 6.5. A congruence ε on a semigroup G is principal if and only if the factor-semigroup G/ε has a disjunctive subset.

It follows that the kernel of a homomorphism of G on a subdirectly irreducible semigroup is a principal congruence generated by at least two different subsets of G (these subsets are inverse images of disjunctive elements).

Another consequence is:

PROPOSITION 6.6. Every congruence on a semigroup is principal if and only if every homomorphic image of this semigroup possesses a disjunctive subset.

7. Completely reductive and *f*-regular semigroups. A semigroup is called *completely right* (*left*) *reductive* if all its homomorphic images are right (left) reductive.

Clearly, every completely right reductive semigroup if right reductive.

PROPOSITION 7.1. A commutative semigroup is not completely reductive if and only if it can be homomorphically mapped on a semigroup of the second kind.

Proof. Semigroups of the second kind are not reductive (they contain two distinct annihilating elements). On the other hand, if G has no homomorphic images of the second kind, then every homomorphic image of G is decomposable as a subdirect product of a family of semigroups of the first two kinds. Semigroups of these kinds have identities and are reductive. G is completely reductive, by Theorem 3.8.

THEOREM 7.2. *A commutative semigroup G is completely reductive if and only if it satisfies one of the following equivalent conditions*:

(1) $AG = A$ *for every ideal A of G.*

(2) *Every element of G has a unit.*

Proof. A semigroup of the second kind does not satisfy (1) if A is the annihilator, and does not satisfy (2) because it has a nonzero annihilating element. If G satisfies (1) or (2) then all homomorphic images of G do. G is completely reductive, by Proposition 7.1.

Suppose G does not satisfy (2). Let g be an element of G having no unit. Then $AG \neq A$ if $A = gG \cup g$. So G does not satisfy (2). Consider the factor-semigroup G/AG. The subset A/AG of this semigroup contains more than one element because AG is a proper subset of A. G/AG is not reductive because it contains different annihilating elements (all elements of A/AG are annihilating). So G is not completely reductive.

COROLLARY 7.2.1. *A commutative periodic semigroup is completely reductive if and only if it satisfies one of the following conditions*:

(1) *The ideal generated by the set of all idempotents coincides with the semigroup.*

(2) *Every element has an idempotent unit.*

Proof. Evidently (2) $\rightarrow$ (1). It is easy to prove (1) $\rightarrow$ (2), so both conditions are equivalent. (2) implies complete reductivity, by Theorem 7.2. If G is completely reductive, periodic and commutative and $g \in G$, then $gh = g$ for some $h \in G$, by condition (2) of Theorem 7.2. So h^n is a unit for g. But h^n is idempotent for some n. So (2) holds.

COROLLARY 7.2.2. *A finite commutative semigroup is completely reductive if and only if it is globally idempotent*, i.e., *if and only if* $G^2 = G$.

Proof. If $G^2 \neq G$, then G does not satisfy condition (1) of Theorem 7.2 when $A = G$. If $G^2 = G$ then every homomorphic image of G is also globally idempotent. So G cannot be mapped on a semigroup of the second kind, because finite semigroups of the second kind are nilpotent. G is completely reductive, by Proposition 7.1.

THEOREM 7.3. *The following properties of a semigroup G are equivalent*:

(1) *The intersection of any two ideals of G is equal to their product.*

(2) *Every ideal is globally idempotent.*

(3) *If (g) is the principal ideal generated by an element $g \in G$, then $g \in (g)^2$.*

(4) *Every subdirectly irreducible homomorphic image of G has a globally idempotent core.*

(5) *Every homomorphic image of G is decomposable as a subdirect product of a family of semigroups with globally idempotent cores.*

Proof. (1) $\rightarrow$ (2) $\rightarrow$ (3) $\rightarrow$ (4) $\rightarrow$ (5) $\rightarrow$ (1). We shall prove those implications that are not evident. (1) $\rightarrow$ (2) because $A \cap A = A$ for every ideal A. (3) is preserved under homomorphisms, so (3) $\rightarrow$ (4) because elements of a nilpotent core do not satisfy (3). Evidently, a subdirect product of semigroups with trivial annihilators has a trivial annihilator. A subdirectly irreducible semigroup with a globally idempotent core has a trivial annihilator. If A and B are ideals such that $A \cap B \neq AB$, then the factor-semigroup G/AB has a nontrivial annihilator (elements of $A \supset B/AB$ are annihilating). So (5) $\rightarrow$ (1).

A semigroup satisfying conditions (1)–(5) of Theorem 7.3 is called *f-regular*. *f*-regular semigroups are an obvious analogue of *f*-regular rings that satisfy these conditions also [3].

Evidently, regular semigroups are *f*-regular. Homomorphic images of *f*-regular semigroups are *f*-regular.

THEOREM 7.4 *Commutative f-regular semigroups are regular.*

Proof. Let A be a right ideal and B a left ideal of a commutative *f*-regular semigroup G. Then A and B are ideals, so $A \cap B = AB$ Therefore G is regular [9].

COROLLARY 7.4.1. *A semigroup is a commutative regular semigroup if and only if each of its homomorphic images is embeddable in a commutative regular semigroup.*

Proof. The "only if" part is evident.

Clearly, subdirectly irreducible commutative regular semigroups are of the first kind (i.e., are periodic groups with or without zeros). Let G be a subdirectly irreducible semigroup embeddable in a commutative regular semigroup G_1. G_1 is decomposable as a subdirect product of a family of periodic groups (with or without zeros). This decomposition induces a decomposition of G. But all decompositions of G are trivial. So G is embeddable in a periodic group (with or without zero). Hence, G is a group (possibly, with a zero). Therefore G has a globally idempotent core. Now let G be a semigroup all of whose homomorphic images are embeddable in commutative regular semigroups. Since G is a homomorphic image of itself, it is commutative. G satisfies condition (4) of Theorem 7.3, so it is *f*-regular. G is a commutative regular semigroup, by Theorem 7.4.

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