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ON A HOMOTOPY CONVERSE TO THE LEFSCHETZ FIXED POINT THEOREM

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Let α be a homotopy class of maps of X, a connected compact metric ANR, into itself and let L_α denote the Lefschetz number of α. A converse to the Lefschetz fixed point theorem is: if $L_\alpha = 0$ then α contains a fixed point free map. The converse is true if X is a compact connected simply-connected topological n-manifold (Fadell) or if X is a compact connected topological n-manifold, with or without boundary, and α contains the identity map (Brown-Fadell). Let $\mu(\alpha)$ denote the fixed point class invariant of α, then every map in α has at least $\mu(\alpha)$ fixed points. The purpose of this paper is to generalize the preceding results by proving that if X is a compact connected topological n-manifold, $n \geqq 3$, with or without boundary, then there is a map in α which has exactly $\mu(\alpha)$ fixed points. It follows that the converse to the Lefschetz theorem will hold whenever α contains a map all of whose fixed points are in a single fixed point class.

Let X be a topological space and let $f: X \to X$ be a map. If $x, x' \in X$ are fixed points of f, then x and x' are in the same *fixed point class* [7], [9] of f if there is a path $w: I \to X$ ($I = [0, 1]$) homotopic to the path fw by a homotopy keeping x and x' fixed, i.e., there exists a map $H: I \times I \to X$ such that $H(s, 0) = w(s)$, $H(s, 1) = f(w(s))$, for all $s \in I$, and $H(0, t) = x$, $H(1, t) = x'$, for all $t \in I$.

In order to state our theorem, we will need the results of Browder's extensive research on fixed point classes and the fixed point index [1], [2]. For the reader's convenience, we will summarize those results which we require. Let X be a connected compact metric ANR. Let $f: X \to X$ be a map and let α denote the homotopy class of maps containing f. The fixed points of f belong to a finite number of fixed point classes $\mathfrak{F}_1, \cdots, \mathfrak{F}_r$. There is a set of mutually disjoint open sets $\mathfrak{G}_1, \cdots, \mathfrak{G}_r$ of X such that $\mathfrak{F}_j \subset \mathfrak{G}_j$, $j = 1, \cdots, r$. The fixed point index $i(f, \mathfrak{G}_j)$ of f on $\mathfrak{G}_j$ is well-defined and independent of the choice of $\mathfrak{G}_j$. Call this integer the *index* of the fixed point class $\mathfrak{F}_j$ and denote it by $i(\mathfrak{F}_j)$. Let $\mu(f)$ denote the number of fixed point classes $\mathfrak{F}_j$ of f such that $i(\mathfrak{F}_j) \neq 0$. If $g \in \alpha$, then $\mu(g) = \mu(f)$ so we may replace $\mu(f)$ by $\mu(\alpha)$. Every map in α has at least $\mu(\alpha)$ fixed points.

THEOREM 1. *Let M be a compact connected topological n-manifold,*

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$n \geqq 3$, with or without boundary, and let α be a homotopy class of maps of M into itself. There is a map $f \in \alpha$ which has exactly $\mu(\alpha)$ fixed points.

In the case of triangulated manifolds, Theorem 1 is a consequence of Theorem 3 of [9]. (See [13] for the announcement of a different extension of Wecken's theorem to topological manifolds.) The restriction on the dimension of the manifold in Theorem 1 is necessary; a two-dimensional counter-example is known [14].

If all the fixed points of a map $g \in \alpha$ are in the same fixed point class $\mathfrak{F}$, then we can take $\mathfrak{G} = M$ and $i(\mathfrak{F}) = i(g, M) = L_g = L_\alpha$ [2, Theorem 4]. Therefore, we have the following homotopy converse to the Lefschetz fixed point theorem.

COROLLARY. *Let M be a compact connected topological n-manifold, $n \geqq 3$, with or without boundary, and let α be a homotopy class of maps on M which contains a map all of whose fixed points lie in a single fixed point class. If $L_\alpha = 0$, then α contains a fixed point free map.*

It is clear that for manifolds of dimension at least three, the converses to the Lefschetz theorem obtained by Fadell [5] and by Brown and Fadell [4] stated above are immediate consequences of the corollary.

Although the Lefschetz fixed point theorem itself holds for very general categories of spaces [2], [6], the converse fails to be true even for finite polyhedra, e.g., for the class of the identity map on $S^2 \vee S^1 \vee S^1$ (Y. H. Clifton).

2. Fixed points of maps on manifolds with boundary.

The results of this section are generalizations of theorems of Weier [12]. (A closely related development is given in [11].).

THEOREM 2. *Let M be a compact connected topological manifold with boundary and let $f: M \to M$ be a map, then there exists a map $f': M \to M$ homotopic to f such that f' has a finite number of fixed points; none of which lie on the boundary of M.*

Proof. If we identify two copies of M by the identity homeomorphism restricted to the boundary B of M, we obtain a compact connected manifold without boundary called the *double* of M and denoted by $2M$. Denote one of the copies of M in $2M$ by M_1 and consider f to be a map on M_1. It follows immediately from [3, Theorem 2] that there is a homeomorphism h of $B \times I$ into M_1 such that $h(b, 0) = b \in B$. Define a family of maps $r^t: M_1 \to M_1, t \in I$, by letting $r^t(x) = x$ for all $x \in [M_1 - h(B \times I)]$ and all $t \in I$ and for $h(b, s) \in h(B \times I)$, let $r^t(h(b, s)) = h(b, (1 - s)t + s)$. The map f induces $F: 2M \to M_1$ in the obvious way so that $F(x) = f(x)$ for all $x \in M_1$. Consider $g = r^1F: 2M \to M_1$, then g is homotopic to F, $g | M_1$ (g restricted to M_1) is homotopic to f, and $g(M_1) \subseteqq [M_1 - h(B \times [0, 1))]$. Let $\varepsilon > 0$ denote the distance from B to $h(B \times \{1\})$. By Theorem 1 of [12], there is a homotopy $g^t: 2M \to 2M, t \in I$, such that $g^0 = g$, $\rho(g^t(x), g(x)) < \varepsilon$ for all $t \in I$ and $x \in 2M$ (ρ is the metric of $2M$) and g^1 has at most a finite number of fixed points. By the definition of ε, it is clear that $f' = g^1 | M_1: M_1 \to M_1$ is homotopic to f and $f'(M_1) \subseteqq M_1 - B$ so f' has no fixed points on B.

REMARK. Suppose $x, x' \in M$ are fixed points of $f: M \to M$ which are in the same fixed point class of f by means of a path w, that is, w is a path in M from x to x' which is homotopic to fw by a homotopy which keeps x and x' fixed. Let $w': I \to M$ be a path from x to x' which is homotopic to w by a homotopy which keeps x and x' fixed, then x and x' are in the same class of f by means of w'.

THEOREM 3. *Let M be a compact connected topological n-manifold, $n \geqq 3$, with boundary B and let $g: M \to M$ be a map with a finite number of fixed points, none of which lie on B. If x_0 and x_1 are fixed points of g in the same fixed point class, then there exists an open set $W \subseteqq M$, containing x_0 and x_1 but no other fixed point of g, and a map $g': M \to M$ such that g' is homotopic to g, $g'(x) = g(x)$ for all $x \in M - W$, and x_0 is the only fixed point of g' in W.*

Proof. We first show that x_0 and x_1 belong to the same fixed point class of g by means of a path $w': I \to M$ such that $w'(I) \cap B = \varnothing$. By hypothesis, x_0 and x_1 are in the same class by means of a path w''. By Theorem 2 of [3], there is a neighborhood U of B in M and a homeomorphism $h : B \times [0, 1) \to U$ (onto) such that $h(b, 0) = b \in B$. Since neither x_0 nor x_1 is in B, we can construct U so that it does not contain these points. Define the path w' by

$$w'(t) = \begin{cases} w''(t) & w''(t) \notin U \\ h\left(b, \dfrac{r+1}{2}\right) & w''(t) = h(b, r) \in U \qquad (b \in B, r \in [0, 1)) . \end{cases}$$

Define $K: I \times I \to M$ by

$$K(t, s) = \begin{cases} w''(t) & w''(t) \notin U , \\ h\left(b, \left[\dfrac{1-r}{2}\right]s + r\right) & w''(t) = h(b, r) \in U , \end{cases}$$

then K is a homotopy connecting w'' and w' keeping x_0 and x_1 fixed, so by the remark, w' is the required path. Now suppose that for some fixed point x_2 of g we have $w'^{-1}(x_2) = J \neq \emptyset$. Let N be a Euclidean neighborhood of x_2 containing no other fixed point of g and let $a: N \to R^n$ be a homeomorphism taking x_2 to the origin. Let $\bar{A}$ be the closed unit ball in R^n centered at the origin and let $\bar{V} = a^{-1}(\bar{A})$. Let $\{C_\gamma\}$ denote the components of $w'^{-1}(\bar{V}) \subset I$, then by the continuity of w', there are only a finite number of such components $\{C_i\}_{i=1}^m$ with the property $C_i \cap J \neq \emptyset$. Note that $C_i = [c_i, d_i] \subset (0, 1)$ for $i = 1, \cdots, m$ and let $\zeta_1 : [c_1, d_1] \to N - V$ such that $\zeta_1(c_1) = w'(c_1)$, $\zeta_1(d_1) = w'(d_1)$, then the path w_1' defined by

$$w_1'(t) = \begin{cases} w'(t) & t \in I - (c_1, d_1) \\ \zeta_1(t) & t \in [c_1, d_1] \end{cases}$$

is homotopic to w' by a homotopy which is constant outside of N and so, in particular, keeps x_0 and x_1 fixed. Thus, by the remark, x_0 and x_1 are in the same fixed point class of g by means of w_1'. Repeating this construction a finite number of times, we obtain a path $w: I \to M$ such that x_0 and x_1 are in the same fixed point class of g by means of w, $w(I) \cap B = \emptyset$, and w intersects no other fixed point of g. Hence there exists an open set W in $M - B$ containing w and disjoint from all fixed points of g except x_0 and x_1. We can now apply the proof of Theorem 5 of [12] to g, W, x_0 the x_1 without any changes whatsoever to obtain the required map $g': M \to M$.

3. **Proof of Theorem 1.** By Theorem 2, there is a map $f' \in \alpha$ with a finite number of fixed points, none of which lie on the boundary B of M. Applying Theorem 3 to f' a finite number of times, we obtain a map $g \in \alpha$ no two of whose fixed points are in the same fixed point class of g. Denote the fixed points of g by $x_1, \cdots, x_r (\varepsilon M - B)$, then there exist Euclidean neighborhoods $U_1, \cdots, U_r$ such that $x_j \in U_j$, $j = 1, \cdots, r$, $\bar{U}_j \cap \bar{U}_k = \emptyset$ for $j \neq k$, and $i(x_j, U_j) = i(\mathfrak{F}_j)$ where $\mathfrak{F}_j$ denotes a fixed point class of g. By a result quoted above (§ 1), $i(\mathfrak{F}_j) \neq 0$ for exactly $\mu(\alpha)$ of the classes $\mathfrak{F}_j$. Let x_j be a fixed point of g such that $i(\mathfrak{F}_j) = 0$. There is a homeomorphism $h: U_j \to R^n$ (onto) taking x_j to the origin. Let $\bar{A}$ be the closed unit ball in R^n centered at the origin and let $\bar{V} = h^{-1}(\bar{A})$. We may obtain a finite triangulation of $\bar{V}$ of mesh small enough so that if P is the closed star of x_j then $g(P) \subset V$. A slight modification of the proof of Proposition 1.1 of [4] permits us to identify O'Neill's index on U_j [8] with the index we have been using in this paper. Therefore, the index of g on U_j as defined in [8] is zero and by Corollary 5.3 of that paper, there is a map $g': M \to M$ such that g' has no fixed point on U_j and g' is suf-

ficiently close to g so that $g'(P) \subset U_j$. Furthermore, from the proof of Theorem 5.2 of [8], it follows that, for $x \in M - P$, $g'(x) = g(x)$. Thus $g' \in \alpha$ and g' has the same fixed points as g except for x_j. If we repeat this construction for each fixed point x_k of g such that $i(\mathfrak{F}_k) = 0$, we obtain in a finite number of steps a map $f \in \alpha$ with exactly $\mu(\alpha)$ fixed points.

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