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UNKNOTTING SPHERES VIA SMALE

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It is shown here that a topological n-sphere which is embedded in Euclidean m-space R^m with a transverse field of $(m-n)$-planes (in the sense of Whitehead) bounds a topological $(n+1)$-disc in R^m, provided $m>n+2>4$ and $n\neq 4$. On the other hand, Haefliger has constructed C^∞ differentiable embeddings of the standard $(4k-1)$-sphere S^{4k-1} in $6k$-space R^{6k} which are differentiably knotted (i.e. they do not bound differentiably embedded $4k$-discs in R^{6k}). However, by using a sharpened form of the h-cobordism theorem of Smale it is possible to topologically unknot these spheres. This is achieved by showing that a differentiably knotted n-sphere in m-space R^m is so knotted because of a single bad point (provided $m>n+2>4$). The topological case is then proved by first approximating the topologically embedded n-sphere by a differentiably embedded homotopy n-sphere, and thus reducing it to the differentiable case.

Differentiable or smooth will mean of class C^∞. An n-disc is a contractible, compact, smooth n-manifold with simply connected boundary. A pair of disc (B^m, B^n) is a pair of discs such that $\partial B^n = B^n \cap \partial B^m$, where ∂M denotes the boundary of a manifold M, and where B^n meets ∂B^m transversally. A theorem of Smale [4] asserts that an n-disc for $n \geqq 6$ is diffeomorphic to the standard n-disc D^n in R^n. Now let (D^m, D^n) be the standard pair of discs.

PROPOSITION 1. *A pair of discs (B^m, B^n) is diffeomorphic to the standard pair (D^m, D^n), provided $m > n + 2 > 7$.*

Proof. This is an easy consequence of Smale [4; Corollary 3.2]. Let $\varphi\colon (D^m, D^n) \to (\operatorname{Int} B^m, \operatorname{Int} B^n)$ be a smooth embedding and consider the exact homology sequence of the pair $(B^n - \operatorname{Int}\varphi(D^n), \varphi(\partial D^n))$. By excision $H_i(B^n - \operatorname{Int}\varphi(D^n), \varphi(\partial D^n)) \approx H_i(B^n, \varphi(D^n)) = 0$ and hence the the inclusion $\varphi(\partial D^n) \to B^n - \operatorname{Int}\varphi(D^n)$ is a homotopy equivalence. To show that the inclusion $\partial B^n \to B^n - \operatorname{Int}\varphi(D^n)$ is also a homotopy equivalence consider the homology sequence of the pair $(B^n - \operatorname{Int}\varphi(D^n), \partial B^n)$. By Poincaré duality

$$H_i(B^n - \operatorname{Int}\varphi(D^n), \partial B^n) \approx H^{n-i}(B^n - \operatorname{Int}\varphi(D^n), \varphi(\partial D^n))$$

and by excision

$$H^{n-i}(B^n - \operatorname{Int}\varphi(D^n), \varphi(\partial D^n)) \approx H^{n-i}(B^n, \varphi(D^n)) .$$

Since $H^{n-i}(B^n, \varphi(D^n)) = 0$, it follows that the inclusion $\partial B^n \to B^n - \operatorname{Int}\varphi(D^n)$

induces isomorphisms of homology and hence is a homotopy equivalence. Therefore, since $n > 5$, [4; Corollary 3.2] implies that $B^n - \operatorname{Int}\varphi(D^n)$ is diffeomorphic to $S^{n-1} \times I$.

Similarly the inclusions $\varphi(\partial D^m) \rightarrow B^m - \operatorname{Int}\varphi(D^m)$ and $\partial B^m \rightarrow B^m - \operatorname{Int}\varphi(D^m)$ are homotopy equivalences and hence by [4; Corollary 3.2] the diffeomorphism $S^{n-1} \times I \approx B^n - \operatorname{Int}\varphi(D^n)$ may be extended to a diffeomorphism $S^{m-1} \times I \approx B^m - \operatorname{Int}\varphi(D^m)$, where, of course, $S^{n-1} \times I$ is embedded in $S^{m-1} \times I$ in the natural way. By using this product structure on $(B^m - \operatorname{Int}\varphi(D^m), B^n - \operatorname{Int}\varphi(D^n))$ it is possible to define a diffeomorphism $(B^m, B^n) \approx (D^m, D^n)$, proving the proposition.

The following theorem is a slight generalization of the topological unknotting of a differentiably knotted S^n in S^m for $m > n + 2 > 6$. Notice that Haefliger [1] has shown that S^n differentiably knots in S^m only if $3n + 3 \geqq 2m \geqq 2n + 4$. Recall that a *homotopy n-sphere* is a closed, oriented, smooth n-manifold with the homotopy type of S^n.

THEOREM A. *Any pair* (V^m, K^n) *of homotopy spheres, with* $m > n + 2 > 6$, *is diffeomorphic to a pair obtained from two copies of* (D^m, D^n) *by identifying boundaries together through some diffeomorphism* $(S^{m-1}, S^{n-1}) \rightarrow (S^{m-1}, S^{n-1})$.

REMARK. If it is assumed that K^n can be obtained by identifying two standard n-discs along their boundaries via a diffeomorphism $S^{n-1} \rightarrow S^{n-1}$, then the theorem is true for $n > 3$.

Proof. The proof is simple; for $n \geqq 6$ even simpler. If $n \geqq 6$, choose an embedding φ: $(D^m, D^n) \rightarrow (V^m, K^n)$. By Proposition 1 the pair $(V^m - \operatorname{Int}\varphi(D^m), K^n - \operatorname{Int}\varphi(D^n))$ is diffeomorphic to (D^m, D^n). (It is easy to see that $(V^m - \operatorname{Int}\varphi(D^m), K^n - \operatorname{Int}\varphi(D^n)$ is a pair of discs; for example, if $B^n = K^n - \operatorname{Int}\varphi(D^n)$, then by Poincaré duality $H_i(B^n - \varphi(\partial D^n) \approx H^{n-i}(B^n, \varphi(\partial D^n))$ and by excision $H^{n-i}(B^n, \varphi(\partial D^n)) \approx H^{n-i}(K^n, \varphi(D^n))$. Since $H^{n-i}(K^n, \varphi(D^n)) \approx H^{n-i}(K^n)$ for $i \neq n$, it follows that $B^n - \varphi(\partial D^n)$ is contractible and hence so is B^n.)

For $n = 5$ choose disjoint smooth embeddings φ_i: $D^5 \rightarrow K^5 (i = 1, 2)$ so that $K^5 - \operatorname{Int}[\varphi_1(D^5) \cup \varphi_2(D^5)]$ is diffeomorphic to $S^4 \times I$ (this is possible because any homotopy 5-sphere is, according to Milnor, h-cobordant to S^5 and hence, by Smale, is diffeomorphic to S^5). The embeddings φ_i may be extended to smooth embeddings φ_i: $(D^m, D^5) \rightarrow (V^m, K^5)$ $(i = 1, 2)$. Now by the previous paragraph $V^m - \operatorname{Int}\varphi_i(D^m)$ is a disc and hence by the proof of Proposition 1 the $\varphi_i(\partial D^m)(i = 1, 2)$ are deformation retracts of $V^m - \operatorname{Int}[\varphi_1(D^m) \cup \varphi_2(D^m)]$. Therefore, by Smale [4; Corollary 3.2] the diffeomorphism $S^4 \times I \approx K^5 - \operatorname{Int}[\varphi_1(D^5) \cup \varphi_2(D^5)]$ may be extended to a diffeomorphism $S^{m-1} \times I \approx V^m - \operatorname{Int}[\varphi_1(D^m) \cup \varphi_2(D^m)$, and the theorem then follows easily.

Let K^n be a homotopy n-sphere smoothly embedded in S^m, $m > n+2 > 6$, and let (S^m, S^n) be the standard pair of spheres, S^n embedded in S^m by the natural inclusion of R^{n+1} in R^{m+1}. A homeomorphism f: $(S^m, K^n) \to (S^m, S^n)$ of pairs, differentiable except possibly at a single point of K^n, is obtained as follows: map one copy of the (D^m, D^n) of Theorem A differentiably onto one pair of hemispheres of (S^m, S^n) and then extend the map radially to the other copy of (D^m, D^n) via the diffeomorphism $(S^{m-1}, S^{n-1}) \to (S^{m-1}, S^{n-1})$ of Theorem A (i.e., the cone map) giving the diffeomorphism up to a point. Thus f unknots K^n in S^m.

COROLLARY (*Hirsch*). *Let N be a closed tubular neighborhood of a homotopy n-sphere K^n smoothly embedded in S^{n+k}. Then for $n \geqq 5$ and $k \geqq 3$ there is a diffeomorphism $N \approx S^n \times D^k$.*

The closed tubular neighborhood N is a neighborhood of K^n in S^{n+k} which is diffeomorphic to a neighborhood of the zero cross-section in the normal bundle of K^n in S^{n+k}, the latter neighborhood being the set of all vectors less than or equal to some fixed $\varepsilon > 0$. The following proof replaces the combinatorial arguments of Hirsch [3] by application of the above theorem.

Proof. Take a closed tubular neighborhood of S^n in S^{n+k}; it is diffeomorphic to $S^n \times D^k$. It may be assumed that the closed normal tube N is embedded in $S^n \times \mathrm{Int}\, D^k$ by the unknotting homeomorphism f: $(S^m, K^n) \to (S^m, S^n)$ constructed above. Moreover, K may be deformed into K' by a differentiable isotopy deforming N into a closed normal tube N' of K', where $N' \subset$ Interior N and N' does not contain the "bad point" of f. Then N is diffeomorphic to N' and N is smoothly embedded in $S^n \times \mathrm{Int}\, D^k$ by f. Now from an argument similar to that in Proposition 1 it follows that $(S^n \times D^k) - \mathrm{Int}\, f(N')$ is diffeomorphic to $S^n \times S^{k-1} \times I$. Consequently the boundary of $f(N')$ may be deformed isotopically onto $S^n \times S^{k-1}$. Since this isotopy may be extended to a differentiable isotopy deforming $f(N')$ onto $S^n \times D^k$, the corollary is proved.

REMARK. Theorem A implies that a smoothly embedded homotopy n-sphere K^n in S^m, where $m > n + 2 > 6$ is topologically unknotted. It can be shown that the pairs (S^m, K^n) and (S^m, S^n) may be smoothly triangulated so that the unknotting homeomorphism f: $(S^m, K^n) \to (S^m, S^n)$ is a combinatorial equivalence. More generally, however, Zeeman [7] has shown that a combinatorially embedded S^n in S^m is combinatorially unknotted if $m > n + 2$. Stallings [5] proves that a locally flat S^n in S^m is unknotted if $n + 3 \leqq m \geqq 5$.

Let $G_{m-n,n}$ be the Grassman manifold of $(m - n)$-planes in R^m. If K^n is a topological n-manifold in R^m, $m > n > 0$, then a *field of*

$(m - n)$-planes transverse to K^n (or a *transverse field*) is a continuous φ: $K^n \to G_{m-n,n}$ such that $\varphi(x)$ is transverse (in the sense of Whitehead [6]) to K^n at x for every $x \in K^n$. A topological n-manifold K^n in S^m is said to have a *transverse field* if K^n has a transverse field in $S^m - \{\infty\}$ as defined above, where $\infty \in S^m - K$.

THEOREM B. *A topological n-sphere K^n embedded in S^m with a transverse field unknots, provided $m > n + 2 > 4$ and $n \neq 4$.*

Of course B follows from Stallings' result since such a K^n is locally flat in S^m. In order to prove B it is necessary to state some facts about transverse fields. So, suppose K^n is a topological n-manifold in R^m with a transverse field φ: $K \to G_{m-n,n}$. The space

$$E(\varphi) = \{(x, y) \mid x \in K, y \in \varphi(x)\}$$

may be considered as the total space of the $(m - n)$-plane bundle over K induced by φ; the fibre over $x \in K$ is the $(m - n)$-plane $\varphi(x)$. Now by Whitehead [6; page 157, second sentence], given a continuous map ε: $K \to R_+$ (R_+ the positive reals), there is a Lipschitz map φ': $K \to G_{m-n,n}$ which is an ε-approximation to φ, and by [6; Theorem 1.3] ε may be chosen so that φ' is a transverse field (which is transversally homotopic to φ). Hence we may assume without loss of generality that the given transverse field φ is Lipschitz.

Define a map

$$\theta: E(\varphi) \to R^m$$

by $\theta(x, y) = x + y$. By [6; Theorem 1.5] there exists a map ρ: $K \to R_+$ (R_+ the positive reals) such that if

$$T'_\rho = \{(x, y) \mid (x, y) \in E(\varphi), |y| < \rho(x)\} ,$$

an open subset of $E(\varphi)$, then $\theta \mid T'_\rho$ is a regular Lipschitz homeomorphism of T'_ρ onto $\theta T'_\rho$. Now define the *φ-projection* π of $\theta T'_\rho$ onto K by

$$\pi\theta(x, y) = \pi(x + y) = x .$$

Then φ is said to be of class C^r $(1 \leqq r \leqq \infty)$ if $\varphi\pi$ is of class C^r in a neighborhood $N \subset \theta T'_\rho$ of K. In this case by [6; Theorem 3] *there exists a smooth C^r submanifold M^n of N such that $\pi \mid M$: $M \to K$ is a homeomorphism and the map $M \to G_{m-n,n}$ sending x into $\varphi\pi(x)$ is a transverse field on M.*

Theorem B is a direct consequence of Theorem A (for $n = 3$ see Remark after Theorem A) and the following.

PROPOSITION 2. A pair (S^m, K^n), where K^n is a closed topological manifold in S^m with a transverse field φ: $K \to G_{m-n,n}$, is homeomorphic to a pair (S^m, M^n), where M^n is a smooth C^∞ submanifold of S^m.

REMARK. The homeomorphism of the pairs (S^m, K^n) and (S^m, M^n) which is defined in the following proof is isotopic (homotopic through homeomorphisms) to the identity map of S^m.

Proof. Let $\rho: K \to R_+$ be as above; by [6; Theorem 1.10] φ may be assumed to be a C^∞ transverse field. Now choose $\rho_0 > 0$ such that $0 < \rho_0 < \operatorname{Inf}\{\rho(x) \mid x \in K\}$ and let

$$T_0' = \{(x, y) \in E(\varphi) \mid |y| < \rho_0\}, \quad T_0 = \theta T_0' .$$

Clearly $T_0' \subset T_\rho'$ and, moreover, the map $\psi: T_0 \to E(\varphi)$ sending $x + y \to (x, (1/(\rho_0 - |y|))y)$ defines a homeomorphism of

$$T_0 = \{x + y \mid x \in K, y \in \varphi(x), |y| < \rho_0\} \quad \text{onto} \quad E(\varphi) .$$

By remarks above there exists a smooth C^∞ submanifold M^n of R^m in T_0 such that $\pi \mid M: M \to K$ is a homeomorphism. The homeomorphism $\pi \mid M$ will be extended to a surjective homeomorphism $f: S^m \to S^m$. The first step is to extend $\pi \mid M$ to a homeomorphism $\bar{\pi}: T_0 \to T_0$ onto T_0 in the following way: the image of M under $\psi: T_0 \to E(\varphi)$ may be described as the set $\{(x, \alpha(x)) \mid x \in K, \alpha(x) \in \varphi(x), \alpha: K \to R^m\}$ and so the map $\beta: E(\varphi) \to E(\varphi)$ defined by $\beta(x, y) = (x, y - \alpha(x))$ is clearly a homeomorphism of $E(\varphi)$ onto itself. Setting $\bar{\pi} = \psi^{-1}\beta\psi$ gives the desired extension of $\pi \mid M$.

It is a tedious but straightforward verification that for $(x + y) \in T_0$, $|\bar{\pi}(x + y) - (x + y)| \to 0$ uniformly for all x as $|y| \to \rho_0$ and hence by defining $f: S^m \to S^m$ to be, for each s in S^m,

$$f(s) = \begin{cases} \bar{\pi}(s) & (\text{if } s \in T_0) , \\ s & (\text{if } s \notin T_0) , \end{cases}$$

it follows that f is a homeomorphism of S^m onto S^m sending M onto K.

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