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A CHARACTERIZATION OF UNIQUELY DIVISIBLE COMMUTATIVE SEMIGROUPS

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Let $(S, +)$ be a commutative semigroup. If, for each $x \in S$, and for each positive integer n, there exists an (unique) element y of S such that $x = ny$, then S is (uniquely) divisible. In this note we present a more or less intrinsic characterization of uniquely divisible commutative semigroups and remark on a special sub-class of these semigroups in which it is possible to discern the fine structure of the addition.

2. The characterization. Let P represent the additive semigroup of positive rational numbers. By a *cone* of a rational vector space we mean a convex subset C such that $PC \subset C$ and $-PC \cap C = 0$. A commutative semigroup is *separative* if $2a = a + b = 2b$ implies $a = b$ for any $a, b \in S$. Let L be the maximal (lower) semilattice homomorphic image of S, and let h be the natural map of S onto L. For $e \in L$, let $h^{-1}(e) = S_e$. The Hewitt-Zuckerman theorem [3; or 1, Th. 4.18] states that, if S is separative, then each S_e is cancellative, and S is isomorphically embeddable in a semilattice of groups, $\{V_e\}$ in such a way that each V_e is the difference group of S_e, and the semilattice is isomorphic to L.

Since an uniquely divisible commutative semigroup is clearly separative, we have immediately that any such entity is isomorphic to a divisible subsemigroup of a semilattice of divisible groups. Indeed, each V_e must be uniquely divisible, and hence a rational vector space (see [4], for example). Furthermore, since each S_e is cancellative, it follows from Hancock's theorem [2, Th. 7] that each S_e is the direct sum of a rational vector space and a cone of a rational vector space. We have now:

THEOREM 1. *Let S be an uniquely divisible commutative semigroup. Then S is a semilattice of subsemigroups S_e, each of which is the direct sum of a rational vector space and a cone of a rational vector space. Furthermore, the addition in S is determined by semigroup homomorphisms between these subsemigroups which are restrictions of homomorphisms (linear maps) between their difference groups.*

3. A special case. We now restrict our attention to the situation in which, for each $e \in L$, $S_e \cong P$. In this case, any $x_e \in S_e$ satisfies

$Px_e = S_e$. By x_α we shall mean an element of S_α.

LEMMA 1. *Let $e, f \in L$, $e \leqq f$; let $x_e + x_f = rx_e$, $r \in P$. Then $r \geqq 1$, and for $s, t \in P$, $sx_e + tx_f = [s + t(r-1)]x_e$.*

Proof. Suppose $r < 1$ and let $z = x_e + (1/(1-r))x_f$. Then

$$z = \left[(x_e + x_f) + \left(\frac{r}{1-r}\right)x_f\right] = \left[rx_e + \left(\frac{r}{1-r}\right)x_f\right] = rz .$$

Hence, $r = 1$, which is a contradiction.

Now, consider S as embedded in a semilattice of rational vector spaces as in the proof of Theorem 1. We have

$$\begin{aligned} sx_e + tx_f &= (sx_e + 0_e) + tx_f \\ &= sx_e + (0_e + tx_f) \\ &= sx_e + t(0_e + x_f) \\ &= sx_e + t([r-1]x_e) \\ &= (s + t[r-1])x_e . \end{aligned}$$

The proof is now complete.

LEMMA 2. *Let $e, f, g \in L$, $e \leqq f \leqq g$. Suppose $x_e + x_f = ax_e$, $x_e + x_g = bx_e$, $x_f + x_g = cx_f$, $a, b, c \in P$. If any two of a, b, c equal 2, then $a = b = c = 2$.*

Proof. Note $[a+(b-1)]x_e = ax_e + x_g = (x_e + x_f) + x_g = x_e + (x_f + x_g) = x_e + cx_f = [1 + c(a-1)]x_e$. By the uniqueness of roots, $a + b - 1 = 1 + c(a-1)$, and proof is complete.

LEMMA 3. *Let $e, f \in L$. If $x_e + x_{ef} = x_f + x_{ef} = 2x_{ef}$,* then *$x_e + x_f = 2x_{ef}$.*

Proof. Let $x_e + x_f = ax_{ef}$. Then $3x_{ef} = x_{ef} + (x_{ef} + x_e) = 2x_{ef} + x_e = (x_{ef} + x_f) + x_e = (1 + a)x_{ef}$. Hence $a = 2$.

THEOREM 2. *Let S be an uniquely divisible commutative semigroup such that $x + y \neq y$, all $x, y \in S$. Then $S \cong P \times L$.*

Proof. Fix $e \in L$, $x_e \in S_e$. For each $f \in L$, choose $x_f \in S_f$ such that:
(1) $x_e + x_f = 2x_f$ if $f \leqq e$,
(2) $x_f + x_{ef} = 2x_{ef}$ otherwise.

Lemma 1 assures the availability of such elements; there is no ambiguity involved provided (1) is accomplished before (2). Fix $f, g \in L$; we shall show $x_f + x_{fg} = x_g + x_{fg} = 2x_{fg}$. To this end, note that $x_e + x_{ef} = 2x_{ef}$ and $x_e + x_{efg} = 2x_{efg}$ by (1) above. Hence, by Lemma 2, $x_{ef} + x_{efg} = 2x_{efg}$. Since $x_e + x_f = x_{ef}$, we have $x_f + 2x_{ef} = 3x_{ef}$; by cancellation in S_{ef}, it follows that $x_f + x_{ef} = 2x_{ef}$. By applying Lemma 2 again, we have $x_f + x_{efg} = 2x_{efg}$. By an argument identical to the one involving f and ef above, $x_{fg} + x_{efg} = 2x_{efg}$. Finally, applying Lemma 2 for the final time, we have $x_f + x_{fg} = 2x_{fg}$. Similarly, $x_g + x_{fg} = 2x_{fg}$; by Lemma 3 it follows that $x_f + x_g = 2x_{fg}$. Finally, if, say $s \geqq t$, then $sx_f + tx_g = t(x_f + x_g) + (s-t)x_f = 2tx_{fg} + (s-t)x_f = (s+t)x_{fg}$ by Lemma 1. The function $\phi \colon S \to P \times L$ defined by $\phi(rx_f) = (r, f)$ is now clearly an isomorphism.

Next, let L be any semilattice, and let ϕ be a homomorphism of L onto a chain B. For each $\beta \in B$, let $L_\beta = \phi^{-1}(\beta)$. For each β, let $S_\beta = P \times L_\beta$, and let $S = \cup \{S_\beta : \beta \in B\}$. Define an addition in S by

$$(r, e) + (s, f) = \begin{cases} (r + s, ef) \text{ if } e, f \in L_\beta\,, \\ (r, ef) \text{ if } e \in L_\beta, f \in L_\gamma, \beta < \gamma\,, \\ (s, ef) \text{ if } e \in L_\beta, f \in L_\gamma, \gamma < \beta\,. \end{cases}$$

With this addition, S is an uniquely divisible commutative semigroup with maximal semilattice image L and with each $S_e \cong P$. The class of semigroups thus defined will be referred to as being of type $\mathscr{E}$.

THEOREM 3. *Let S be an uniquely divisible commutative semigroup such that each S_e is isomorphic to P. Then S is isomorphic to a semigroup of type $\mathscr{E}$.*

Proof. Define a relation $\sim$ on S by $x \sim y$ if and only if $x + (x + y) \neq x + y \neq y + (x + y)$. To check transitivity, let $x \sim y, y \sim z$. In particular, let $x + (x + y) = r(x + y)$, $y + (y + z) = s(y + z)$, with $r, s > 1$. Then $x + (x + y + z) = r(x + y) + z = rx + (r-1)y + (y + z) = rx + [1 + (r-1)(s-1)](y + z) \neq x + y + z$. Hence $x + (x + z) \neq x + z$. Similarly, $z + (x + z) \neq x + z$.

It follows by arguments similar to the above that $\sim$ is a congruence on S and that $S/\sim$ is a chain. Let j be the natural map of S onto $S/\sim$; note that j factors into the composition of h and an induced map from L to $S/\sim$. For $\beta \in S/\sim$, $j^{-1}(\beta)$ satisfies the conditions of Theorem 2. Specifically, $j^{-1}(\beta) \cong P \times h\, j^{-1}(\beta)$. Thus any $x \in j^{-1}(\beta)$ has an unique representation, $x = rx_e$, with $e \in hj^{-1}(\beta), r \in P$, and x_e selected from $h^{-1}(e)$ in line with the proof of Theorem 2. Suppose $\beta, \gamma \in S/\sim \gamma, \beta < \gamma$, and let $rx_e \in j^{-1}(\beta)$, $sx_f \in j^{-1}(\gamma)$. Then $x_e + x_f \in j^{-1}(\beta)$ and $x_f + (x_e + x_f) = x_e + x_f$. Let $x_e + x_f = tx_{ef}$. By Lemma 1, $x_f + x_{ef} = x_{ef}$; since $x_e, x_{ef} \in j^{-1}(\beta)$, $x_e + x_{ef} = 2x_{ef}$. Hence $(1 + t)x_{ef} = x_{ef} +$

$(x_e + x_f) = (x_{ef} + x_e) + x_f = 2x_{ef} + x_f = 2x_{ef}$; hence $t = 1$. Now, if, say $r \leqq s$, then $rx_e + sx_f = r(x_e + x_f) + (s - r)x_f = rx_{ef} + (s - r)x_f = rx_{ef}$ by Lemma 1. If, on the other hand, $s < r$, then $rx_e + sx_f = s(x_e + x_f) + (r - s)x_e = sx_{ef} + (r - s)x_e = rx_{ef}$ by Lemma 1. We have now shown that the addition of S satisfies:

$$rx_e + sx_f = \begin{cases} (r + s)x_{ef} \text{ if } jh^{-1}(e) = jh^{-1}(f)\ , \\ rx_{ef} \text{ if } jh^{-1}(e) < jh^{-1}(f)\ , \\ sx_{ef} \text{ if } jh^{-1}(f) < jh^{-1}(e)\ . \end{cases}$$

The mapping $rx_e \rightarrow (r, e)$ now establishes that S is isomorphic to a semigroup of type $\mathscr{E}$.

In closing, we remark that the relations used in proving Theorems 2 and 3 can be reformulated in terms of the homomorphisms guaranteed by Theorem 1. In Theorem 3 in particular, if $e \leqq f$, then $x_e \sim x_f$ if and only if the addition homomorphism is an isomorphism. Furthermore, if x_e and x_f are not equivalent, then the addition homomorphism is the zero mapping.

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