PACIFIC JOURNAL OF MATHEMATICS
Vol. 18, No. 1, 1966

THE SUM OF TWO INDEPENDENT EXPONENTIAL-TYPE RANDOM VARIABLES

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Let X_1, X_2 be nondegenerate, independent, exponential-type random variables (r.v.) with probability density functions, (p.d.f.) $f_1(x_1; \theta), f_2(x_2; \theta)$, (not necessarily with respect to the same measure), where $f_i(x_i; \theta) = \exp\{x_i p_i(\theta) + q_i(\theta)\}$ for $\theta \in (a, b)$ and $p_i(\theta)$ is an analytic function of θ (for $Re\, \theta \in (a, b)$) with $p_i'(\theta)$ never equal to zero on (a, b). If X_1, X_2 are neither both normal nor both Poisson type r.v.'s, then $X_1 + X_2$ is an exponential-type r.v. if and only if $p_1'(\theta) = p_2'(\theta)$.

2. Lemmas. It follows from Patil's result ([3]) that a r.v. X is of exponential type if and only if the cumulants, $\lambda_j(\theta)$, exist and satisfy

$$\lambda_j'(\theta) = p'(\theta)\lambda_{j+1}(\theta) \qquad \text{for } j = 1, 2, 3, \cdots . \tag{1}$$

Lehmann ([2], p. 52) has shown that $q(\theta)$ and hence also $\lambda_j(\theta)$ are analytic functions of $p(\theta)$. Then $\lambda_j(\theta)$ is an analytic function of θ for $Re\, \theta \in (a, b)$.

Let $\lambda_{j,i}(\theta)$ be the j^{th} cumulant of X_i and $\lambda_j(\theta)$ the j^{th} cumulant of Y. Then

$$\lambda_j(\theta) = \lambda_{j,1}(\theta) + \lambda_{j,2}(\theta) \tag{2}$$

$$\lambda_{j,i}'(\theta) = p_i'(\theta)\lambda_{j+1,i}(\theta) \qquad \text{for } j = 1, 2, 3, \cdots . \tag{3}$$

Let $h_j(\theta) = \lambda_{j,1}(\theta)\lambda_{2,2}(\theta) - \lambda_{j,2}(\theta)\lambda_{2,1}(\theta)$ and $c(\theta) \equiv \lambda_{2,2}(\theta)/\lambda_{2,1}(\theta)$.

LEMMA 1. *If $h_3(\theta) \equiv 0$ and if $c'(\theta) \equiv 0$, then either X_1 and X_2 are both normal or $p_1'(\theta) \equiv p_2'(\theta)$.*

Proof. Since $h_3(\theta) \equiv 0$,

$$\lambda_{3,2}(\theta) = c(\theta)\lambda_{3,1}(\theta) . \tag{4}$$

Since $c'(\theta) \equiv 0$,

$$\lambda_{2,2}'(\theta) = c(\theta)\lambda_{2,1}'(\theta) . \tag{5}$$

From (3), (4) and (5) it follows that

$$p_2'(\theta)\lambda_{3,2}(\theta) = c(\theta)p_1'(\theta)\lambda_{3,1}(\theta) = p_1'(\theta)\lambda_{3,2}(\theta) .$$

If $\lambda_{3,2}(\theta) \equiv 0$, then $\lambda_{3,1}(\theta) \equiv 0$ and X_1, X_2 are both normal. If there is a point θ_0 such that $\lambda_{3,2}(\theta) \neq 0$, then there is a neighborhood, $N(\theta_0)$, in which $\lambda_{3,2}(\theta) \neq 0$. For $\theta \in N(\theta_0)$, $p_1'(\theta) = p_2'(\theta)$. By analyticity, $p_1'(\theta) = p_2'(\theta)$ for $\theta \in (a, b)$.

LEMMA 2. *If $h_j(\theta) \equiv 0$ for $j > 2$ and if $c'(\theta) \not\equiv 0$, then X_1 and X_2 are Poisson type r.v.'s.*

Proof. Since $h_j(\theta) \equiv 0$,

$$\lambda_{j,2}(\theta) = c(\theta)\lambda_{j,1}(\theta) . \tag{6}$$

Differentiating (6) and using (3), we get

$$c(\theta)\lambda_{j,1}'(\theta) + c'(\theta)\lambda_{j,1}(\theta) = p_2'(\theta)\lambda_{j+1,2}(\theta) .$$

Then,

$$c(\theta)p_1'(\theta)\lambda_{j+1,1}(\theta) + c'(\theta)\lambda_{j,1}(\theta) = p_2'(\theta)c(\theta)\lambda_{j+1,1}(\theta) . \tag{7}$$

In particular,

$$c(\theta)p_1'(\theta)\lambda_{3,1}(\theta) + c'(\theta)\lambda_{2,1}(\theta) = p_2'(\theta)c(\theta)\lambda_{3,1}(\theta) . \tag{8}$$

Multiplying (7) by $\lambda_{3,1}(\theta)$ and (8) by $\lambda_{j+1,1}(\theta)$, we find that

$$c'(\theta)[\lambda_{2,1}(\theta)\lambda_{j+1,1}(\theta) - \lambda_{3,1}(\theta)\lambda_{j,1}(\theta)] = 0 \qquad \text{for } j \geqq 2 . \tag{9}$$

Since $c'(\theta) \not\equiv 0$, there is a sub-interval M of (a, b) in which $c'(\theta) \neq 0$. For $\theta \in M$,

$$\lambda_{2,1}(\theta)\lambda_{j+1,1}(\theta) - \lambda_{3,1}(\theta)\lambda_{j,1}(\theta) = 0 ,$$

or

$$\lambda_{j+1,1}(\theta) = \frac{\lambda_{3,1}(\theta)}{\lambda_{2,1}(\theta)}\lambda_{j,1}(\theta) . \tag{10}$$

By analyticity, (10) is true for all $\theta \in (a, b)$. Now let $a(\theta) = \lambda_{3,1}(\theta)/\lambda_{2,1}(\theta)$. Then, by (3),

$$\begin{aligned} p_1'(\theta)\lambda_{4,1}(\theta) = \lambda_{3,1}'(\theta) &= a'(\theta)\lambda_{2,1}(\theta) + a(\theta)\lambda_{2,1}'(\theta) \\ &= a'(\theta)\lambda_{2,1}(\theta) + a(\theta)p_1'(\theta)\lambda_{3,1}(\theta) . \end{aligned}$$

Since $\lambda_{4,1}(\theta) = a(\theta)\lambda_{3,1}(\theta)$, it follows that

$$a'(\theta)\lambda_{2,1}(\theta) = 0 .$$

So $a'(\theta) = 0$ and $a(\theta) = d$. Then (10) becomes

$$\lambda_{j+1,1}(\theta) = d\lambda_{j,1}(\theta) \qquad \text{for } j \geqq 2 . \tag{11}$$

This implies

$$\lambda_{j,1}(\theta) = d^{j-2}\lambda_{2,1}(\theta) \qquad \text{for } j \geqq 2 . \tag{12}$$

By (6),

$$\lambda_{j,2}(\theta) = d^{j-2}c(\theta)\lambda_{2,1}(\theta) \qquad \text{for } j \geqq 2 . \tag{13}$$

Now,

$$p_1'(\theta) = \lambda_{1,1}'(\theta)/\lambda_{2,1}(\theta) ,$$
$$p_1'(\theta) = \lambda_{2,1}'(\theta)/\lambda_{3,1}(\theta) = \lambda_{2,1}'(\theta)/d\lambda_{2,1}(\theta) .$$

So

$$\lambda_{1,1}(\theta) = d^{-1}\lambda_{2,1}(\theta) + k_1 . \tag{14}$$

Similarly,

$$\lambda_{1,2}(\theta) = d^{-1}c(\theta)\lambda_{2,1}(\theta) + k_2 . \tag{15}$$

Using (12), (13), (14) and (15), we find that

$$\log M_1(t;\theta) = k_1 t + d^{-2}\lambda_{2,1}(\theta)(e^{dt} - 1)$$
$$\log M_2(t;\theta) = k_2 t + d^{-2}c(\theta)\lambda_{2,1}(\theta)(e^{dt} - 1) ,$$

where $M_i(t;\theta)$ is the moment generating function corresponding to $f_i(x_i;\theta)$.

This concludes the proof of Lemma 2.

3. The sum of two independent exponential-type random variables.

THEOREM 1. *If X_1, X_2 are neither both normal nor both Poisson type r.v.'s, then $X_1 + X_2$ is an exponential-type r.v. if and only if $p_1'(\theta) = p_2'(\theta)$.*

Proof. If $p_1'(\theta) = p_2'(\theta)$, then if follows from (2) and (3) that

$$\begin{aligned}\lambda_{j+1}(\theta) &= \lambda_{j+1,1}(\theta) + \lambda_{j+1,2}(\theta)\\ &= [p_1'(\theta)]^{-1}\lambda_{j,1}'(\theta) + [p_1'(\theta)]^{-1}\lambda_{j,2}'(\theta)\\ &= [p_1'(\theta)]^{-1}\lambda_j'(\theta) .\end{aligned}$$

Conversely, assume $X_1 + X_2$ is an exponential-type r.v.. Then, using (1), (2), and (3), we find that

$$p'(\theta)[\lambda_{j,1}(\theta) + \lambda_{j,2}(\theta)] = p_1'(\theta)\lambda_{j,1}(\theta) + p_2'(\theta)\lambda_{j,2}(\theta) . \tag{16}$$

In particular,

$$p'(\theta)[\lambda_{2,1}(\theta) + \lambda_{2,2}(\theta)] = p_1'(\theta)\lambda_{2,1}(\theta) + p_2'(\theta)\lambda_{2,2}(\theta) \,. \tag{17}$$

Multiplying (16) by $\lambda_{2,1}(\theta)$ and (17) by $\lambda_{j,1}(\theta)$ and then subtracting, we get

$$[p'(\theta) - p_2'(\theta)]h_j(\theta) \equiv 0 \qquad \text{for } j \geqq 2 \,. \tag{18}$$

Now, if for some $j_0 \geqq 2$, $h_{j_0}(\theta) \not\equiv 0$, then there is a subinterval, M, of (a, b) in which $h_{j_0}(\theta) \neq 0$. Then, for $\theta \in M$, $p_2'(\theta) = p'(\theta)$. By analyticity, $p_2'(\theta) = p'(\theta)$ for all $\theta \in (a, b)$. Substitution in (16) yields $p_1'(\theta) = p'(\theta)$ for $\theta \in (a, b)$. If, on the other hand, $h_j(\theta) \equiv 0$, for $j \geqq 2$, the result follows from Lemmas 1 and 2 since we assumed that X_1, X_2 are neither both normal nor both Poisson type r.v.'s.

It should be noted that Girshick and Savage [1] proved that if X_1 and X_2 are independent identically distributed r.v.'s such that their sum is of exponential-type, then X_1 and X_2 are also of exponential-type.

The following theorem gives necessary and sufficient conditions for the sum of two Poisson-type r.v.'s to be exponential-type.

THEOREM 2. *If* $\log M_i(t; \theta) = C_i t + A_i(\theta)[l^{b_i t} - 1]$, *then* $X_1 + X_2$ *is an exponential-type r.v. if and only if either* $b_1 = b_2$ *or* $p_1'(\theta) = p_2'(\theta)$.

Proof. If $X_1 + X_2$ is an exponential-type r.v., then, as in the proof of the preceding theorem,

$$[p'(\theta) - p_2'(\theta)]h_j(\theta) \equiv 0 \qquad \text{for } j \geqq 2 \,.$$

Equivalently,

$$\begin{aligned} &[\lambda_{j,1}(\theta)\lambda_{2,2}(\theta) - \lambda_{j,2}(\theta)\lambda_{2,1}(\theta)] \\ &\quad = p_2'(\theta)[p'(\theta)]^{-1}[\lambda_{j,1}(\theta)\lambda_{2,2}(\theta) - \lambda_{j,2}(\theta)\lambda_{2,1}(\theta)] \qquad \text{for } j \geqq 2 \,. \end{aligned} \tag{19}$$

Since, for $j \geqq 2$, $\lambda_{j,i}(\theta) = b_i^j A_i(\theta)$, (19) becomes

$$[b_1^j b_2^2 - b_2^j b_1^2]A_1(\theta)A_2(\theta) = p_2'(\theta)[p'(\theta)]^{-1}[b_1^j b_2^2 - b_2^j b_1^2]A_1(\theta)A_2(\theta) \,.$$

But $A_1(\theta)A_2(\theta) > 0$, so that

$$[b_1^j b_2^2 - b_2^j b_1^2] = p_2'(\theta)[p'(\theta)]^{-1}[b_1^j b_2^2 - b_2^j b_1^2] \,.$$

Now, if $b_1^j b_2^2 = b_2^j b_1^2$ for all $j \geqq 2$, then $b_1^3 b_2^2 = b_2^3 b_1^2$, so that $b_1 = b_2$. On the other hand, if, for some j_0, $b_1^{j_0} b_2^2 - b_2^{j_0} b_1^2 \neq 0$, then $p_2'(\theta) = p'(\theta)$ and it follows that $p_1'(\theta) = p_2'(\theta)$.

Conversely, if $p_1'(\theta) = p_2'(\theta)$, then $X_1 + X_2$ is an exponential-type r.v. since (1) is satisfied. If $b_1 = b_2$, let

$$p'(\theta) = [A_1'(\theta) + A_2'(\theta)]/b_1[A_1(\theta) + A_2(\theta)] .$$

It is easy to see that (1) is again satisfied.

The author wishes to thank William L. Harkness for his help in the preparation of this paper.

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Received August 17, 1964, and in revised form February 26, 1965.

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