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A CONVERGENT GRADIENT PROCEDURE IN PREHILBERT SPACES

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In this paper, we present a new method of approximating the minimum of a functional, J, defined on a prehilbert space and subject to constraints of the form $\psi_i(x) = 0$, $1 \leqq i \leqq p$, where the ψ_i are also functionals on the space. The method generates a convergent sequence of approximations using the gradients of J and ψ_i. However, it is not a steepest descent procedure with respect to J. A theorem is proven which establishes the convergence of the approximating sequence to the minimum.

The literature on extremal problems in abstract spaces is fairly extensive. We refer to the bibliographies in [1], [5] for a partial list. That part of the literature which deals with approximation procedures for finding extrema subject to various constraints is also extensive. Again, see [1], [5] and also [3] for bibliographies. In particular, gradient-type methods have received considerable attention recently in a variety of contexts as in [2], [7], [6] to mention a few. In this paper, we present a new method of approximating the minimum of a functional, J, defined on a prehilbert space and subject to equality constraints. The method generates a convergent sequence of approximations using gradients but it is not of the steepest descent type with respect to J.

2. Preliminary remarks. Let H be a prehilbert space. For $u, v, \in H$, we denote their inner product by $\langle u, v\rangle$ and $\| u \|^2 = \langle u, u\rangle$.

Let J and ψ_i, $1 \leqq i \leqq p$, be real functionals defined on some subset in H. We shall say that $u^* \in H$ is a *solution of the minimization problem defined by* $\{J, \psi_i\}$ if $\psi_i(u^*) = 0$, $1 \leqq i \leqq p$ and $J(u^*) \leqq J(u)$ for those u in a neighborhood of u^* which satisfy the constraints $\psi_i(u) = 0$, $1 \leqq i \leqq p$.

By the *gradient of J at u* we mean an element of H, designated by $\nabla J(u)$, such that for all Δu in some neighborhood of $0 \in H$,

$$J(u + \Delta u) - J(u) = \langle \nabla J(u), \Delta u\rangle + \varepsilon(u, \Delta u) ,$$

where $| \varepsilon(u, \Delta u) | / \| \Delta u \| \rightarrow 0$ as $\Delta u \rightarrow 0$. Similarly, $\nabla \psi_i(u)$ denotes the gradient of ψ_i at u.

If f is a real functional defined in a neighborhood of $u \in H$ and if $df(u; h) = \lim_{s \to 0} (f(u + sh) - f(u))/s$ exists for all $h \in H$, we call $df(u; h)$ the *weak differential* of f at u with respect to h. The relation between $df(u; h)$ and $\langle \nabla f(u), h \rangle$ is well-known [8].

Now suppose that $\nabla J(u)$ and $\nabla \psi_i(u)$, $1 \leqq i \leqq p$, exist at u. Suppose further that the $p \times p$ Gram matrix,

$$D(u) = (\langle \nabla \psi_i(u), \nabla \psi_j(u) \rangle) ,$$

is nonsingular. Let γ be the p-dimensional row vector

$$(\langle \nabla J(u), \nabla \psi_1(u) \rangle, \cdots, \langle \nabla J(u), \nabla \psi_p(u) \rangle)$$

and define $\lambda = (\lambda_1, \cdots, \lambda_p)$ as the vector $\lambda = \gamma D^{-1}$. Then, as is well-known, the "projection" of ∇J on the subspace $G \subset H$ spanned by $\{\nabla \psi_i(u) \mid 1 \leqq i \leqq p\}$ is given by $\nabla J_G(u) = \sum_1^p \lambda_j \nabla \psi_j(u)$. The component of ∇J orthogonal to G is

$$\nabla J_T(u) = \nabla J(u) - \nabla J_G(u) .$$

If u^* is a solution of the minimization problem defined by $\{J, \psi_i\}$ and $\nabla J(u)$, $\nabla \psi_i(u)$ exist as continuous functions of u in a neighborhood of u^* with $D(u^*)$ nonsingular, then by the Lagrange multiplier rule [4], [5], [8] it follows that $\nabla J_T(u^*) = 0$.

In the next section, we shall use this necessary condition as the basis for a gradient method of obtaining successive approximations which converge to u^* when certain regularity conditions hold in a neighborhood of u^*. With obvious modifications, the method can also be applied to the simpler problem of finding a solution of the system $\{\psi_i(u) = 0\}$. In this context, it generalizes a method given in [7] for finite-dimensional spaces.

3. The gradient method. For any $u \in H$, we shall use the notation "$\bar{u}$" to denote the normalized vector $(1/\| u \|)u$.

DEFINITION. Let u^* be a solution of the minimization problem defined by $\{J, \psi_i\}$. u^* is a *regular minimum* if there is a neighborhood, $N = N(u^*)$, of u^* in which the following conditions are satisfied:

(1) $\nabla J(u)$ exists as a continuous function of u and $\nabla J_G(u) \neq 0$ for $u \in N$;

(2) $\nabla \psi_i(u)$, $1 \leqq i \leqq p$, exists as a continuous function of u and $\nabla \psi_i(u) \neq 0$ for $u \in N$;

(3) For each $u \in N$, the matrix $D(u)$ is nonsingular;

(4) For $\theta(u) = \arcsin (\| \nabla J_\tau(u) \| / \| \nabla J(u) \|)$, the gradient $\nabla \theta(u)$ exists and $\| \nabla \theta(u) \| > a > 0, u \neq u^*$. At u^* the weak differential $d\theta(u^*; h)$ exists and $\langle \nabla \theta(u), h \rangle \to d\theta(u^*, h)$ as $u \to u^*$.

(5) For $u = (u^* + \Delta u) \in N$, $\Delta u \neq 0$, let

$$\begin{aligned} \alpha_i &= \alpha_i(u) = \arccos \langle \overline{\nabla \psi_i(u)}, \overline{\Delta u} \rangle, \ 1 \leqq i \leqq p \ , \\ \alpha_0 &= \alpha_0(u) = \arccos \langle \overline{\nabla J_T(u)}, \overline{\Delta u} \rangle \ , \\ \beta &= \beta(u) = \arccos \langle \overline{\nabla \theta(u)}, \overline{\Delta u} \rangle \ , \\ \gamma &= \gamma(u) = \arccos \langle \overline{\nabla \theta(u)}, \overline{\nabla J_T(u)} \rangle \ . \end{aligned}$$

There exist constants $a_1 > 0$ and $a_2 > 0$ such that

$$\sum_1^p \cos^2 \alpha_i + \cos \alpha_0 \cos \beta / | \cos \gamma | > a_1$$

and $| \cos \gamma | > a_2$ for all $u \in N$ except possibly u^*. A neighborhood such as N is called a *regular* neighborhood of u^*.

REMARK. $\theta(u^*) = 0$ by the necessary condition for a minimum. If u^* is the unique minimum, this condition is also sufficient under the assumptions required in the multiplier rule.

LEMMA. *Let u^* be the unique solution of the minimization problem defined by $\{J, \psi_i\}$ and let N be a regular neighborhood of u^*. For $u = (u^* + \Delta u) \in N$, $\Delta u \neq 0$, let*

$$h_G = h_G(u) = - \sum_{i=1}^p (\psi_i(u) / \| \nabla \psi_i(u) \|) \overline{\nabla \psi_i(u)} \ , \tag{6}$$

$$h_T = h_T(u) = - (1 / \| \nabla J_G(u) \| \cdot | \langle \overline{\nabla \theta(u)}, \overline{\nabla J_T(u)} \rangle |) \nabla J_T(u) \ , \tag{7}$$

and $h = h_G + h_T$. Then there exists positive constants k, d and r, with $k < 1$, such that for all u with $0 < \| \Delta u \| < r$, u is in N and $\| u + sh - u^* \| < k \| \Delta u \|$ whenever $d/2 < s < d$.

Proof. By the definition of gradient,

$$\psi_i(u) = \psi_i(u) - \psi_i(u^*) = \langle \nabla \psi_i(u^*), \Delta u \rangle + \varepsilon_i, \ 1 \leqq i \leqq p \ ,$$

where $\varepsilon_i = \varepsilon_i(u^*, \Delta u)$ is such that $\| \varepsilon_i \| / \| \Delta u \| \to 0$ as $\Delta u \to 0$.

(For convenience, we shall adopt the notational convention that throughout the proof any quantity designated by ε with appropriate subscripts or superscripts is such that $\| \varepsilon \| / \| \Delta u \| \to 0$ as $\Delta u \to 0$. We shall make no further mention of this property.)

Now, by the continuity of $\nabla \psi_i$ at u^* (condition 2 of the definition), $\psi_i(u) = \langle \nabla \psi_i(u), \Delta u \rangle + \tilde{\varepsilon}_i$. Using these relations in (6), we obtain

$$h_G = - \sum_1^p \langle \overline{\nabla \psi_i}(u), \Delta u \rangle \overline{\nabla \psi_i(u)} + \varepsilon_G \ . \tag{8}$$

Since $u + sh - u^* = \Delta u + sh_G + sh_T$, we have

$$(9)\quad \|u+sh-u^*\|^2 = \|\Delta u\|^2 + 2s\langle \Delta u, h_G\rangle + 2s\langle \Delta u, h_T\rangle + s^2(\|h_G\|^2 + \|h_T\|^2)\ .$$

From (8) it follows that

$$(10)\qquad \langle h_G, \Delta u\rangle = -\|\Delta u\|^2 \sum_1^p \cos^2\alpha_i + \langle \varepsilon_G, \Delta u\rangle\ ,$$

where the α_i are given in condition 5 of the definition. Applying the Schwarz inequality to (8) yields

$$(11)\qquad \|h_G\|^2 \leqq \|\Delta u\|^2 \left(p\sum_1^p \cos^2\alpha_i + w_G(\Delta u)\right)\ .$$

where $w_G(\Delta u) \to 0$ as $\Delta u \to 0$.

Using (7), we find

$$\langle h_T, \Delta u\rangle = -\frac{\|\nabla J_T(u)\|}{\|\nabla J_G(u)\|}\,\frac{\|\Delta u\|\cos\alpha_0}{\|\nabla\theta(u)\|\,|\cos\gamma|} = -\tan\theta(u)\frac{\|\Delta u\|\cos\alpha_0}{\|\nabla\theta(u)\|\,|\cos\gamma|},$$

where α_0 and γ are given in condition 5. By the properties of $\theta(u)$ in condition 4 and the definition of the weak differential, we have

$$\tan\theta(u) = \tan\theta(u^*) + \sec^2\theta(u^*)\langle\nabla\theta(u), \Delta u\rangle + \hat\varepsilon = \langle\nabla\theta(u), \Delta u\rangle + \hat\varepsilon\ .$$

Since $\langle\nabla\theta(u), \Delta u\rangle = \|\nabla\theta(u)\|\,\|\Delta u\|\cos\beta$, we obtain

$$(12)\qquad \langle h_T, \Delta u\rangle = -\|\Delta u\|^2(\cos\alpha_0\cos\beta/|\cos\gamma| + w(\Delta u))\ ,$$

where $w(\Delta u) \to 0$ as $\Delta u \to 0$. Similarly, from (7) we see that

$$(13)\quad \|h_T\| = \frac{\|\nabla J_T(u)\|}{\|\nabla J_G(u)\|}\,\frac{1}{\|\nabla\theta(u)\|\,|\cos\gamma|} = \|\Delta u\|(\cos\beta/\cos\alpha + w_T)$$

where $w_T \to 0$ as $\Delta u \to 0$.

Combining (9)–(13), we obtain

$$(14)\qquad \|u+sh-u^*\|^2 \leqq \|\Delta u\|^2(\Phi_1(s) + 2sw_1(\Delta u) + s^2w_2(\Delta u))\ ,$$

where $w_1(\Delta u) \to 0$ and $w_2(\Delta u) \to 0$ as $\Delta u \to 0$ and

$$\Phi_1(s) = 1 - 2bs + c_1s^2\ ,$$

$$b = \sum_1^p \cos^2\alpha_i + \cos\alpha_0\cos\beta/|\cos\gamma|\ ,$$

$$c_1 = p\sum_1^p \cos^2\alpha_i + (\cos\beta/\cos\gamma)^2\ .$$

If we replace $\Phi_1(s)$ by $\Phi(s) = 1 - 2bs + cs^2$, where

$$c = c_1 + \left(2\textstyle\sum_1^p \cos^2\alpha_i + 1 + 1/|\cos\gamma|\right)/|\cos\gamma|\ ,$$

the inequality in (14) remains valid. Now, $0 < \Phi(s) < 1$ for

$0 < s < 2b/c$. Since $c \leqq p^2 + (2p+3)/|\cos\gamma|^2$ and, by condition 5 of the definition, $|\cos\gamma| > a_2 > 0$ and $b > a_1 > 0$, it follows that $0 < \Phi(s) < 1$ for $0 < s < d$, where

$$d = 2a_1a_2^2/(a_2^2p^2 + 2p + 3) . \tag{15}$$

For $d/2 < s < d$ and $0 < \|\varDelta u\| < r$ with r sufficiently small, there exists a constant k^2 such that $0 < \Phi(s) + 2sw_1(\varDelta u) + s^2w_2(\varDelta u) < k^2 < 1$. Hence, $\| u + sh - u^* \| < k \|\varDelta u\|$ whenever $d/2 < s < d$, which completes the proof.

The lemma serves as the basis of a convergent procedure for approximating u^*, as described in the following theorem.

THEOREM. *Let u^*, k, d and r be as in the lemma. Let $u_0 \in H$ be such that $\| u_0 - u^* \| < r$. For every integer $n \geqq 0$ define*

$$\begin{cases} h_G^{(n)} = -\sum_{i=1}^{p} (\psi_i(u_n)/\|\nabla\psi_i(u_n)\|)\, \overline{\nabla\psi_i(u_n)} , \\ h_T^{(n)} = -(1/\|\nabla J_G(u_n)\| \cdot |\langle\nabla\theta(u_n), \overline{\nabla J_T(u_n)}\rangle|)\, \nabla J_T(u_n) , \\ h_n = h_G^{(n)} + h_T^{(n)} , \end{cases} \tag{16}$$

if $\nabla J_T(u_n) \neq 0$. If $\nabla J_T(u_n) = 0$, take $h_T^{(n)} = 0$. Further, define for $n \geqq 0$,

$$u_{n+1} = u_n + s_nh_n, \text{ where } d/2 < s_n < d . \tag{17}$$

Then $\lim_{n\leftarrow\infty} u_n = u^*$ and in fact,

$$\| u_n - u^* \| < k^n \| u_0 - u^* \| . \tag{18}$$

Proof. Since $\| u_0 - u* \| < r$, u_0 satisfies the conditions on the point u of the lemma. Comparing h_0 with h and s_0 with s of the lemma, we see immediately that $\| u_1 - u^* \| < k \| u_0 - u^* \|$. Since $k < 1$, $\| u_1 - u^* \| < r$ and the lemma can be applied at u_1 to obtain $\| u_2 - u^* \| < k \| u_1 - u^* \| < k^2 \| u_0 - u^* \|$. By induction, (18) is immediate and this establishes the theorem.

The application of this theorem to numerical procedures for solving minimization problems with equality constraints will be the subject of a forthcoming paper.

REFERENCES

1. H. Antosiewicz and W. Rheinboldt, *Numerical analysis and functional analysis*, Survey of Numerical Analysis, McGraw-Hill, 1962.

2. A. V. Balakrishnan, *An operator theoretic formulation of a class of control problems and a steepest descent method of solution*, J.S.I.A.M. Control, Ser. A, Vol. 1, No. 2, 1963.

3. A. V. Balakrishnan and L. W. Neustadt, ed., *Computing Methods in Optimization Problems*, Academic Press, 1964.
4. E. K. Blum, *Minimization of functionals with equality constraints*, Abstract 64T-381, Notices A. M. S., August 1964.
5. ———, *Minimization of functionals with equality constraints*, J. S. I. A. M. Control. Ser. A, vol. 3, No. 2, 1965, pp. 299-316.
6. A. Goldstein, *Convex programming in Hilbert space*, Bull. Amer. Math. Soc. **70**, No. 5, September 1964, 709-710.
7. W. Hart and T. S. Motzkin, *A composite Newton-Raphson gradient method for the solution of systems of equations*, Pacific J. Math. **6** (1956), 691-707.
8. L. Liusternik and V. Sobolev, *Elements of Functional Analysis*, Ungar, 1961.

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