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ON AN ENTIRE FUNCTION OF AN ENTIRE FUNCTION DEFINED BY DIRICHLET SERIES

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In this note we prove the following theorem which seems to exhibit an essential property of the order (R) of entire function defined by Dirichlet series.

THEOREM **If $h(s)$ and $g(s)$ are entire functions defined by Dirichlet series and $g(\log h(s))$ is an entire function of finite order (R), then there are only two possible cases: either**
(a) the internal function $h(s)$ is a Dirichlet polynomial and the external function $g(s)$ is of finite order (R); or
(b) the internal function $h(s)$ is of finite order (R) and the external function $g(s)$ is of order zero.

Here $h(s)$ and $g(s)$ are entire functions defined by the Dirichlet series

$$h(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s} ,$$

$$g(s) = \sum_{n=0}^{\infty} b_n e^{ns} ,$$

satisfying the relations

$$\underset{-\infty<t<\infty}{\text{l.u.b.}} \mid h(\sigma + it) \mid = H(\sigma) ,$$

$$\underset{-\infty<t<\infty}{\text{l.u.b.}} \mid g(\sigma + it) \mid = G(\sigma) ,$$

for any real value of (in particular, every Dirichlet series absolutely convergent in the whole plane will have this property).

For this type of function Ritt [2] defines the order in the following way:

$$\rho = \limsup_{\sigma=\infty} \frac{\log \log H(\sigma)}{\sigma}$$

will be called the order (R) of $h(s)$; we shall also express it by saying that $h(s)$ is of order (R) equal to ρ.

It is to be noted that since $g(\log h(s))$ is simply a power series in $h(s)$, it is a single valued function. Since $h(s)$ is an entire function, there will be at least one term of the series which is greater than all other terms in its absolute value. In the case when there are more than one such terms we regard the term with the highest rank as the maximum term. With this convention the maximum term of $h(s)$ will be denoted by $\mu(\sigma, h)$.

2. For the proof of the above theorem we require following lemmas.

LEMMA 1. *If $h(s)$ is an entire function defined by Dirichlet series with least upper bound $H(\sigma)$ and maximum term $\mu(\sigma, h)$, then*

$$H(\sigma) \geqq \mu(\sigma, h) \geqq | a_n | e^{\lambda_n \sigma} .$$

Sugimura [3] has proved this result.

LEMMA 2. *Suppose that θ is any number $0 < \theta < \infty$ and*

$$w = \phi(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s}, \ (s = \sigma + it, \ 0 < \lambda_1 < \lambda_2 \cdots \rightarrow \infty)$$

is any function regular for $Re(s) \leqq \sigma$ and is representable by an absolutely convergent Dirichlet series. Furthermore suppose that $\phi(s)$ satisfies the conditions

$$(1) \qquad \phi(-\infty) = 0, \ \underset{\substack{Re(s)=\sigma-\theta \\ -\infty<t<\infty}}{\text{l.u.b.}} | \phi(s) | = 1 .$$

Let r_ϕ denote the radius of the largest circle $| w | = r_\phi$ all of whose points represent the values taken by $\phi(s)$ in the domain $Re(s) \leqq \sigma$. Then r_ϕ is not less than C, $C = C(\lambda_1, \theta)$ being a positive number which depends on λ_1 and θ.

This lemma is a generalization of a result of Bohr [1] which runs as follows:

Suppose that μ is any number, $0 < \mu < 1$, and $w = \phi(z)$ is any function which is regular for $| z | \leqq 1$ and satisfies the conditions

$$(2) \qquad \phi(0) = 0, \ \underset{|z|=\mu}{\text{Max}} | \phi(z) | = 1 .$$

Let r_ϕ be the radius of the largest circle $| w | = r_\phi$ all of whose points represent the values taken by $\phi(z)$ in the circular domain $| z | \leqq 1$. Then r_ϕ is not less than a positive number C, $C = C(\mu)$ which depends on μ.

The simplest proof that I can present is based on a theorem of Ritt [2] which is a generalization of Landau-Schottky's theorem.

Proof of Lemma 1. The theorem of Ritt which we need runs as follows: Let

$$(3) \qquad \psi(s) = a_0 + \sum_{n=1}^{\infty} a_n e^{\lambda_n s}, \ (s = \sigma + it, \ 0 < \lambda_1 < \lambda_2 \cdots \infty)$$

be analytic for $Re(s) \leqq \sigma$ and for $Re(s) \leqq T \leqq \sigma$ be representable by an absolutely convergent Dirichlet series. Let θ be a variable assuming all values greater than zero, and $\psi(s)$ is nowhere zero or unity for $Re(s) \leqq \sigma$. Then there exists a function $S(a_0, \lambda_1, \theta)$, such that

$$| \psi(s) | < S(a_0, \lambda_1, \theta) \text{ for } Re(s) \leqq \sigma - \theta .$$

A simple modification will show that if $|\psi(-\infty)| \leqq 1$ then there exists a function $S(\lambda_1, \theta)$, such that $|\psi(s)| < S(\lambda_1, \theta)$. In particular this is true, if $a_0 = -1$.

From the result stated above follows the existence of a number $C = C(\lambda_1, \theta)$ in the sense of the lemma, if we take $C(\lambda_1, \theta) = 1/1 + 3\,S(\lambda_1, \theta)$. For if we suppose that the lemma is false, then for $Re(s) \leqq \sigma$ there exists a regular function $\phi(s)$ with $\phi(-\infty) = 0$ and $\text{l.u.b.}_{-\infty<t<\infty}^{Re(s)=\sigma-\theta} |\phi(s)| = 1$, for which the radius $r_\phi < C$. This means that $\phi(s)$ does not take all values on the circle $|w| = C$ and also on the circle $|w| = 2C$. In particular it does not assume the values $a = C.e^{i\alpha}$ and $b = 2C.e^{i\beta}$ respectively. Let

$$\psi(s) = \frac{\phi(s) - a}{b - a}\,. \tag{4}$$

$\psi(s)$ is regular for $Re(s) \leqq \sigma$ and is representable by an absolutely convergent Dirichlet series and does not take the values zero or unity. Further

$$|\psi(-\infty)| = \left|\frac{\phi(-\infty) - a}{b - a}\right| \leqq 1\,.$$

Hence from the theorem of Ritt, there exists a function $S(\lambda_1, \theta)$ such that for $Re(s) \leqq \sigma - \theta$

$$|\psi(s)| < S(\lambda_1, \theta)\,. \tag{5}$$

Hence for $Re(s) \leqq \sigma - \theta$

$$|\phi(s)| = |a + (b - a)\,\psi(s)| < C + 3C.S(\lambda_1, \theta) < 1\,, \tag{6}$$

which is contrary to the hypothesis that

$$\underset{\substack{Re(s)=\sigma-\theta \\ -\infty<t<\infty}}{\text{l.u.b.}} |\phi(\sigma + it)| = 1\,.$$

LEMMA 3. *Suppose that $f(s)$, $g(s)$ and $h(s)$ are entire functions connected by the relation*

$$f(s) = g(\log h(s))\,, \tag{7}$$

suppose further that

$$h(-\infty) = 0\,. \tag{8}$$

Let $F(\sigma)$, $G(\sigma)$ and $H(\sigma)$ denote the least upper bounds of $f(s)$, $g(s)$ and $h(s)$ respectively for $Re(s) = \sigma$. Then there exists a definite number C independent of $g(s)$, $h(s)$ and σ, such that

$$F(\sigma) \geqq G\{\log\,(C.H(\sigma - \theta))\} \tag{9}$$

where θ is a fixed constant.

Proof of Lemma 3. To fix the ideas, we take θ to be a fixed positive constant, and apply Lemma 2 to the function

$$z = \phi(s) = \frac{h(s)}{H(\sigma - \theta)}$$

which satisfies the conditions (1). We see that this function maps the domain $Re(s) \leqq \sigma$ on a Riemann surface over the z-plane whose various sheets cover the whole periphery of a certain circle of centre $z = 0$ and radius R, R being not less than $C.H(\sigma - \theta)$. Furthermore, the function $w = \log z$ maps a Riemann surface consisting of infinity of sheets, each cut along the negative real axis from $-R$ to zero and the upper half of each sheet is joined to the lower half of the next, on the w-plane extending from $-\infty$ to $Re(w) = \log R$. Thus a strip of width 2Π corresponds to one complete sheet of the Riemann surface and every point of Riemann surface corresponds to just one point of w-plane. R being not less than $C.H(\sigma - \theta)$.

Suppose that w_0 is a point on the line $Re(w) = \log R$, such that

$$g(w_0) = G\{Re(w_0)\} = G(\log R) ,$$

then there is at least one point s_0 inside $Re(s) \leqq \sigma$, such that

$$w_0 = \log h(s_0) .$$

Hence it follows that

$$G[\log\{C.H(\sigma - \theta)\}] \leqq G\{Re(w_0)\} = | g(w_0) | = | g(\log h(s_0)) | \leqq F(\sigma) .$$

3. Proof of the theorem. We observe that $F(\sigma)$, $G(\sigma)$ and $H(\sigma)$ are increasing functions. We may express the hypothesis that $f(s)$ is of finite order (R) 'a' by the inequality

$$F(\sigma) < \exp[\exp\{\sigma(a + \varepsilon)\}] . \tag{10}$$

We have

$$h(s) = a_1 e^{\lambda_1 s} + a_2 e^{\lambda_2 s} + \cdots + a_m e^{\lambda_m s} + \cdots .$$

From Lemma 1, we have

$$H(\sigma) \geqq \mu(\sigma, h) \geqq | a_m | e^{\lambda_m \sigma} . \tag{11}$$

In virtue of (9), (10) and (11)

$$G[\log\{C. | a_m | e^{-\lambda_m \theta}\} + \lambda_m \sigma] \leqq G[\log\{C.H(\sigma - \theta)\}] < \exp[\exp\{\sigma(a + \varepsilon)\}]$$
$$G[\log\{C. | a_m | e^{-\lambda_m \theta} + \sigma] < \exp[\exp\{(a + \varepsilon)\sigma/\lambda_m\}] .$$

That is to say, the order (R) of $g(s)$ does not exceed a/λ_m. If $h(s)$ is not a Dirichlet polynomial, λ_m can be chosen arbitrarily large and in this case the order (R) of $g(s)$ is zero.

In any case there is an inequality for $g(s)$, analogous to (11), let us say

$$G(\sigma) \geqq |b_n| e^{n\sigma} \; (n \geqq 1) .$$

Combining this with (9) and (10), we obtain

$$\begin{aligned} |b_n| \exp[n \log\{C.H(\sigma - \theta\}] &\leqq G[\log\{C.H(\sigma - \theta)\}] \\ &\leqq F(\sigma) < \exp[\exp\{(a + \varepsilon)\sigma\}] . \end{aligned}$$

Thus the order (R) of $h(s)$ is not greater than 'a'.

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References

1. H. Bohr, *Uber einen Staz von Edmund Landau*, Scripta Univ atque Bibl Hierosolymitanarum, (2) **1** (1923).
2. J. M. Ritt, *On certain points in the theory of Dirichlet series*, Amer. J. Math, **50** (1928), 73.
3. K. Sugimura, *Ubertragung einiger Satze and der Theorie der ganzen Funktionen auf Dirichletschen Reihen*, Math. Zeit **29** (1929), 264.

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