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A CONVEXITY PROPERTY

RAYMOND W. FREESE

There exist a variety of conditions yielding convexity of a set, dependent upon the nature of the underlying space. It is the purpose here to define a particular restriction involving n-tuples (the n-isosceles property) on subsets of a straight line space and study the effect of this restriction in establishing convexity. By a straight line space is meant a finitely compact, convex, externally convex metric space in which the linearity of two triples of a quadruple implies the linearity of the remaining two. The principal theorem states that the n-isosceles property is a sufficient condition for a closed and arcwise connected subset of a straight line space to be convex if and only if n is two or three.

In such a space S we use two of the definitions stated by Marr and Stamey (4).

DEFINITION 1. If p, q, r are distinct points of S such that at least two of the distances pq, pr, qr are equal, then the points p, q, r are said to form an isosceles triple in S.

DEFINITION 2. A subset M of S is said to have the double-isosceles three-point property if two connecting segments of each of its isosceles triples belong to M.

A proof of (2) together with (4) shows that if M is a closed connected subset of S and possesses the double isosceles property, then M is convex.

DEFINITION 3. A subset M of S is said to have the n-isosceles property ($n \geqq 2$) provided for every $(n+1)$-tuple $p_1, p_2, \cdots, p_{n+1}$ of distinct points of M such that $p_ip_{i+1} = p_{i+1}p_{i+2}$, $i = 1, 2, \cdots, n-1$, at least n of the connecting segments lie in M.

A comparison of the double isosceles property and n-isosceles property shows that in S the two are equivalent for $n = 2$. For n greater than 2, the double isosceles property clearly implies the n-isosceles property but it is not immediately evident whether the two are equivalent. The question may be raised concerning the conditions under which the n-isosceles property is sufficient to replace the double-isosceles property in the above-mentioned theorem yielding convexity.

This question is answered in part by the following theorem.

THEOREM 1. *Let M be a closed subset of S such that every pair of points of M can be joined by a rectifiable arc in M. If M has the three-isosceles property, then M is convex.*

Proof. Let p, q be any two points of M and let A denote a rectifiable arc in M with endpoints p, q. Then there exists a shortest arc in M joining p, q, say A^γ. Let r, s be points of A^γ such that $pr = rs = sq$. Then since M possesses the three-isosceles property and A^γ is a geodesic arc in M, consideration of the cases reveals $S(p, q) \subset M$ or A^γ is the union of a finite number of noncollinear metric segments, or all three connecting segments of triples p, r, s or q, r, s are contained in M.

We shall suppose $S(p, q) \not\subset M$. If A^γ is the union of a finite number of noncollinear metric segments, then by the metric transitivities of the space, r or s is noncollinear with p, q. Hence for at least one of the points r, s say r, that point is the terminal and initial point, respectively, [when traversing A^γ from p to q] of metric segments $S_1 \subset M \cap A^\gamma$, $S_2 \subset M \cap A^\gamma$, which in turn contain point pairs u_1, u_2 and v_1, v_2, respectively, such that $u_1u_2 = u_2v_1 = v_1v_2$ while u_2v_1 is strictly less than $u_2r + rv_1$. Applying the three-isosceles property to the points u_1, u_2, v_1, v_2, it follows that $S(u_i, v_j) \subset M$ for some $i, j = 1, 2$ which violates the shortest arc hypothesis for A^γ.

Now suppose all three connecting segments of a triple (say p, r, s) are contained in M. If ps is less than $pr + rs$ and p, r, s are met in this or reverse order, a contradiction is encountered. A similar argument holds if the order is p, s, r. We may then assume the labeling such that p, r, s are encountered in this order and $ps = pr + rs$. Consider the longest segment containing $S(p, s)$ with one endpoint p and contained within M and denote its remaining terminal point by s'. Considering the subarc $A'(s', q)$, it follows as above that it consists of a finite number of metric segments (and hence A^γ, which was discussed previously) or else there exists a metric segment contained in $A' \cap M$ with either s' or q as endpoint.

Repeating this latest procedure at most once, it follows that either A^γ consists of a finite number of metric segments or there exist two noncollinear metric segments contained in $A^\gamma \cap M$ with a common endpoint. Applying the three-isosceles condition to the appropriate four points of these two segments results again in a contradiction.

We conclude M is convex.

The following sequence of lemmas will lead to a strengthening of the above theorem. In each of these lemmas, S is assumed to be a straight line space and M to be a closed, arcwise connected subset of S possessing the three-isosceles property.

LEMMA 1. *Let A denote an arc in M with endpoints p, q. If p, q are not joined by a rectifiable arc, then one of the two points (to be termed 'exceptional') has the property that every arc joining it to other points is nonrectifiable.*

Proof. Let a, b be points of A such that $pa = ab = bq$. Then since M possesses the three-isosceles property, the existence of three of the six segments within M implies that either there exists an arc with endpoints p, q consisting of one, two, or three segments each contained within M (and hence there exists a rectifiable arc with endpoints p, q) or all three connecting segments of some triple (say a, b, q) of the quadruple are contained within M.

In the latter case, given a positive ε less than $ab/4$, by the method of proof of Lemma 23.1 (1), there exists a finite sequence $p_1, p_2, \cdots, p_n$ of distinct points of the arc such that $p_ip_{i+1} = \varepsilon$, $p_ip_j \geqq \varepsilon$ for $i \neq j$, $p = p_1$, and $0 < p_na \leqq \varepsilon$ for $i, j = 1, 2, 3, \cdots, n$, where p_{n+1} is defined as follows. If none of the p_i, $i = 1, 2, \cdots, n$ are elements of $S(a, b)$ let p_{n+1}, p_{n+2} be two points of $S(a, b)$ such that $p_np_{n+1} = p_{n+1}p_{n+2} = \varepsilon$.

Applying the three-isosceles property to $p_{n-1}, p_n, p_{n+1}, p_{n+2}$, it follows that at least one other connecting segment of the quadruple must form with $S(p_{n+1}, p_{n+2})$ a connected set and be contained within M. Hence there exists a rectifiable arc from b to p_{n-1} or p_n contained within M and consisting of a finite union of metric segments. Suppose p_n [or p_{n-1}] is the endpoint of this arc. Then there exists a point, which may be denoted by r_n [s_{n-1}] such that $p_nr_n = \varepsilon$ and $S(p_n, r_n) \subset M$. [$p_{n-1}s_{n-1} = \varepsilon$ and $S(p_{n-1}, s_{n-1}) \subset M$]. Then applying the three-isosceles condition to the appropriate quadruple, it follows that there exists a rectifiable arc from b to p_{n-1} or p_{n-2}. Repeating this process a finite number of times shows the existence of a rectifiable polygonal arc contained in M with one endpoint b and the other endpoint p_1 or p_0 where p_0 is any point of M with $p_0p_1 = \varepsilon$. Hence the lemma is valid, for in the contrary case, if p_1 and some point u are the endpoints of a rectifiable arc $A(p_1, u)$, then, given that all segments of a, b, q are contained in M, the above method of proof can be followed for a positive δ less than min $[ab/4, up]$ and hence there exists a rectifiable arc $A(p_1, t)$ where t is in $A(p_1, u)$ such that $p_1t = \delta$. Then by the preceding it is not possible for t to be p_i for any $i = 1, 2, \cdots, n + 2$ for then there exists a rectifiable arc with endpoints b, p_1, whereas if t is distinct from these points we may set $t = p_0$ and observe that there exists a rectifiable arc with endpoints p_1, b which implies the existence of a rectifiable arc contained in M and joining p, q, contrary to hypothesis.

If $S(a, b) \cap \{p_i\}$ is not null, let p_j denote the point with minimum index and delete the members of the sequence with higher index. Then relabel as p_{j+1} a point of $S(a, b)$ such that $p_j p_{j+1} + \varepsilon$, and in the above proof replace n by $j - 1$.

LEMMA 2. *There exists at most one 'exceptional' point.*

Proof. Suppose the contrary, and let x, y denote two such points. The method of proof of the preceding lemma involving p, q and now applied to x, y shows that there exists a rectifiable arc $A(x, y_0) \subset M$ where $x \neq y_0$ or a rectifiable arc $A(y, x_0) \subset M$, where $y \neq x_0$, which violates our supposition.

LEMMA 3. *The set of points of M that is not 'exceptional' is convex.*

Proof. Denote by x the 'exceptional' point of M if such exists. Given any two points p, q of $M - \{x\}$, $p \neq q$, it follows from Lemma 2 that neither p nor q is 'exceptional' and hence by Lemma 1 they are the endpoints of a rectifiable arc in M. As in Theorem 1, considering M as a finitely compact metric space it follows that there exists in M a geodesic arc A joining p, q. Since x is an 'exceptional' point, x is not in A. Again as in Theorem 1, there exist two points of A which, with p, q, form a quadruple to which the three-isosceles condition can be applied. Again x is not a point of any of the connecting segments in M whose existence is determined since it is 'exceptional'. Hence the proof proceeds as in Theorem 1, yielding a contradiction unless the segment joining p, q is contained in $M - \{x\}$.

LEMMA 4. *The set M is convex.*

Proof. In view of Lemma 3, it suffices to show that if x denotes the 'exceptional' point and p is a point of $M - \{x\}$, there exists a point of M between p and x.

Since M is connected, let $\{x_n\}$ denote a sequence of points of $M - \{x\}$ such that $\lim x_n = x$. Denote by m_n the midpoint of y, x_n for $n = 1, 2, \cdots$. Since M is finitely compact, there exists a point m of M such that m is the limit of a subsequence $\{m_{i_n}\}$ of $\{m_n\}$. Hence $\lim x_{i_n} = x$ and $pm_{i_n} + m_{i_n}x_{i_n} = px_{i_n}$ for all n implies $pm + mx = px$.

From these lemmas, it follows that the theorem below is valid.

THEOREM 2. *Let M be a closed arcwise connected subset of a straight line space S. If M has the three-isosceles property, then M is convex.*

The above theorem is not valid when the condition that M possess the three-isosceles property is replaced by the demand that M possess the n-isosceles property with $n \geqq 4$. This may be observed by considering any nonlinear isosceles triple q, r, s of the euclidean plane. Let M_0 be the union of the equal segments $S(q, r)$, $S(r, s)$. Since M_0 clearly is not convex, it suffices to show that M_0 possesses the n-isosceles property for all n greater than three.

Let $p_1, p_2, \cdots, p_{n+1}$ be any $n+1$ distinct points of M_0 such that $p_i p_{i+1} = p_{i+1} p_{i+2}$, $i = 1, 2, \cdots, n-1$. If n is even the minimum number of segments lying entirely within M_0 will occur when $n/2$ points lie on one of the two segments comprising M_0 and $(n+2)/2$ points on the other segment. Hence there always exist at least $n(n-2)/8 + n(n+2)/8$ connecting segments contained within M_0 which is greater than or equal to n for $n \geqq 4$. If n is odd, the minimum number of segments lying entirely within M_0 will occur when $(n+1)/2$ points lie on each segment. Hence since $(n^2-1)/8 + (n^2-1)/8 \geqq n$ for $n \geqq 5$, it follows that M_0 has the n-isosceles property for $n \geqq 4$.

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ST. LOUIS UNIVERSITY