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A NOTE ON LOOPS

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An Associative Element of a quasigroup is defined to be an element a with the property that $x(yz) = a$ implies $(xy)z = a$. It is then shown that

(i) a quasigroup which contains an associative element is a loop,

(ii) if a loop contains an associative element then the nuclei coincide,

(iii) if a loop is weak inverse then the set of associative elements coincides with the nucleus,

(iv) if a loop is not weak inverse then no associative element is a member of the nucleus and the product of any two associative elements is not associative.

In [2] Osborn defines a Weak Inverse Loop to be a loop with the property that $x(yz) = 1$ implies $(xy)z = 1$. More generally we will define an *Associative Element* of a quasigroup to be an element a with the property that $x(yz) = a$ implies $(xy)z = a$. In this note some of the properties of associative elements will be considered.

LEMMA 1. *If a is an associative element of a quasigroup G then $(xy)z = a$ implies $x(yz) = a$.*

Proof. Assume that $(xy)z = a$. Since G is a quasigroup there exists an element v such that $v(yz) = a$. Hence, since a is associative $(vy)z = a$. Thus $(vy)z = (xy)z$ and so $x = v$ since G is a quasi-group.

THEOREM 2. *A quasigroup which contains an associative element is a loop.*

Proof. Let a be an associative element and y any element of the quasigroup, then there exist elements z and b such that $(ay)z = a$ and $ba = a$. Thus $a = b[(ay)z] = [b(ay)]z$, since a is associative. But $a = (ay)z$ and so $b(ay) = ay$. However y is any element of the quasigroup and so $bx = x$ for all x in the quasigroup. Thus b is a left unit and similarly there exists a right unit and hence a unit element.

Not all loops contain associative elements, for example the loop given by the following multiplication table.

	1	2	3	4	5
1	1	2	3	4	5
2	2	5	4	1	3
3	3	1	2	5	4
4	4	3	5	2	1
5	5	4	1	3	2

The loop given by the following multiplication table contains an associative element 2, but the unit element 1 is not associative, i.e., the loop is not weak inverse.

	1	2	3	4	5
1	1	2	3	4	5
2	2	1	4	5	3
3	3	5	2	1	4
4	4	3	5	2	1
5	5	4	1	3	2

Bruck [1] defines the Left Nucleus, N_L of a loop to be the set of those elements n satisfying $(nx)y = n(xy)$ for all x and y. The Middle and Right Nuclei, N_M and N_R, are similarly defined. The Nucleus, $N = N_L \cap N_M \cap N_R$. Bruck shows that N is a group. Osborn shows that the nuclei of a weak inverse loop coincide. More generally we have the following result.

THEOREM 3. *If a loop contains an associative element then the nuclei coincide.*

Proof. In a loop $N_M \neq \varnothing$ since $1 \in N_M$.

Let n belong to N_M, x and y be any elements of the loop and a be an associative element of the loop.

There exists an element z such that

$$\begin{aligned} a &= [x(yn)]z \\ &= x[(yn)z], \ a \text{ is associative}, \\ &= x[y(nz)], \ n \in N_M, \\ &= (xy)(nz), \ a \text{ is associative}, \\ &= [(xy)n]z, \ a \text{ is associative}. \end{aligned}$$

Thus $[x(yn)]z = [(xy)n]z$ and so $x(yn) = (xy)n$. Hence $n \in N_R$, and so $N_M \subseteqq N_R$. Reversing the argument shows that $N_R \subseteqq N_M$ and hence $N_R = N_M$. Similarly $N_L = N_M$.

Writing A for the set of associative elements we have the following relationship between A and N.

THEOREM 4. *If a loop is weak inverse then the set of associative elements coincides with the nucleus. If a loop is not weak inverse then no associative element is a member of the nucleus and the product of any two associative elements is not associative.*

Proof. We show first that if, in a loop, $A \neq \emptyset$, then $a \in A$ and $n \in N$ implies $An = A$ and $aN = A$.

Let $nm = 1$. Then since N is a group $m \in M$ and $mn = 1$. Also $(an)m = a(nm) = a$.

Let $an = (xy)z$. Then

$$\begin{aligned} a &= [(xy)z]m, \\ &= (xy)(zm), \quad a \in A \\ &= x[y(zm)], \quad a \in A \\ &= x[(yz)m], \quad m \in N \\ &= [x(yz)]m, \quad a \in A\,. \end{aligned}$$

Thus $[(xy)z]m = [x(yz)]m$ and so $(xy)z = x(yz)$ and hence an is associative, i.e., $an \in A$. Thus $An \subseteqq A$ and $aN \subseteqq A$.

It follows that $Am \subseteqq A$ and so $(Am)n \subseteqq An$. But $(Am)n = A(mn) = A$ and so $A \subseteqq An$. Thus $An = A$.

To show that $aN \supseteqq A$ let $b \in A$ and $ak = b$. Given elements y and z there exists an element x such that $b = x[(yz)k] = [x(yz)]k$ since $b \in A$ and as $b = ak$ we have $a = x(yz)$ and so $a = (xy)z$.

$$\begin{aligned} \text{Thus } b &= [(xy)z]k \\ &= (xy)(zk) \\ &= x[y(zk)] \text{ since } b \in A\,. \end{aligned}$$

Hence $x[(yz)k] = x[y(zk)]$ and so $(yz)k = y(zk)$. Thus $k \in N$ and so $b \in aN$ and $A \subseteqq aN$. Hence $A = aN$. In a weak inverse loop $1 \in A$ and so $N = 1N = A$.

If $A \cap N \neq \emptyset$, say $y \in A \cap N$ then $yN = A$ and $yN = N$ since N is a group and so $A = N$. But $1 \in N$ and hence $1 \in A$, i.e., the loop is weak inverse.

If $AA \cap A \neq \emptyset$, then there exist $a, b, c \in A$ such that $ab = c$. But $aN = A$ and so $an = c$ for some $n \in N$. Thus $b = n$, i.e., $b \in N$ and so $A \cap N \neq \emptyset$ and hence the loop is weak inverse. This completes the proof of Theorem 4.

REFERENCES

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2. J. M. Osborn, *Loops with the weak inverse property*, Pacific J. Math. **10** (1960), 295–304.

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