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CHARACTERIZATIONS OF DIRECT SUMS AND COMMUTING SETS OF VOLTERRA OPERATORS

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Theorem 1 contains an abstract characterization and unitary invariants of operators T which are finite direct sums of n Volterra operators $(\alpha_j Vf)(x) = \alpha_j \int_0^x f(y)dy$ with real nonzero α_j defined on a Hilbert space $\mathscr{H}$ which is a direct sum of n $\mathscr{L}_2(I_1)$ spaces on the unit interval I_1. This is done by demanding that the dimension of $(T + T^*)\mathscr{H}$ be n; that the subspaces $\mathscr{H}_j$ of $\mathscr{H}$ generated by T and the eigenvectors e_j of $T + T^*$ be orthogonal to all e_k for $k \neq j$; and that the spectrum of T be 0. Theorem 2 contains an abstract characterization and unitary invariants of finite commuting sets $\{W_j\}_1^n$ of Volterra operators which are real nonzero multiples of integration in the various coordinate axis directions on a Hilbert space $\mathscr{H}$ which is the $\mathscr{L}_2$ space on the unit cube in n real dimensions. The characterization is given by demanding that the W_j commute with all W_k and W_k^* for $k \neq j$; that $\prod (W_j + W_j^*)\mathscr{H} = \mathscr{E}$ have dimension 1; that $\mathscr{H}$ be spanned by the W_j's and $\mathscr{E}$; and that the W_j's have spectrum 0.

The simplest bounded Hermitian operators are the *simple* or *cyclic* operators which are defined as "multiplication by the independent variable on a suitable $\mathscr{L}_2(\mu)$ space" where μ is a Borel measure with compact support on the real line. The simplest bounded Volterra operators are of the form αV for real nonzero α defined on $\mathscr{L}_2(I_1)$. In general, we mean by *Volterra operator* a linear transformation T_F defined on a space of functions so that $(T_F f)(x) = \int_0^x F(x, y)f(y)dy$. It is a remarkable fact that the simple Hermitian operators depend unitarily on the measure μ, that is, two such operators are unitarily equivalent if and only if the corresponding measures μ are equivalent, while two Volterra operators V defined on different $\mathscr{L}_2(\mu)$ spaces *are* unitarily equivalent so long as both measures μ are nonatomic and have the same total mass. Thus there is no loss of generality in our paper if we confine ourselves to the Lebesgue spaces $\mathscr{L}_2(I_1)$, etc. The presence of atoms brings our different phenomena which we hope to develop in the future.

The most general Hermitian operators are direct sums of the simple ones. This motivates our aim to characterize direct sums of real multiples of the simplest Volterra operators. This work follows the spirit of [3] in that we seek to establish concrete analytic representations. Our Theorem 1 is a first steps in this direction. While these operators fall into the class of operators considered in Livšic's theory [1, 4], the conclusions of our theorem have to be deduced from that theory in approximately the same way we proceed, namely by basing it upon the characterization of V itself [3].

Another representation theorem for Hermitian operators says that commuting families can be *simultaneously* represented as families of multiplication operators by functions on some suitable $\mathscr{L}_2(\mu)$ space with Borel measure μ of finite mass in I_1; see for example [3]. This motivates our interest in the corresponding situation for Volterra operators which is explored in our Theorem 2. The proof is again based on the characterization of V itself [3]; in addition it uses a lemma due to Livšic in the form stated on p. 354 of [3] and referred to here as "Livšic's Lemma". The proof then proceeds by establishing explicit formulas for two analytic functions in n and $2n$ variables respectively determined by products of the resolvents of the W's; one of these functions is the joint characteristic function of the set $\{W_j\}$ [3].

We write I_n for the unit cube in the space R_n of n real dimensions and $\mathscr{L}_2(I_n)$ for the Lebesgue $\mathscr{L}_2$ space on I_n. We define the operator V_j on $\mathscr{L}_2(I_n)$ as $(V_jf)(x_1, \cdots, x_n) = \int_0^{x_j} f(x_1, \cdots, \xi_j, \cdots, x_n)d\xi_j$. We say that the Hilbert space $\mathscr{H}$ is *generated* by the set $\{T_j\}$ of operators and the subset $\mathscr{S} \subset \mathscr{H}$ if $\mathscr{H}$ is the least closed subspace of $\mathscr{H}$ containing $\mathscr{S}$ and invariant under each T_j. We say that the set $\{S_j\}$ of operators on the Hilbert space $\mathscr{H}$ is *isomorphic* (more precisely *isometrically isomorphic*) with the set $\{T_j\}$ of operators on the Hilbert space $\mathscr{K}$ if there is an isometry U of $\mathscr{H}$ onto $\mathscr{K}$ such that $US_j = T_jU$ for all j.

The following two theorems state easily verified *necessary* conditions; it is their *sufficiency* we are concerned with here.

THEOREM 1. *Let the operator T be defined on the Hilbert space $\mathscr{H}$ such that*

(i) $\dim (T + T^*)\mathscr{H} = n$ *and* $(T + T^*)e_j = \alpha_j e_j$ *for* $j = 1, \cdots, n$ *where* $\{e_j\}$ *is orthonormal and the* α_j *are nonzero real numbers;*

(ii) *if* $\mathscr{H}_j$ *is the subspace of* $\mathscr{H}$ *generated by* T *and* e_j *then* $\mathscr{H}_j \perp e_k$ *for all distinct* j *and* k;

(iii) $\mathscr{H}$ *is generated by* T *and the set* $\{e_j\}$;

(iv) *the spectrum of* $T = 0$.

Then T *on* $\mathscr{H}$ *is isomorphic with the direct sum* $\sum_{j=1}^{n} \oplus \alpha_j V$ *defined on the direct sum of* n *copies of* $\mathscr{L}_2(I_1)$. *Two operators* T *as above are unitarily equivalent if and only if they have the same* α*'s as defined in* (i) *above.*

REMARKS. (1) Our demanding that the α's in (i) be *real* reflects the crucial role that the hypothesis that $\dim (V + V^*)\mathscr{L}_2(I_1) = 1$ plays in the proof of the representation theorem for V [3]; it is in fact an obvious necessary condition. A trivial extension of our theorem is of course possible if there exists a complex number β such that βT satisfies condition (i). The case for general complex α_j does not fall within the scope of our technique.

(2) Turning now to hypothesis (ii), the example $S = iV^2$ shows that $\dim (S + S^*)\mathscr{H} = \dim \mathscr{E} = 2$ while any single nonzero vector $e \in \mathscr{E}$ has the property that the subspace $(S; e)$ of $\mathscr{H}$ generated by S and e is all of $\mathscr{H}$ (see for example [2], Lemma 7). Thus S is certainly not isomorphic with any operator of the form $\alpha V \oplus \beta V$.

Proof of Theorem 1. We base the proof on the case $n = 1$, the characterization of αV, found in [3]. We show below that for distinct j and k, we have $\mathscr{H}_j \perp \mathscr{H}_k$; then (iii) implies that $\mathscr{H} = \sum \oplus \mathscr{H}_j$. The definition of $\mathscr{H}_j$ implies that $T\mathscr{H}_j \subset \mathscr{H}_j$ and the mutual orthogonality of the $\mathscr{H}_j$ implies that $T^*\mathscr{H}_j \subset \mathscr{H}_j$. Therefore the restriction T_j of T to $\mathscr{H}_j$ has the property $(T_j)^* = (T^*)_j$ so that T_j satisfies the hypotheses of our theorem for $n = 1$ and T_j on $\mathscr{H}_j$ is isomorphic with $\alpha_j V$ on $\mathscr{L}_2(I_1)$ by [3]; thus the proof is complete. It remains to show that $\mathscr{H}_j \perp \mathscr{H}_k$ for distinct j and k. Note first that (i) implies that for all $x \in \mathscr{H}$ we have $(T + T^*)x = \sum \alpha_j(x, e_j)e_j$ and hence (ii) implies that $(T + T^*)T^n e_j = T^{n+1}e_j + T^*T^n e_j = \alpha_j(T^n e_j, e_j)e_j$ so that, still by (ii), we have $T^*T^n e_j \in \mathscr{H}_j$ for all nonnegative integers n, whence $T^*\mathscr{H}_j \subset \mathscr{H}_j$ and $T^{*n}\mathscr{H}_j \subset \mathscr{H}_j$ for all nonnegative integers n. But then (ii) implies that $e_k \perp T^{*n}\mathscr{H}_j$, i.e., $T^n e_k \perp \mathscr{H}_j$, which implies that $\mathscr{H}_k \perp \mathscr{H}_j$ as desired. The fact that the set $\{\alpha_j\}$ determines T unitarily is then an immediate consequence of our representation.

THEOREM 2. *Let a finite set of n operators* $\{W_j\}$ *be defined on the Hilbert space* $\mathscr{H}$ *such that*

(i) $(\prod_{j=1}^n (W_j + W_j^*))\mathscr{H}$ *has dimension one and is spanned by the element e of norm one;*

(ii) *all* W_j *commute with all* W_k *and* W_k^* *for all* $k \neq j$;

(iii) $\mathscr{H}$ *is generated by the set* $\{W_j\}$ *and e;*

(iv) *the spectrum of every* W_j *is zero.*

Then the set $\{W_j\}$ *on* $\mathscr{H}$ *is isomorphic with the set* $\{\alpha_j V_j\}$ *on* $\mathscr{L}_2(I_n)$ *where the nonzero real numbers* α_j *are related to* $\{W_j\}$ *by*

$$(W_j + W_j^*)e = \alpha_j e \;. \tag{1}$$

Two sets of operators $\{W_j\}$ *as above are unitarily equivalent if and only if they have the same* α*'s as defined above in* (1).

Proof. The proof is based on the following formulas:

$$(2) \qquad \left(\prod_{j=1}^{n} \alpha_j (W_j - z_j)^{-1} e, e\right) = \prod_{j=1}^{n} (1 - \exp(\alpha_j z_j^{-1})) ,$$

$$(2') \qquad \left(\prod_{j=1}^{n} \alpha_j (Z_j - z_j)^{-1} e, e\right) = \prod_{j=1}^{n} (1 - \exp(\alpha_j z_j^{-1})) ,$$

$$(3) \qquad \prod_{j=1}^{n} (z_{j1} + \overline{z_{j2}})\left(\prod_{j=1}^{n} \alpha_j (W_j - z_{j1})^{-1} e, \prod_{j=1}^{n} \alpha_j (W_j - z_{j2})^{-1} e\right)$$
$$= \prod_{j=1}^{n} (\exp(\alpha_j (z_{j1}^{-1} + \overline{z_{j2}^{-1}})) - 1) ,$$

where the z's are arbitrary nonzero complex numbers and we write $W - z$ instead of $W - zI$ with the identity operator I; where Z_j is either W_j or W_j^*; and where the left side of (2) is the *joint characteristic function of the sets* $\{W_j\}$ (see [3]). These formulas and the isomorphism of the sets $\{W_j\}$ and $\{\alpha_j V_j\}$ will be proved by induction on n. The case $n = 1$ is again, as in Theorem 1, the characterization of αV and may be found in [3].

We begin by justifying (1). Define the commuting set of nonzero Hermitian operators $\{E_j\}$ by $E_j = W_j + W_j^*$ and set $E = \prod_{j=1}^{n} E_j$. Assumption (i) implies that for all $x \in \mathscr{H}$ we have

$$(4) \qquad Ex = \alpha(x, e)e$$

for some real nonzero α. This implies that $E_j e = (E_j e, e) = \alpha_j e$ and so

$$(5) \qquad E_{j_1} \cdots E_{j_s} e = \alpha_{j_1} \cdots \alpha_{j_s} e$$

so that $\alpha = \prod_{j=1}^{n} \alpha_j$ and the α_j are not zero.

In order to simplify the exposition, we replace the W_j's by $\alpha_j^{-1} W_j$ and then establish the theorem and the relevant formulas (1)–(3) for the special case where all the α's are 1. The results for the original W_j's are then obtained by replacing the z's by the corresponding $\alpha^{-1}z$'s.

The induction hypothesis uses the conclusions of the theorem and the formulas (1)–(3) for all $j < n$. We first establish (2), then (3), then the isomorphism of the sets $\{W_j\}$ and $\{V_j\}$, and then (2′).

We apply the identity

$$(6) \qquad \begin{aligned} &(z_1 + \bar{z}_2)(W^* - \bar{z}_2)^{-1}(W - z_1)^{-1} \\ &\quad = -(W - z_1)^{-1} - (W^* - \bar{z}_2)^{-1} \\ &\qquad + (W^* - \bar{z}_2)^{-1}(W + W^*)(W - z_1)^{-1} \end{aligned}$$

to the left side L of (3) and obtain

$$L \equiv \left(\prod_{j=1}^{n}\left[-(W_j - z_{j1})^{-1} - (W_j^* - \overline{z_{j2}})^{-1} + (W_j^* - \overline{z_{j2}})^{-1} E_j (W_j - z_{j1})^{-1}\right] e, e\right)$$
$$= \left(\prod_{j=1}^{n}\left[-A_j - B_j + B_j E_j A_j\right] e, e\right) \equiv R$$

where $A_j = (W_j - z_{j1})^{-1}$ and $B_j = (W_j^* - \overline{z_{j2}})^{-1}$. In the expansion of R we separate out the last factor so that

$$R = \left(\left[\prod_{j=1}^{n-1}(-A_j - B_j + B_jE_jA_j)\right](-A_n - B_n + B_nE_nA_n)e, e\right)$$

and we write $R = \sum X + Y$ where the terms designated by X are of the form

$$\begin{aligned} X = (-1)^{n-t}&\left(\left(\prod_{j=1}^{n-1} C_j\right)A_nF_1 \cdots F_tD_1 \cdots D_te, e\right) \\ &+ (-1)^{n-t}\left(\left(\prod_{j=1}^{n-1} C_j\right)B_nF_1 \cdots F_tD_1 \cdots D_te, e\right) \\ &+ (-1)^{n-t+1}\left(\left(\prod_{j=1}^{n-1} C_j\right)B_nF_1 \cdots F_tE_nD_1 \cdots D_tA_ne, e\right) \end{aligned}$$

where: C_j is either A_j or B_j; $F_k = E_{j(k)}$ and $D_k = C_{j(k)}$ for $j(k)$ a suitable permutation of a subset T of $\{1, \cdots, n-1\}$ containing $t \leqq n-2$ elements (or the F's and D's are absent and we set $t = 0$); and where

$$Y = \left(\left[\prod_{j=1}^{n-1}(B_jE_jA_j)\right](-A_n - B_n + B_nE_nA_n)e, e\right) .$$

Now (4) and (5) imply that

$$(7) \qquad F_1 \cdots F_tD_1 \cdots D_te = ED_1 \cdots D_te = (D_1 \cdots D_te, e)e$$

and similarly

$$(8) \qquad F_1 \cdots F_tE_nD_1 \cdots D_tA_ne = (D_1 \cdots D_tA_ne, e)e .$$

We now wish to use our induction hypothesis in order to calculate the right sides of (7) and (8). Relabel the indices so that $T = \{1, \cdots, t\}$ with $t \leqq n-1$ for our present purpose and set $\mathscr{H}_t = E_n \cdots E_{t+1}\mathscr{H}$. Assumption (ii) implies that $\mathscr{H}_t$ is invariant under W_j and W_j^* for all $j \in T$ and clearly $e \in \mathscr{H}_t$. In $\mathscr{H}_t$ we consider the closed linear subspace $\mathscr{K}$ generated by $\{W_j\}_{j\in T}$ and e. Clearly $\mathscr{K}$ is invariant under $\{W_j\}_{j\in T}$. We wish to show that this set of operators restricted to $\mathscr{K}$ satisfies the hypotheses of our theorem; it clearly suffices to establish (ii): to that end we show that $\mathscr{K}$ is invariant with respect to all W_j^*, $j \in T$. We observe that $\mathscr{K}'$, the orthogonal complement of $\mathscr{K}$ in $\mathscr{H}_t$, is invariant under all E_j: take $x' \in \mathscr{K}'$, then $(x', W_1^{p_1} \cdots W_t^{p_t}e) = 0$ for all nonnegative integral exponents p and $(E_jx', W_1^{p_1} \cdots W_t^{p_t}e) = (x', W_1^{p_1} \cdots W_j^0 \cdots W_t^{p_t}E_jW_j^{p_j}e) = (x', W_1^{p_1} \cdots W_j^0 \cdots W_t^{p_t})(W_j^{p_j}e, e) = 0$ since $E_jW_j^{p_j}e = EW_j^{p_j}e = (W_j^{p_j}e, e)e$ by (4) and (5); we have used the convention $W^0 = I$, the identity operator. Thus $\mathscr{K}'$ and hence $\mathscr{K}$ is invariant under E_j and hence $\mathscr{K}$ is invariant under W_j^*: take $x \in \mathscr{K}$, then $W_j^* x = (E_j - W_j)x \in \mathscr{K}$. Thus all our formulas are applicable and we have in particular for the right side of (7)

$$(D_1 \cdots D_t e, e) = \prod_{j=1}^{t} (1 - \exp(w_j^{-1})) = P$$

where w_j is z_{j1} or $\overline{z_{j2}}$ depending on whether the corresponding D_j is an A_j or a B_j; note that we have made use of (2′). In order to calculate the right side of (8) note that since $t \leqq n - 2$, our induction hypothesis is applicable and we obtain

$$(D_1 \cdots D_t A_n e, e) = (1 - \exp(z_{n1}^{-1}))P .$$

Hence (7) and (8) imply that

$$X = (-1)^{n-t} P\left[\left(\left(\prod_{j=1}^{n-1} C_j\right) A_n e, e\right) + \exp(z_{n1}^{-1})\left(\left(\prod_{j=1}^{n-1} C_j\right) B_n e, e\right)\right] .$$

If in (6) we set $-\bar{z}_2 = z_1 = z_{j1}$ and $W = W_j$, we obtain $(W_j - z_{j1})^{-1} = (W_j^* + z_{j1})^{-1} E_j (W_j - z_{j1})^{-1} - (W_j^* + z_{j1})^{-1}$ so that after a little calculation and using $E_j A_j e = (A_j e, e)e = (1 - \exp(z_{j1}^{-1}))e$ we have

$$\begin{aligned} (9) \quad & (D_1 \cdots D_{j-1} A_j D_{j+1} \cdots D_n e, e) \\ & = -\exp(z_{j1}^{-1})(D_1 \cdots D_{j-1}(W_j^* + z_{j1})^{-1} D_{j+1} \cdots D_n e, e) . \end{aligned}$$

In a similar way, if we set in (6) $-\bar{z}_2 = z_1 = z_{j2}$ and $W = W_j^*$, we obtain

$$\begin{aligned} (9') \quad & (D_1 \cdots D_{j-1} B_j D_{j+1} \cdots B_n e, e) \\ & = -\exp(\overline{z_{j2}}^{-1})(D_1 \cdots D_{j-1}(W_j + \overline{z_{j2}})^{-1} D_{j+1} \cdots D_n e, e) \end{aligned}$$

so that if we set $-\overline{z_{n2}} = z_{n1}$, then X is seen to be identically 0. We now turn to Y. A little calculation shows that

$$Y = -(A_1 \cdots A_{n-1} e, e)[(B_1 \cdots B_{n-1} A_n e, e) + (B_1 \cdots B_n e, e)] + ab$$

where $a = (A_1 \cdots A_n e, e)$ and $b = (B_1 \cdots B_n e, e)$. We now use (9), set $-\overline{z_{n2}} = z_{n1}$, and use our induction hypothesis to conclude that

$$Y = b\left[a - \prod_{j=1}^{n-1} (1 - \exp(z_{j1}^{-1}))(1 - \exp(z_{n1}^{-1}))\right] .$$

Since now $L = R = \sum X + Y$, the substitution $-\overline{z_{n2}} = z_{n1}$ implies that L and R are identically zero; X is also identically zero and therefore Y must be identically zero and therefore Y must be identically zero. Since, however, b is not identically zero, we can conclude that $a - \prod_{j=1}^{n} (1 - \exp(z_{j1}^{-1}))$ must be identically zero. Thus (2) is established.

To prove (3), we turn once again to $L = R = \sum X + Y$; (9′) shows that in the expansion of R we can successively replace B's by A's (and exponentials), so that finally L can be expressed entirely in terms of exponentials and terms like the right side R' of (2). Now the set $\{V_j\}$ on $\mathscr{L}_2(I_n)$ satisfies the hypotheses of our theorem (all the α's still being 1); therefore (2) is true if we replace $\{W_j\}$ by $\{V_j\}$. The left side of (3) equals $\sum X + Y$; these terms can be expressed entirely in terms of A's and other exponentials. The way these A's and other exponentials occur is based only on the hypotheses (i)–(iii) of the theorem which are satisfied by $\{W_j\}$ as well as $\{V_j\}$. Hence, since the A's are the same for these two sets, i.e., since (2) with all the α's equal to 1 is true for both sets, the left side of (3) (with the α's equal to 1) is also the same for both sets. Thus if we calculate L for $\{V_j\}$, we must get same thing as if we do it for $\{W_j\}$; the former calculation is elementary and yields the desired right side of (3) which is thus established for $\{W_j\}$.

In order to prove the isomorphism of the sets $\{W_j\}$ and $\{V_j\}$ we observe that (3) is valid and equal for both sets. Thus the infinite power series expansion deduced from (3) implies that for all nonnegative exponents, we have

$$(W_1^{p_{11}} \cdots W_n^{p_{n1}}e, W_1^{p_{12}} \cdots W_n^{p_{n2}}e) = (V_1^{p_{11}} \cdots V_n^{p_{n1}}e, V_1^{p_{12}} \cdots V_n^{p_{n2}}e) .$$

Since e and $\{W_j\}$ generate $\mathscr{H}$ and the function identically equal to 1 and the set $\{V_j\}$ generate $\mathscr{L}_2(I_n)$, we can apply Livšic's Lemma and conclude the desired isomorphism.

The last step is to check (2′). In view of the isomorphism of $\{W_j\}$ and $\{V_j\}$, it suffices to verify it for this latter set. If u is the function identically equal to 1, it is easy to verify by induction that

$$\prod_{j=1}^{k} (V_j - z_j)^{-1}u = (-1)^k \prod_{j=1}^{k} z_j^{-1} \exp(z_j^{-1}x_j) .$$

We now rewrite the left side of (2′) and obtain after eventual relabelling of indices and still keeping the α's equal to 1 the expression $(\prod_{j=1}^{k} (V_j - z_j)^{-1}u, \prod_{j=k+1}^{n} (V_j - \bar{z}_j)^{-1}u)$. A simple calculation then shows that this equals the right side of (2′); in view of the preceding paragraph u and e are identified.

Just as was the case in Theorem 1, the fact that the set $\{\alpha_j\}$ determines $\{W_j\}$ unitarily is an immediate consequence of our representation. This completes the proof of the theorem.

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