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ON THE BOUNDED SLOPE CONDITION

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Let Ω be a bounded open set in R^n and let $\varphi(x)$, $x \in \partial\Omega$, satisfy a "bounded slope condition". The latter reduces to the classical "3-point condition" if $n=2$ and occurs in papers on partial differential equations. The properties of $\varphi(x)$ are studied. It is shown, for example, that if $\partial\Omega \in C^1$ or $C^{1,\lambda}$, $0 < \lambda \leqq 1$, then $\varphi(x) \in C^1$ or $C^{1,\lambda}$. Hence, if $\partial\Omega \in C^{1,1}$ is uniformly convex, then $\varphi(x)$, $x \in \partial\Omega$, satisfies a bounded slope condition if and only if $\varphi(x) \in C^{1,1}$. The proofs use generalized convex functions of Beckenbach and, if $n > 2$, the equivalence of the bounded slope condition and an "$(n+1)$-point condition".

Let $n \geqq 2$, $x = (x^1, \cdots, x^n)$ denote a point of R^n and $z \in R^1$, so that $(x, z) \in R^{n+1}$. Let Ω be a bounded open set in R^n with boundary $\Gamma = \partial\Omega$.

DEFINITION (BSC). A real valued function $\varphi(x)$ defined for $x \in \Gamma$ is said to satisfy a bounded slope condition (BSC) [with constant K] if, for every $x_0 \in \Gamma$, there exist two linear functions $\lambda_\pm(x) = \lambda_\pm(x, x_0)$ of x,

$$\lambda_\pm(x, x_0) = a_\pm \cdot (x - x_0) + \varphi(x_0) = \sum_{k=1}^{n} a_\pm^k (x^k - x^k) + \varphi(x_0) \ , \tag{1.1}$$

where the constants $a_\pm^k = a_\pm^k(x_0)$ depend only on x_0,

$$\lambda_-(x, x_0) \leqq \varphi(x) \leqq \lambda_+(x, x) \quad \text{for } x \in \Gamma \ , \tag{1.2}$$

$$\| a_\pm(x_0) \| = \Big(\sum_{k=1}^{n} | a_\pm^k |^2 \Big)^{1/2} \leqq K \quad \text{for } x_0 \in \Gamma \ . \tag{1.3}$$

The definition of a BSC occurs in [4] and is used in [9], [2], [5]. The name "bounded slope condition" was introduced in [9]. This paper is concerned with characterizations and properties of functions φ satisfying a BSC. Section 4 dealing with the smoothness of φ uses generalized convex functions of Beckenbach [1].

It has been pointed out to me by Professor Nirenberg that if $n = 2$, a BSC is equivalent to a "3-point condition" occurring in the calculus of variations and the theory of elliptic partial differential equations; cf. [7, 49-51 and 62-63] for references to Hilbert, Lebesgue, Haar, Rado and von Neumann. In Section 3, an "$(n+1)$-point condition" will be defined and shown to be equivalent to a BSC. This fact will be used in Section 4 on smoothness properties of φ.

Note that, whether or not Ω is convex, any linear function

$$(1.4)\qquad \varphi(x) = a.x + c = \sum_{k=1}^{n} a^k x^k + c \qquad \text{for } x \in \Gamma$$

satisfies a BSC (with the choices $\lambda_\pm(x, x_0) = a.(x - x_0) + \varphi(x_0) = a.x + c$). If however $\varphi(x)$ satisfies a BSC and is not the restriction of a linear function to Γ, then Ω is convex. For, in this case, the linear functions $\lambda_\pm(x, x_0)$ of x are not identical and (1.1), (1.3) imply that

$$(a_+ - a_-)\cdot(x - x_0) \geqq 0 \qquad \text{for } x \in \Gamma\ ,$$

hence for $x \in \Omega$. Thus, through every boundary point x_0 of Ω, there is a supporting plane $0 \not\equiv (a_+ - a_-)\cdot(x - x_0) = 0$.

In what follows, it will be assumed that Ω is convex. *It should be remarked that, even if Ω is uniformly convex, it does not follow* that $a_\pm$ can be chosen so that $a_+\cdot(x - x_0) \geqq 0$ [and/or $a_-\cdot(x - x_0] \leqq 0$] for $x \in \Gamma$. For example, let $n = 2$, Ω be the disk $(x^1)^2 + (x^2 - 1)^2 < 1$ and $\varphi(x) = x^1$ for $(x^1, x^2) \in \Gamma : (x^1)^2 + (x^2 - 1)^2 = 1$. By the remark concerning (1.4), $\varphi(x) = x^1$ satisfies a BSC. The unique supporting line of *Ω through the origin is $x^2 = 0$. But it is clear that no choice of the* constant a^2 satisfies $a^2x^2 \geqq \varphi(x) = x^1 = \pm[2x^2 - 2(x^2)^2]^{1/2}$ for all $(x^1, x^2) \in \Gamma$ (e.g., for small $x^2 > 0$ and $x^1 = [2x^2 - 2(x^2)^2]^{1/2} > 0$).

2. Characterizations of $\varphi(x)$. Let $x^* \in \Omega$ and z^* be a real number. Let $C(x^*, z^*)$ denote the conical surface consisting of the set of points $(x, z) \in R^{n+1}$ of the form

$$(2.1)\qquad C(x^*, z^*):\ \ x = x^* + t(x_0 - x^*),\ z = z^* + t[\varphi(x_0) - z^*]\ ,$$

$t \geqq 0$ and $x_0 \in \Gamma$, so that $C(x^*, z^*)$ is the union of the sets of points on the half-lines from (x^*, z^*) directed towards $(x_0, \varphi(x_0))$, $x_0 \in \Gamma$.

THEOREM 2.1. *Let $\Omega \in R^n$, $n \geqq 2$, be a bounded open convex set, $\Gamma = \partial\Omega$, $x^* \in \Omega$ (fixed), and $\varphi(x)$ a function defined for $x \in \Gamma$. Then $\varphi(x)$ satisfies a* BSC *if and only if the conical surface $C(x^*, z^*)$ bounds a convex set $\Omega(x^*, z^*) \subset R^{n+1}$ for large $|z^*|$ (say, for $|z^*| \geqq N$; in which case, N can be chosen independent of x^*).*

It will be clear from the proof that $\varphi(x)$ satisfies a BSC if and only if there exists a convex function $\rho_-(x)$ and a concave function $\rho_+(x)$ defined for all $x \in R^n$ such that the restrictions of $\rho_\pm(x)$ to $\Gamma = \partial\Omega$ are identical with $\varphi(x)$.

Proof. "If". Let $z^* > 0$ be so large that $|\varphi(x_0)| < z^*$ for $x_0 \in \Gamma$ and that $C(x^*, \pm z^*)$ bound convex sets $\Omega(x^*, \pm z^*)$. Let $z = \lambda_\pm(x, x_0)$ be a supporting hyperplane of $\Omega(x^*, z^*)$ at the point $(x_0, \varphi(x_0)) \in C(x^*, \pm z^*)$, *corresponding to $t = 1$ in (2.1). It is clear that $\lambda_\pm(x, x_0)$ are of the* form (1.1) and satisfy (1.2). The conical surfaces $C(x^*, \pm z^*)$ have representations of the form

$$z = \tau_+(x) \quad \text{and} \quad z = \tau_-(x)$$

defined for all $x \in R^n$ such that $-\tau_+(x), \tau_-(x)$ are convex functions. In particular, $\tau_\pm(x)$ are uniformly Lipschitz continuous on compacts, say, on $\Omega \cup \Gamma$. It follows that there exists a constant K satisfying (1.3).

Proof. "*Only if*". Let $\varphi(x)$ satisfy a BSC. For fixed $x_0 \in \Gamma$, let $\lambda_\pm(x, x_0)$ be the linear functions of x in (1.1)—(1.3). Then

$$|\lambda_\pm(x, x_0)| \leqq K\|x - x_0\| + |\varphi(x_0)| \leqq K\|x\| + K_1 ,$$

where K_1 is a constant independent of $x \in R^n$ and $x_0 \in \Gamma$. Thus

$$\rho_-(x) = \sup \lambda_-(x, x_0) , \qquad \rho_+(x) = \inf \lambda_+(x, x_0) \qquad \text{for } x_0 \in \Gamma \tag{2.2}$$

exist (finite) for all x and satisfy

$$|\rho_\pm(x)| \leqq K\|x\| + K_1 \qquad \text{for } x \in R^n , \tag{2.3}$$

$$\rho_\pm(x_0) = \varphi(x_0) \qquad \text{for } x_0 \in \Gamma , \tag{2.4}$$

and $\mp\rho_\pm(x)$ are convex functions of x. Since $\rho_-(x) - \rho_+(x)$ is a convex function and vanishes on Γ.

$$\rho_-(x) \leqq \rho_+(x) \qquad \text{for } x \in \Omega . \tag{2.4}$$

The convexity of $\mp\rho_\pm(x)$ and (2.3) imply that $\rho_\pm(x)$ are uniformly Lipschitz continuous with a Lipschitz constant K on R^n.

Let $\Omega^\pm \in R^{n+1}$ denote the convex sets

$$\Omega^- = \{(x, z) : z > \rho_-(x)\} , \quad \Omega^+ = \{(x, z) : z < \rho_+(x)\} .$$

For $x_0 \in \Gamma$, let the linear function $\lambda^\pm(x, x_0)$ of x,

$$\lambda^\pm(x, x_0) = a^\pm(x_0)\cdot(x - x_0) + \varphi(x_0) , \tag{2.5}$$

be chosen so that $z = \lambda^\pm(x, x_0)$ is a supporting plane of $\Omega^\pm$ at the boundary point $(x, z) = (x_0, \varphi(x_0))$. In particular,

$$\|a^\pm(x_0)\| \leqq K , \tag{2.6}$$

$$\lambda^-(x, x_0) \leqq \varphi(x) \leqq \lambda^+(x, x_0) \qquad \text{for } x \in \Gamma . \tag{2.7}$$

Let $\lambda(x, x_0) \equiv a(x_0)\cdot(x - x_0)$ be a linear function of x such that $\lambda(x, x_0) = 0$ is a supporting plane for $\Omega \subset R^n$ with the normalization

$$\lambda(x, x_0) > 0 \text{ for } x \in \Omega, \ \|a(x_0)\| = 1 . \tag{2.8}$$

In view of (2.5) and (2.6), there exists numbers $N > 0$ such that $|\lambda^{\pm}(x, x_0)| \leqq N$ for $x \in \Omega$, $x_0 \in \Gamma$. Let $z^* \geqq N$ and choose numbers $\mu^{\pm}(x_0) \geqq 0$ with the property that the linear functions

$$\sigma_{\pm}(x, x_0) = \lambda^{\pm}(x, x_0) \pm \mu^{\pm}(x_0)\lambda(x, x_0)$$

of x satisfy $\sigma_{\pm}(x^*, x_0) = \pm z^*$. It is clear that $|\mu^{\pm}(x_0)\lambda(x^*, x_0)|$ and $1/|\lambda(x^*, x_0)|$ are bounded for all $x_0 \in \Gamma$ (and $x^* \in \Omega$ fixed). Thus if $\sigma_{\pm}(x, x_0)$ is written in the form

$$\sigma_{\pm}(x, x_0) = b^{\pm}(x_0)\cdot(x - x_0) + \varphi(x_0) ,$$

there is a constant $K_0 = K_0(x^*)$ such that

$$\|b^{\pm}(x_0)\| \leqq K_0 \qquad \text{for } x_0 \in \Gamma .$$

Also,

$$\sigma_{-}(x, x_0) \leqq \varphi(x) \leqq \sigma_{+}(x, x_0) \qquad \text{for } x \in \Gamma .$$

Thus, $\sigma_{\pm}(x, x_0)$ satisfy conditions analogous to (1.1) (1.3) with $\lambda_{\pm}, a^{\pm}, K$ replaced by $\sigma_{\pm}, b^{\pm}, K_0$. Corresponding to (2.2), put

$$\text{(2.9)} \qquad \tau_{-}(x) = \sup \sigma_{-}(x, x_0) , \quad \tau_{+}(x) = \inf \sigma_{+}(x, x_0) \qquad \text{for } x_0 \in \Gamma .$$

The functions $\mp\tau_{\pm}(x)$ are convex. Since $\tau_{\pm}(x_0) = \varphi(x_0)$ for $x_0 \in \Gamma$, $\tau_{\pm}(x^*) = \pm z^*$, and for $x = x^* + t(x - x_0)$, $t \geqq 0$,

$$\sigma_{-}(x, x_0) \leqq -z^* + t[\varphi(x_0) + z^*] ,$$
$$\sigma_{+}(x, x_0) \geqq z^* + t[\varphi(x_0) - z^*] ,$$

it follows that

$$\tau_{\pm}(x) = z^* + t[\varphi(x_0) \mp z^*] \qquad \text{for } x = x^* + t(x_0 - x^*) ,$$

$t \geqq 0$. Thus $z = \tau_{\pm}(x)$ are the conical surfaces $C(x^*, \pm z^*)$. Since these surfaces are convex, Theorem 2.1 is proved.

For applications, it will be convenient to reformulate Theorem 2.1 in different terms. Let $x^* \in \Omega$ be fixed and $x_0, x_1 \in \Gamma$. Suppose that the half-lines

$$\text{(2.10)} \qquad x^* + tx_0 \quad \text{and} \quad x^* + tx_1 \quad \text{for } t \geqq 0$$

in R^n are not on the same line and so determine a 2-dimensional plane $\pi_2(x_0, x_1) \in R^n$ and a convex sector $S(x_0, x_1)$ of $\pi_2(x_0, x_1)$ with vertex at x^*. Let Γ_{01} be the 2-dimensional plane convex curve $\Gamma_{01} = \pi_2(x_0, x_1) \cap \Gamma$. By a point x_{01} *of* Γ *between* x_0 *and* x_1 is meant a point x_{01} of the arc $\Gamma(x_0, x_1) = S(x_0, x_1) \cap \Gamma_{01}$. Introduce rectangular coordinates (ξ, η) in the plane $\pi_2(x_0, x_1)$ with x^* as origin such that the ξ-axis, η-axis, and the half-line $(x, z) = (x^*, t)$, $t \geqq 0$, form a right-hand system. It will be supposed that the enumeration of x_0, x_1 is chosen so that the arc

$\Gamma(x_0, x_1)$ in $\pi_2(x_0, x_1)$ is positively oriented in going from x_0 to x_1. Let (ξ_0, η_0), (ξ_1, η_1), (ξ_{01}, η_{01}) be the (ξ, η)-coordinates of x_0, x_1, x_{01}, respectively.

When the half-lines (2.10) are on the same line in R^n, the notion of a point x_{01} between x_0 and x_1 will not be defined.

COROLLARY 2.1. *Let $\Omega \subset R^n$ be a bounded open convex set, $\varphi(x)$ a function defined for $x \in \Gamma = \partial\Omega$, and $x^* \in \Omega$. Then $\varphi(x)$ satisfies a* BSC *if and only if there exists a number N such that, for $|z^*| \geqq N$, the inequality*

$$(2.11) \qquad z^* \begin{vmatrix} \xi_0 & \eta_0 & \varphi(x_0) - z^* \\ \xi_{01} & \eta_{01} & \varphi(x_{01}) - z^* \\ \xi_1 & \eta_1 & \varphi(x_1) - z^* \end{vmatrix} \leqq 0$$

holds for all points $x_0, x_1 \in \Gamma$ and points $x_{01} \in \Gamma$ between them.

See Lemma 3.1 and part (b) of the proof of Theorem 3.1 for analogous necessary and sufficient conditions.

Proof. It has to be verified that (2.11) is equivalent to the "convexity" of the cones (2.1). As the case $z^* < 0$ is similar to that of $z^* > 0$, consider only the latter. For $z^* > 0$, it will be shown that (2.11) is equivalent to the concavity of z in (2.1) as a function of x.

To verify that z is concave (i.e., that $-z$ is convex), it suffices to consider the situation when x varies along a line in R^n. If x varies along a line which passes through x^*, the concavity of the function z is clear. Consider a line L in R^n which does not pass through x^*. After a suitable translation and rotation of coordinates in the x-space, it can be supposed that $x^* = 0$ and that the line L and the point $x^* = 0$ are in the (x^1, x^2)-plane, $x^3 = \cdots = x^4 = 0$. We now ignore the trivial coordinates $x^3 = \cdots = x^4 = 0$ and write (ξ, η) in place of (x^1, x^2).

It can be supposed that L is the line $L: \xi = c > 0$. Consider two points $\pi_0 = (c, u_0)$, $\pi_1 = (c, u_1)$ on L, $u_0 < u_1$, and the condition

$$(2.12) \qquad z(\pi_{01}) \geqq \theta z(\pi_0) + (1 - \theta) z(\pi_1)$$

for z to be a concave function of $\pi_{01} = (c, u_{01})$, $u_{01} = \theta u_0 + (1 - \theta) u_1$, $0 < \theta < 1$.

Let the half-line from x^* toward π_0, π_1, π_{01} meet Γ at $x_0 = (\xi_0, \eta_0)$, $x_1 = (\xi_1, \eta_1)$, $x_{01} = (\xi_{01}, \eta_{01})$, respectively, and let t_0, t_1, t_{01} denote the unique positive numbers such that

$$(2.13) \qquad \pi_j = t_j x_j, \text{ i.e., } (c, u_j) = t_j(\xi_j, \eta_j), \text{ for } j = 0,1, \text{ and } 01 .$$

Correspondingly,

$$(2.14) \qquad z(\pi_j) = z^* + t_j[\varphi(x_j) - z^*] \text{ for } j = 0, 1, \text{ and } 01 .$$

From (2.13), $t_j = c/\xi_j$, so that, by (2.14), (2.12) is equivalent to

$$(2.15) \qquad [\varphi(x_{01}) - z^*]/\xi_{01} \geqq \theta[\varphi(x_0) - z^*]/\xi_0 + (1-\theta)[\varphi(x_1) - z^*]/\xi_1$$

Also, $u_j = t_j\eta_j = c\eta_j/\xi_j$ and $u_{01} = \theta(u_0 - u_1) + u_1$, so that

$$\theta = (\eta_1/\xi_1 - \eta_{01}/\xi_{01})/(\eta_1/\xi_1 - \eta_0/\xi_0) ,$$
$$1 - \theta = (\eta_{01}/\xi_{01} - \eta_0/\xi_0)/(\eta_1/\xi_1 - \eta_0/\xi_0) .$$

Since $\xi_0\eta_1 - \xi_1\eta_0 > 0$, (2.15) is equivalent to

$$(\xi_0\eta_1 - \xi_1\eta_0)[\varphi(x_{10}) - z^*]$$
$$\geqq (\xi_{01}\eta_1 - \xi_1\eta_{01})[\varphi(x_0) - z^*] - (\xi_{01}\eta_0 - \xi_0\eta_{01})[\varphi(x_1) - z^*]$$

which, in turn, is equivalent to (2.11) when $z^* > 0$. This completes the proof.

3. The $(n+1)$-point condition.

Let $n \geqq 2$ and $\Omega \subset R^n$ be a bounded open convex set and $\varphi(x)$ a function defined for $x \in \Gamma = \partial\Omega$.

DEFINITION (I). *$(n+1)$-point condition.* $\varphi(x)$ is said to satisfy an $(n+1)$-point condition [with constant K] if, for every set of $n+1$ points $x_0, \cdots, x_n$ of Γ, there is a hyperplane

$$(3.1) \qquad z = a\cdot x + c = \sum_{h=1}^{n} a^h x^h + c$$

in R^{n+1} which passes through the points $(x, z) = (x_j, \varphi(x_j))$ for $j = 0, 1, \cdots, n$ and satisfies

$$(3.2) \qquad ||a|| = (\sum_{k=1}^{n} |a^k|^2)^{1/2} \leqq K .$$

In deciding whether or not φ satisfies an $(n+1)$-point condition, continuity considerations show that it suffices to consider only sets of $n+1$ points $x_0, \cdots, x_n$ of Γ such that $(x, z) = (x_j, \varphi(x_j))$, $j = 0, \cdots, n$, determine a unique hyperplane in R^{n+1}. In particular, if $x_0, \cdots, x_n$ are on an $(n-1)$-dimensional plane $\pi_{n-1} \subset R^n$, then the restriction of $\varphi(x)$ to $\Gamma \cap \pi_{n-1}$ is the restriction of a linear function of x.

THEOREM 3.1. *Let $\Omega \subset R^n$ be a bounded open convex set and $\varphi(x)$ a function defined for $x \in \Gamma = \partial\Omega$. Then $\varphi(x)$ satisfies a* BSC *if and only if φ satisfies an $(n+1)$-point condition.*

In the proof, it will be convenient to have the following auxiliary definition.

DEFINITION (II). Let $n \geqq 2$, $\Omega \subset R^n$ a bounded, open convex set, $\Gamma = \partial\Omega$, $\varphi(x)$ a function on Γ, and $2 \leqq m \leqq n$. The function φ is said to satisfy an $(m+1)$-point condition with constant K if, for every m-dimensional plane $\pi_m \subset R^n$ containing an interior point of Ω, the restriction of $\varphi(x)$ to the boundary of $\Omega \cap \pi_m$ satisfies an $(m+1)$-point condition with a constant K (in the sense of Definition (I) where $n = m$).

The proof of Theorem 3.1 will be given in several steps: (a), Lemma 3.1, (b), (c), (d), in which Ω, Γ, φ are as in Theorem 3.1.

(a) $\varphi(x)$ satisfies an $(n+1)$-point condition if and only if there exists a number N with the property that, for every set of $n+1$ points $x_0, \cdots, x_n$ of Γ, there is a hyperplane (3.1) passing through $(x, z) = (x_j, \varphi(x_j))$ for $j = 0, \cdots, n$ and satisfying

$$|a \cdot x + c| \leqq N \qquad \text{for } x \in \Omega \,. \tag{3.3}$$

In fact, if (3.1) is the hyperplane satisfying (3.1) and (3.2), then, for $x \in \Omega$,

$$|a \cdot x + c| = |a \cdot (x - x_0) + \varphi(x_0)| \leqq K \operatorname{diam} \Omega + \text{const} \,.$$

Conversely, if (3.1) is a hyperplane satisfying (3.3) and $a \neq 0$ then there is a number $c_0 > 0$ (independent of a) and a pair of points $y_0, y_1 \in \Omega$ such that

$$y_0 - y_1 = ta/\|a\| \,, \quad t = c_0 > 0 \,.$$

Thus, from

$$a \cdot (y_0 - y_1) = (a \cdot y_0 + c) - (a \cdot y_1 + c)$$

and (3.3), $|a \cdot (y_0 - y_1)| \leqq 2N$, and so $\|a\| \leqq 2N/c_0$.

LEMMA 3.1. *Let Ω, Γ, φ be as in Theorem 3.1. Let $x_j = (x_j^1, \cdots, x_j^n)$ for $j = 0, 1, \cdots, n$ be $n+1$ points of Γ,*

$$\Delta_0(x_0, \cdots, x_n) = \begin{vmatrix} x_0^1 & \cdots & x_0^n & 1 \\ x_1^1 & \cdots & x_1^n & 1 \\ \cdots & \cdots & \cdots & \cdots \\ x_n^1 & \cdots & x_n^n & 1 \end{vmatrix} \,, \tag{3.4}$$

$$\Delta(x, z) = \begin{vmatrix} x^1 & \cdots & x^n & z & 1 \\ x_0^1 & \cdots & x_0^n & \varphi(x_0) & 1 \\ x_1^1 & \cdots & x_1^n & \varphi(x_1) & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_n^1 & \cdots & x_n^n & \varphi(x_n) & 1 \end{vmatrix} \,. \tag{3.5}$$

Then φ satisfies an $(n+1)$-point condition if and only if there exists a number N such that

$$(3.6)\qquad (-1)^n z\Delta(x, z)\Delta_0(x_0, \cdots, x_n) \geqq 0 \qquad \textit{for } |z| \geqq N\,,$$

$x \in \Omega$, and all sets of $n+1$ points $x_0, \cdots, x_n$ of Γ.

In fact, if there is a unique hyperplane of the form (3.1) passing through $(x_j, \varphi(x_j))$ for $j = 0, 1, \cdots, n$, then

$$\Delta(x, z) = (-1)^n \Delta_0(x_0, \cdots, x_n)[z - (a \cdot x + c)]\,.$$

Thus (3.6) for $|z| \geqq N, x \in \Omega$, is equivalent to (3.3).

(b) Let m, $2 \leqq m \leqq n$, be fixed. If $\varphi(x)$ satisfies an $(n+1)$-point condition with constant K in the sense of Definition (I), then it satisfies an $(m+1)$-point condition with constant K in the sense of Definition (II).

This is clear. Theorem 3.1 and its proof will show that the converse is correct.

(c) BSC $\Rightarrow$ $(n+1)$-point condition.

Let $\varphi(x)$ satisfy a BSC and let $x_0, \cdots, x_n$ be $n+1$ points of Γ such that there is a unique hyperplane passing through the points $(x_j, \varphi(x_j))$ for $j = 0, \cdots, n$. This hyperplane necessarily has an equation of the form (3.1). It will be shown that there exists a number N satisfying (3.3).

Let N be so large that the conical surfaces $C(x^*, z^*)$ in Theorem 2.1 bound open convex sets $\Omega(x^*, z^*)$ for every $x^* \in \Omega$ and $|z^*| \geqq N$. It will be shown that (3.3) holds for the arbitrary (but fixed) point $x = x^* \in \Omega^*$.

Suppose first that x^* is in the convex closure of the set of points $x_0, \cdots, x_n$. Consider a supporting hyperplane $\pi^+ \colon z = a_+ \cdot x + c_+$ of the convex set $\Omega(x^*, N)$ through the boundary point $(x_0, \varphi(x_0)) \in C(x^*, N)$. Then $(x, z) \in C(x^*, N)$ implies that $z \leqq a_+ \cdot x + c$. Hence

$$(3.7)\qquad a \cdot x + c \leqq a_+ \cdot x + c_+$$

holds for $x = x_0, \cdots, x_n$ and hence for all x in the convex closure of the set of point $x_0, \cdots, x_n$. In particular, $a \cdot x^* + c \leqq a_+ \cdot x^* + c_+ = N$. Similarly, $a \cdot x^* + c \geqq -N$.

Consider now the case where x^* is not in the convex closure of the set of points $x_0, \cdots, x_n$. Let B denote the convex closure of x^* and $x_0, \cdots, x_n$, so that B is bounded by a polyhedron. Since $x_0, \cdots, x_n$ are not contained in an $(n-1)$-dimensional plane π_{n-1}, the set $B \subset R^n$ has interior points. Thus there are n edges on the boundary of B terminating at x^*. Suppose that the enumeration of $x_0, \cdots, x_n$ is such that the line segments $[x^* x_j]$, where $j = 1, \cdots, n$, are on the boundary of B. Thus B contains the closed "simplex" B^* with vertices $x^*, x_1, \cdots, x_n$.

Suppose, if possible, that $x_0 \in B^*$. Let π_{n-1} be a supporting $(n-1)$-dimensional plane (in R^n) of Ω through the point x_0. Then the face $x_1, \cdots, x_n$ of B^* is not on π_{n-1} (for otherwise $x_0, \cdots, x_n \in \pi_{n-1}$). Also, $x^* \notin \pi_{n-1}$ since $x^* \in \Omega$. Thus no face of B^* is on π_{n-1} and, since x_0 is not a vertex of B^*, π_{n-1} is not a supporting plane of B^*. Hence π_{n-1} separates at least one pair of vertices of B^*. But this is impossible since π_{n-1} supports Ω. Hence $x_0 \notin B^*$.

Consequently, B is the union of two simplices, B^* with vertices $x^*, x_1, \cdots, x_n$ and B_0 with vertices $x_0, x_1, \cdots, x_n$, with the common face $x_1, \cdots, x_n$. Thus the diagonal $[x_0 x^*]$ of B meets the face $x_1, \cdots, x_n$ of B at some point. Consequently, x^* is in the convex closure of the set of points on the n half-lines $x_0 + t(x_j - x_0)$, where $t \geqq 0$ and $j = 1, \cdots, n$. Let $\pi^+ : z = a_+ \cdot x + c_+$ be a supporting hyperplane of $\Omega(x^*, N)$ through the boundary point $(x_0, \varphi(x_0)) \in C(x^*, N)$. Then (3.7) holds for $x = x_0, \cdots, x_n$, hence on the half-lines $x = x_0 + t(x_j - x_0)$, $t \geqq 0$ and $j = 1, \cdots, n$, and consequently for all points (including $x = x^*$) in the convex closure of the set of points on these half-lines. Thus, as before, $a \cdot x^* + c \leqq a_+ \cdot x^* + c_+ = N$. Similarly $a \cdot x^* + c \geqq -N$. By (a), this proves that φ satisfies an $(n+1)$-point condition.

(d) $(n+1)$-point condition $\Rightarrow$ BSC.

Let $\varphi(x)$ satisfy an $(n+1)$-point condition with a constant K. Then $\varphi(x)$ satisfies a 3-point condition with constant K by (b). Let $\pi_2 \subset R^n$ be a 2-dimensional plane containing an interior point $x^* \in \Omega$, and (ξ, η) rectangular coordinates on π_2. Let (ξ_0, η_0), (ξ_1, η_1), (ξ_{01}, η_{01}) be the (ξ, η)-coordinates of points x_0, x_1, x_{01} of $\Gamma \cap \pi_2$,

$$\delta_0(x_0, x_{01}, x_1) = \begin{vmatrix} \xi_0 & \eta_0 & 1 \\ \xi_{01} & \eta_{01} & 1 \\ \xi_1 & \eta_1 & 1 \end{vmatrix}, \tag{3.8}$$

$$\delta(\xi, \eta, z) = \begin{vmatrix} \xi_0 - \xi & \eta_0 - \eta & \varphi(x_0) - z \\ \xi_{01} - \xi & \eta_{01} - \eta & \varphi(x_{01}) - z \\ \xi_1 - \xi & \eta_1 - \eta & \varphi(x_1) - z \end{vmatrix}. \tag{3.9}$$

Then, by Lemma 3.1, there exists a constant N such that

$$z\delta_0(x_0, x_{01}, x_1)\delta(\xi, \eta, z) \leqq 0 \tag{3.10}$$

for $|z| \geqq N$ and all points $(\xi, \eta) \in \Omega \cap \pi_2$. It follows from Corollary 2.1 that $\varphi(x)$ satisfies a BSC (for if the origin of the (ξ, η)-coordinate system is chosen at $x^* \in \Omega \cap \pi_2$, then (2.11) and (3.10) with $(\xi, \eta) = 0$ are equivalent). This completes the proof.

4. Smoothness of $\varphi(x)$. It will be shown that if φ satisfies a BSC, then its smoothness (in some sense) is similar to that of a convex function.

COROLLARY 4.1. *Let $\Omega \subset R^n$ be a bounded open convex set and $\varphi(x)$ a function on $\Gamma = \partial\Omega$ satisfying a* BSC. *Let π_2 be a 2-dimensional plane in R^n containing an interior point of Ω, $\Gamma_{01} = \pi_2 \cap \Gamma$, and*

$$\Gamma_{01} : x = x(s) \tag{4.1}$$

an arclength parametrization of Γ_{01}. Then

$$\psi(s) = \varphi(x(s)) \tag{4.2}$$

has a derivative $d\psi/ds$ except on a set of s-values which is at most countable.

It turns out that when one imposes additional smoothness conditions on Γ, the required smoothness on a function φ satisfying a BSC is correspondingly increased.

COROLLARY 4.2. *Let $\Omega \subset R^n$ be a bounded open convex set and $\varphi(x)$ a function on $\Gamma = \partial\Omega$ satisfying a* BSC.

(i) *If $\Gamma \in C^1$, then $\varphi(x) \in C^1$.*

(ii) *If $\Gamma \in C^{1,\lambda}$, then $\varphi(x) \in C^{1,\lambda}$.*

A function on an open set $A \subset R^n$ is said to be of class $C^{1,\lambda}$ if it has continuous, first order, partial derivatives which are uniformly Hölder [or Lipschitz] continuous of order λ, $0 < \lambda < 1$ [or $\lambda = 1$] on closed spheres in A. The definition of a hypersurface $\Gamma \subset R^{n+1}$ of class $C^{1,\lambda}$ or of a function $\varphi(x)$ on Γ of class $C^{1,\lambda}$ is analogous.

REMARK. Let $\varphi(x)$ satisfy a BSC and let the conical surfaces $C(x^*, z^*)$, $z^* = \pm N$, have the equations

$$C(x^*, \pm N): \quad z = \tau_\pm(x) \qquad \text{for all } x$$

[cf. the proof of Theorem 2.1]; so that $\tau_\pm(x) = \varphi(x)$ for $x \in \Gamma$. Then, in case (i) of Corollary 4.2, $\tau_\pm(x)$ has continuous partial derivatives except at $x = x^*$; in case (ii), these partial derivatives are uniformly Hölder continuous of order λ on compacts not containing $x = x^*$. Thus suitable modifications of $\tau_\pm(x)$ near $x = x^*$ give functions on R^n which are respectively of class $C^1, C^{1,\lambda}$ and which are identical with φ on Γ.

The arguments in [5] show that if Ω is uniformly convex (whether or not $\Gamma \in C^{1,1}$) and if φ is the restriction to Γ of a function on R^n of class $C^{1,1}$, then φ satisfies a BSC; cf. [8, 625–628] and [2], where

Γ and φ are of class C^2. Conversely, if $\Gamma \in C^{1,1}$ (whether or not Ω is uniformly convex), then, by Corollary 4.2 (ii), a necessary condition for φ to satisfy a BSC in that $\varphi \in C^{1,1}$. Thus we have

COROLLARY 4.3. *Let $\Omega \in R^n$ be a bounded, open set with a uniformly convex boundary $\Gamma = \partial\Omega$ of class $C^{1,1}$. Then a necessary and sufficient condition for a function $\varphi(x)$, $x \in \Gamma$, to satisfy a* BSC *is that $\varphi(x) \in C^{1,1}$.*

Γ is called *uniformly convex* if there is a constant $c > 0$ such that through every $x_0 \in \Gamma$, there is a hyperplane $\pi_{n-1} \subset R^n$ satisfying *dist* $(x, \pi_{n-1}) \geqq c \| x - x_0 \|^2$ for $x \in \Gamma$.

The "sufficiency" does not hold if Ω is not uniformly convex, but is only strictly convex. For example, let $n = 2$ and let the "lower" portion of Γ be on the curve $x^2 = (x^1)^4$ near the origin and let $\varphi(x) = (x^1)^2$ for $(x^1, x^2) \in \Gamma$. Then there is no choice of constants a^1, a^2 such that $a^1x^1 + a^2x^2 \geqq (x^1)^2$ for small $|x^1|$ and $(x^1)^2 = |x^2|^{1/2} \geqq 0$.

The following remark will not be used below but it may be of interest to note that if $\Gamma \in C^2$, $\varphi \in C^2$ and if Γ_{01}: $\xi = \xi(s)$, $\eta = \eta(s)$ and $\psi(s)$ are is in the proof of Corollary 4.1 below, then condition (2.11) is equivalent to

$$z^* \begin{vmatrix} \psi(s) - z^* & \xi(s) & \eta(s) \\ \psi'(s) & \xi'(s) & \eta'(s) \\ \psi''(s) & \xi''(s) & \eta''(s) \end{vmatrix} \leqq 0$$

for all 2-dimensional plane sections Γ_{01} of Γ. This fact makes it clear, for example, that if $\Gamma \in C^2$, $\varphi \in C^2$ and Ω is uniformly convex, than φ satisfies a BSC.

Proof of Corollary 4.1. Choose a coordinate system in R^n such that π_2 is the plane $x^3 = \cdots = x^n = 0$ and with the origin at a point x^* in $\pi_2 \cap \Omega$. Write (ξ, η) in place of (x^1, x^2). Let $\xi = \xi(s)$, $\eta = \eta(s)$, where $0 \leqq s \leqq s_0$, be an arclength parametrization of Γ_{01}.

Choose an s-interval, say $0 \leqq s \leqq \alpha < s_0$, such that the radius vector [the line from the origin to $(\xi(s), \eta(s))$] moves through an angle less than π as s varies from 0 to α. Then, if ξ, η is a pair of arbitrary numbers and $0 \leqq s_1 < s_2 \leqq \alpha$, the linear equations

$$c_1\xi(s_1) + c_2\eta(s_1) = \xi \ , \quad c_1\xi(s_2) + c_2\eta(s_2) = \eta$$

have a unique solution for c_1, c_2. In the terminology of Beckenbach [1], this means that the linear family F of functions $c_1\xi(s) + c_2\eta(s)$ is a 2-parameter family on the interval $0 \leqq s \leqq \alpha$.

Let $\psi(s)$ be defined by (4.2), i.e.,

$$\psi(s) = \varphi(\xi(s), \eta(s), 0, \cdots, 0) .$$

Then (2.11) implies that $\psi(s) - z^*$ is F-concave and $\psi(s) + z^*$ is F-convex, where $z^* = N > 0$. In other words, if

$$f_{\pm}(s) = c_{1\pm}\xi(s) + c_{2\pm}\eta(s) \tag{4.3}$$

is a linear combination of $\xi(s), \eta(s)$ such that $f_{\pm}(s) = \psi(s) \mp z^*$ at $s = s_1, s_2$ where $0 \leqq s_1 < s_2 \leqq \alpha$, then $f_+(s) \geqq \psi(s) - z^*, f_-(s) \leqq \psi(s) + z^*$ for $s_1 \leqq s \leqq s_2$; cf. the proof of Corollary 4.2, $n > 2$, below.

Let $0 < s_0 < \alpha$. By [6], there exist elements (4.3) of F which support $\psi(s) \mp z^*$ in the sense that

$$f_{\pm}(s_0) = \psi(s_0) \mp z^* , \tag{4.4}$$

$$f_-(s) - z^* \leqq \psi(s) \leqq f_+(s) + z^* \quad \text{for } 0 \leqq s \leqq \alpha ; \tag{4.5}$$

see also [3] for generalizations and references to Bonsall, J. W. Green and Reid.

Since Γ_{01} is a (plane) convex curve, $\xi(s)$ and $\eta(s)$ are differentiable except possibly on a countable set of s-values. Choose $s = s_0$ so that $\xi' = d\xi/ds$, $\eta' = d\eta/ds$ exist at $s = s_0$. Note that

$$[f_+(s) + z^*] - [f_-(s) - z^*]$$

is nonnegative for $0 \leqq s \leqq \alpha$ and vanishes at $s = s_0$. Hence $f'_+(s_0) = f'_-(s_0)$. Consequently (4.5) implies that $\psi'(s_0)$ exists (and is $f'_+(s_0) = f'_-(s_0)$). This proves Corollary 4.1.

Proof of Corollary 4.2, $n = 2$. It is clear from the proof of Corollary 4.1 that if $\Gamma \in C^1$, then $\psi'(s)$ exists for all s. It is also clear that the coefficients $c_{1\pm}, c_{2\pm}$ in (4.3)-(4.5) are determined by the linear equations

$$c_{1\pm}\xi (s_0) + c_{2\pm}\eta (s_0) = \psi (s_0) \mp z^* ,$$
$$c_{1\pm}\xi'(s_0) + c_{2\pm}\eta'(s_0) = \psi'(s_0) .$$

The determinant $\xi(s_0)\eta'(s_0) - \xi'(s_0)\eta(s_0)$ is bounded away from zero for $0 \leqq s_0 \leqq \alpha$. Also $\psi(s_0) \mp z^*, \psi'(s_0)$ are bounded (in fact, the boundedness of $\psi'(s_0)$ follows from the fact that $\varphi(x)$ is uniformly Lipschitz continuous). Thus there exists a constant M such that the functions $|c_{1\pm}|, |c_{2\pm}|$ of s_0 are majorized by M.

For $\delta > 0$, let

$$\omega(\delta) = \sup (|\xi'(s) - \xi'(s_1)| + |\eta'(s) - \eta'(s_1)|)$$

for $|s - s_1| \leqq \delta$, $0 \leqq s < s_1 \leqq \alpha$. Thus

$$|f'_{\pm}(s) - f'_{\pm}(s_0)| \leqq M\omega(|s - s_0|)\ .$$

Consequently, by (4.5),

$$|\psi(s) - \psi(s_0) - \psi'(s_0)(s - s_0)| \leqq M\omega(|s - s_0|)\,|s - s_0|\ .$$

Interchanging s, s_0 and adding gives

$$|\psi'(s) - \psi'(s_0)| \leqq 2M\omega(|s - s_0|)\ .$$

This proves Corollary 4.2 if $n = 2$.

Proof of Corollary 4.2, $n > 2$. When $n > 2$, it is necessary to estimate the degree of continuity of the directional derivatives of φ not only in the direction of the derivative but also in directions orthogonal to it.

It suffices to deal with $\varphi(x)$ in a neighborhood of a given point of Γ. Choose a coordinate system in R^n with the origin at such a point and such that Ω is in the half-space $x^n \geqq 0$. Then, in the neighborhood of the origin, Γ has a parametrization of the form

$$(4.6) \qquad x^n = \zeta(x^1, \cdots, x^{n-1}) \quad \text{for} \quad |x^j| \leqq \varepsilon,\ j = 1, \cdots, n-1\ ,$$

where ζ is of class C^1 or $C^{1,\lambda}$ in case (i) or (ii).

Write ξ for $(\xi^1, \cdots, \xi^{n-1})$, where $\xi^i = x^i$ for $i = 1, \cdots, n-1$ and $\psi(\xi) = \varphi(x) = \varphi(\xi, \zeta(\xi))$ for $x \in \Gamma$. It has to be shown that ψ is correspondingly of class C^1 or $C^{1,\lambda}$. The proof of Corollary 4.1 shows that $\psi_i = \partial\psi/\partial\xi^i$, $i = 1, \cdots, n-1$, exist at every point.

Let $\gamma > 0$ be chosen so that

$$(4.7) \qquad x^* = (0, \cdots, 0, \gamma) \in \Omega\ .$$

Let $x_j = (\xi_j, \zeta(\xi_j)) = (x_j^1, \cdots, x_j^n)$, where $j = 0, 1, \cdots, n$, be $n+1$ points of Γ. The analogue of (3.4) is

$$(4.8) \qquad \Delta_0(x_0, \cdots, x_n) = (-1)^n \begin{vmatrix} x_1^1 - x_0^1 & \cdots & x_1^n - x_0^n \\ x_2^1 - x_0^1 & \cdots & x_2^n - x_0^n \\ \cdots\cdots & \cdots & \cdots\cdots \\ x_n^1 - x_0^1 & \cdots & x_n^n - x_0^n \end{vmatrix}$$

and that of (3.5) is

$$(4.9) \qquad \Delta(x^*, z) = (-1)^{n+1} \begin{vmatrix} x_0^1 & x_0^2 & \cdots & x_0^{n-1} & x_0^n - \gamma & \tau \\ x_1^1 & x_1^2 & \cdots & x_1^{n-1} & x_1^n - \gamma & \varphi(x_1) - z \\ \cdot\cdot & \cdot\cdot & \cdots & \cdots & \cdots & \cdots\cdots \\ x_n^1 & x_n^2 & \cdots & x_n^{n-1} & x_n^n - \gamma & \varphi(x_n) - z \end{vmatrix},$$

where $\tau = \varphi(x_0) - z$ and $x^* = (0, \cdots, 0, \gamma)$. Thus (3.6) holds for $z =$

$\pm N$, $|x_j^k| \leqq \varepsilon$, and $j, k = 0, 1, \cdots, n-1$.

In (4.8) and (4.9), consider $\gamma, z, x_1, \cdots, x_n$ as fixed for the moment and x_0 (or rather $\xi_0 = (\xi_0^1, \cdots, \xi_0^{n-1})$) as variable. In (4.9), let τ be replaced by

$$f_\pm(\xi_0) = \sum_{k=1}^{n-1} c_{k\pm}\xi_0^k + c_{n\pm}[\zeta(\xi_0) - \gamma] , \tag{4.10}$$

where $c_{1\pm}, \cdots, c_{n\pm}$ are chosen, if possible, so that

$$f_\pm(\xi_j) = \varphi(x_j) - z, \; z = \pm N, \; j = 1, \cdots, n . \tag{4.11}$$

Then the analogue of the determinant (4.9) vanishes and so, $\Delta(x^*, z)$ is not changed if the last column is replaced by $\tau - f_\pm(\xi_0), 0, \cdots, 0$. In this case, we conclude from (3.6) that $\tau = \varphi(x_0) \mp N$ satisfies

$$\mp N\Delta_0(x_0, x_1, \cdots, x_n)\Delta_0(x^*, x_1, \cdots, x_n)[(\varphi(x_0) \mp N) - f_\pm(\xi_0)] \geqq 0 .$$

Thus according as

$$\Delta_0(x_0, x_1, \cdots, x_n)\Delta_0(x^*, x_1, \cdots, x_n) > 0 \text{ or } <0 , \tag{4.12}$$

we have

$$\varphi(x_0) - N \leqq f_+(\xi_0) \quad \text{or} \quad \varphi(x_0) - N \geqq f_+(\xi_0) \tag{4.13}$$

and

$$\varphi(x_0) + N \geqq f_-(\xi_0) \quad \text{or} \quad \varphi(x_0) + N \leqq f_-(\xi_0) . \tag{4.14}$$

If the points $x_1, \cdots, x_n$ are not in an $(n-2)$-dimensional plane π_{n-2}, then $\Delta_0(x^*, x_1, \cdots, x_n) \neq 0$. It will be supposed that $x_1, \cdots, x_n$ are enumerated so that

$$\Delta_0(x^*, x_1, \cdots, x_n) > 0 . \tag{4.15}$$

Then the coefficients $c_{k\pm}$ of (4.10) can be uniquely determined so that (4.11) holds. The alternative (4.12) is now equivalent to

$$\Delta_0(x_0, x_1, \cdots, x_n) > 0 \quad \text{or} \quad <0 . \tag{4.16}$$

This, in turn, is equivalent to

(4.17) the line segment $[x^*x_0]$ does not or does meet the π_{n-1} determined by $x_1, \cdots, x_n$; i.e., x_0 is or is not on the same side of π_{n-1} as x^*.

Let $x_1 = (\xi_1, \zeta(\xi_1))$ be fixed and $h > 0$ small. Choose $\xi_j = he_{j-1} + \xi_1$, for $j = 2, \cdots, n$, where $e_j = (0, \cdots, 0, 1, 0, \cdots, 0)$ and the 1 is in the j-th place. Correspondingly, $x_j = (\xi_j, \zeta(\xi_j))$. Then (4.15) holds (e.g., if $x_1 = 0$, then $\Delta(x^*, x_1, \cdots, x_n)$ reduces to $\gamma h^{n-1} > 0$). The equations

(4.11) for $c_{k\pm}$ are equivalent to

$$(4.18)\qquad \begin{gathered}\sum_{k=1}^{n-1} c_{k\pm}\xi_1^k + c_{n\pm}[\zeta(\xi_1) - \gamma] = \psi(\xi_1) \mp N\,,\\ c_{j\pm}h + c_{n\pm}[\zeta(\xi_{j+1}) - \zeta(\xi_1)] = \psi(\xi_{j+1}) - \psi(\xi_1)\,,\end{gathered}$$

for $j = 1, \cdots, n-1$. For these choices of $c_{k\pm}$, the first inequality in both (4.13), (4.14) hold if the segment $[x^*x_0)$ does not meet π_{n-1} containing $x_1, \cdots, x_n$.

If the last $n-1$ equations of (4.18) are divided by h and $h \to 0$, it follows that the solutions $c_{1\pm}, \cdots, c_{n\pm}$ of (4.18) tend to the unique solutions of the equations

$$(4.19)\qquad \begin{gathered}\sum_{k=1}^{n-1} c_{k1}\xi_1^k + c_{n\pm}[\zeta(\xi_1) - \gamma] = \psi(\xi_1) \mp N\,,\\ c_{j\pm} + c_{n\pm}\zeta_j(\xi_1) = \psi_j(\xi_1)\,,\end{gathered}$$

for $j = 1, \cdots, n-1$, where $\zeta_j = \partial\zeta/\partial\xi^j$. Also, the π_{n-1} containing $x_1, \cdots, x_n$ tends to the tangent plane of Γ at x_1.

Thus, if $c_{1\pm}, \cdots, c_{n\pm}$ are chosen as the solution of (4.19), then, in addition, to (4.19),

$$(4.20)\qquad f_-(\xi) - N \leqq \psi(\xi) \leqq f_+(\xi) + N$$

for all ξ, $|\xi^j| \leqq \varepsilon$. Actually, one first obtains (4.20) for all ξ such that $x = (\xi, \zeta(\xi))$ is not on the tangent plane π_{n-1} to Γ at x_1. By continuity considerations, (4.20) holds also for the limits of such points. On the other hand, if $\pi_{n-1} \cap \Gamma$ contains interior points, then $\varphi(x)$ is a linear function of $x \in \pi_{n-1} \cap \Gamma$ and (4.20) is trivial for those ξ for which $x = (\xi, \zeta(\xi)) \in \pi_{n-1} \cap \Gamma$.

Let $\omega(\delta)$ be a monotone majorant for the degree of continuity of $\zeta_j = \partial\zeta/\partial\xi_j$, $j = 1, \cdots, n-1$. Then arguing as at the end of the proof of the case $n = 2$, it is seen that there is a constant M such that the degree of continuity of the partial derivatives of $f_\pm(\xi) - N$ is majorized by $M\omega(\delta)$. (For in the matrix of coefficients of (4.19), the first row is the vector $x_1 - x^*$ from the point x^* to the point $x_1 \in \Gamma$, the second row is the vector $(1, 0, \cdots, 0, \zeta_1(\xi_1))$ which is a tangent vector to Γ at x_1, etc., so that the determinant of this matrix is bounded away from zero). Thus,

$$(4.21)\qquad \left|\psi(\xi) - \psi(\xi_1) - \sum_{k=1}^{n-1} \psi_k(\xi_1)(\xi^k - \xi_1^k)\right| \leqq M\omega(\delta)\delta\,,$$

where

$$(4.22)\qquad \delta = \max\left(|\xi^1 - \xi_1^1|, |\xi^2 - \xi_1^2|, \cdots, |\xi^{n-1} - \xi_1^{n-1}|\right).$$

As in the proof in the case $n = 2$, this implies that

$$\left|\sum_{k=1}^{n-1}[\psi_k(\xi)-\psi_k(\xi_1)](\xi^k-\xi_1^k)\right| \leqq 2M\omega(\delta)\delta\ .$$

In particular,

$$|\psi_k(\xi)-\psi_k(\xi+\delta e_k)| \leqq 2M\omega(\delta)\ . \tag{4.23}$$

The relations (4.19), (4.20) show that

$$|[f_+(\xi)+N]-[f_-(\xi)-N]| \leqq 2M\omega(\delta)\delta\ . \tag{4.24}$$

Let $k \neq j$ and let

$$\xi_1,\ \xi_2=\xi_1+\delta e_j,\ \ \xi_3=\xi_1+\delta e_j+\delta e_k,\ \ \xi_4=\xi_1+\delta e_k$$

be the vertices of a square. By (4.20), the quantity

$$\psi(\xi_1)+\psi(\xi_3)-\psi(\xi_2)-\psi(\xi_4)=\sum_{m=1}^{4}(-1)^{m+1}\psi(\xi_m)$$

is bounded from above by

$$[f_+(\xi_1)+N]+[f_+(\xi_3)+N]-[f_-(\xi_2)-N]-[f_-(\xi_4)-N]$$

and there is an analogous bound from below. Hence, by (4.24),

$$\left|\sum_{m=1}^{4}(-1)^{m+1}\psi(\xi_m)\right| \leqq \left|\sum_{m=1}^{4}(-1)^{m+1}[f_+(\xi_m)+N]\right| + 4M\omega(2\delta)\delta\ .$$

Since (4.23) implies that

$$|\psi(\xi_4)-\psi(\xi_1)-\delta\psi_k(\xi_1)| \leqq 2M\omega(\delta)\delta$$
$$|\psi(\xi_3)-\psi(\xi_1)-\delta\psi_k(\xi_2)| \leqq 2M\omega(\delta)\delta$$

and similar relations hold if ψ is replaced by f_+, it follows that

$$\delta\,|\psi_k(\xi_1)-\psi_k(\xi_2)| \leqq \delta\,|f_{+k}(\xi_1)-f_{+k}(\xi_2)| + 16M\omega(2\delta)\delta\ .$$

Consequently

$$|\psi_k(\xi_1)-\psi_k(\xi_1+\delta e_j)| \leqq 18M\omega(2\delta) \qquad k\neq j\ .$$

This, together with (4.23), proves Corollary 4.2 for $n>2$.

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