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EVERYWHERE DEFINED LINEAR TRANSFORMATIONS AFFILIATED WITH RINGS OF OPERATORS

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Let M be a ring of operators on a Hilbert space H. This paper considers conditions under which an operator T affiliated with M is bounded (or can be unbounded). In particular, we consider operators whose domain is the entire space H. We prove: THEOREM 3. If M has no type I factor part, then T is bounded. THEOREM 4. T is bounded if and only if T is bounded on each minimal projection in M. THEOREM 6. In order that every linear mapping from H into H which commutes with M be bounded, it is necessary and sufficient that M should contain no minimal projection whose range is an infinite dimensional subspace of H. These results were suggested by a theorem of J. R. Ringrose: THEOREM 8. If $M = M'$ then T is bounded.

In a paper on triangular algebras ([4], Lemma 2.12) J. R. Ringrose encountered the following situation: he was given a linear operator T with domain equal to an entire Hilbert space H and a ring of operators M commuting with T. In the case $M = M'$ (M maximal abelian) he was able to show that T had to be bounded. (For the relevant background theory, see [1, 2].) The purpose of this paper is to consider other types of rings of operators commuting with T and conditions under which T can be unbounded.

2. Since the projections in M commute with T, the ranges of these projections are invariant under T; and consequently operators are induced thereby on such subspaces. We begin by considering orthogonal families of such operators.

LEMMA 1. *If $\{E_\gamma \mid \gamma \in \Gamma\}$ is an orthogonal family of projections in M, then the norms $\{\| TE_\gamma \| \mid \gamma \in \Gamma\}$ are almost uniformly bounded; that is, there exists a finite subset Γ_0 of Γ and a positive number b such that $\| TE_\gamma \| \leqq b$ for $\gamma \in \Gamma - \Gamma_0$.*

Proof. Assume lemma false. We first choose a E_{γ_1} such that $\| TE_{\gamma_1} \| > 1$. (If $\| TE_\gamma \| \leqq 1$ for all $\gamma \in \Gamma$; then $\Gamma_0 =$ null set, $b = 1$ fulfills the lemma.) Now assume for a positive integer n that $\{E_{\gamma_k} \mid k = 1, 2, 3, \cdots, n\}$ have been chosen so that $\| TE_{\gamma_k} \| > k$ for each k. If $\| TE_\gamma \| \leqq n+1$ for $\gamma \in \Gamma - \{\gamma_k \mid k = 1, 2, 3, \cdots, n)$, then $b = n + 1$ leads to the conclusion of the lemma. Thus we can pick a γ_{n+1} such that $\| TE_{\gamma_{n+1}} \| > n+1$. Finally, our induction produces a sequence E_{γ_k} with $\| TE_{\gamma_k} \| > k$ for all integers k.

Next, select x_k in E_kH such that $\| x_k \| = 1$ and $\| Tx_k \| = \| TE_{\gamma_k} x_k \| \geqq k$ for all k, using $\| TE_k \| > k$. Let $y_n = \sum_{k=1}^{n} x_k/k$. Now, since

$$\| y_n \|^2 = \sum_{k=1}^{n} \frac{\| x_k \|^2}{k^2} = \sum_{k=1}^{n} \frac{1}{k^2} < \infty .$$

y_n converges to a vector y in H. Clearly $y_n \in (\sum_{k=1}^{n} E_{\gamma_k})H$ and $y - y_n \in (\sum_{k>n} E_{\gamma_k})H$. Since these two subspaces are orthogonal and invariant under T, Ty_n is orthogonal to $T(y - y_n)$ for each integer n. But, by Bessel's inequality

$$\| Ty \|^2 \geqq \| Ty_n \|^2 = \left\| \sum_{k=1}^{n} \frac{TE_{\gamma_k} x_k}{k} \right\|^2 = \sum_{k=1}^{n} \frac{\| TE_{\gamma_k} x_k \|^2}{k^2} \geqq \sum_{k=1}^{n} \frac{k^2}{k^2} = n .$$

The contradiction $\| Ty \|^2 \geqq n$ for all n completes the proof of the lemma.

COROLLARY 1. *If all the* TE_γ *are bounded, then the set* $\{\| TE_\gamma \| \mid \gamma \in \Gamma\}$ *is uniformly bounded.*

COROLLARY 2. *If* Γ *is infinite, at least one of the* TE_γ *is bounded.*

LEMMA 2. *If the* $\{E_\gamma \mid \gamma \in \Gamma\}$ *of Lemma* 1 *is such that the* $\{\| TE_\gamma \| \mid \gamma \in \Gamma\}$ *are uniformly bounded by the number* $b > 0$, *then* $\| T(\Sigma_\gamma E_\gamma) \| \leqq b$.

Proof. For x in H, $\| T(\Sigma_\gamma E_\gamma)x \|^2 = \| (\Sigma_\gamma E_\gamma)Tx \|^2 = \Sigma_\gamma \| E_\gamma Tx \|^2 = \Sigma_\gamma \| (TE_\gamma)E_\gamma x \|^2 \leqq \Sigma_\gamma b^2 \| E_\gamma x \|^2 \leqq b^2 \| x \|^2$.

LEMMA 3. *If* E, F *are projections which are equivalent relative to* M; *then* $\| TE \| = \| TF \|$.

Proof. Let V be a partial isometry in M with initial domain EH and terminal domain FH. (or $V^*V = E$, $VV^* = F$.) Now $V(TE)V^* = T\,VEV^* = T\,VV^*VV^* = TF^2 = TF$ and $\| TF \| = \| V(TE)V^* \| \leqq \| V \| \cdot \| TE \| \cdot \| V^* \| \leqq \| TE \|$. Interchanging $V - V^*$ yields $\| TE \| \leqq \| TF \|$ and completes the proof.

3. DEFINITION 1. Let T be a everywhere defined operator on a Hilbert space. T is said to be *totally unbounded* with respect to the ring of operators M if T commutes with the elements of M and TE is *unbounded* for each nonzero projection E in M.

THEOREM 1. *Let T be a everywhere defined operator on H commuting with a ring of operators M. If T is unbounded then there exists a central projection P in M such that PT is bounded and $P^{\perp}T$ is totally unbounded with respect to $MP^{\perp}$.*

Proof. If T totally unbounded with respect to M, then the projection 0 qualifies as our P. Thus we may restrict ourselves to the cases in which nonzero projections E exist in M such that TE is bounded.

Choose a maximal collection $\{E_\gamma \mid \gamma \in \Gamma\}$ of nonzero orthogonal projections of M with each TE_γ bounded. It then follows from Corollary 1 to Lemma 1 that the $\{\| TE_\gamma \| \mid \gamma \in \Gamma\}$ are uniformly bounded. Then, Lemma 2 shows that T is bounded on $\Sigma_{\gamma \in \Gamma} E_\gamma$.

Now, let $P = \Sigma_{\gamma \in \Gamma} E_\gamma$. Obviously TP is bounded; and if TE is bounded for $0 \neq E \leqq P^{\perp}$, $E \in M$, then E could be added to our maximal collection $\{E_\gamma \mid \gamma \in \Gamma\}$. Thus, it only remains to prove that P is central.

Let Q be the central cover of P in M. If $Q - P \neq 0$, apply the projection comparison lemma ([1], p. 227, Lemma 1) to $(Q - P, P)$ to get two nonzero projections E, F in M such that $E \leqq P$, $F \leqq Q - P$ and $E \sim F$. But Lemma 3 shows that $\| TF \| = \| TE \| < \infty$ contradicting the first part of the proof of this lemma. Thus $Q - P = 0$ and P is central in M.

4. It is clear from Theorem 1 that the problem of classifying everywhere defined unbounded operators commuting with rings of operators can be reduced to the study of totally unbounded operators. Thus we consider the following theorem.

THEOREM 2. *If T is totally unbounded with respect to the ring M, then M is a finite direct sum of finite factors of type I. In each factor direct summand, M' is an infinite factor of type I.*

Proof. Let E be any nonzero projection in M and assume that E does not contain a minimal projection of M. If so, then there exists a nonzero projection E_1 in M such that $E_1 \lneqq E$ and E_1 not minimal. Similarly an $E_2(\neq 0)$ in M with $E_2 \lneqq E_1$ and E_2 not minimal exists. Continuing in this fashion by induction we obtain a decreasing sequence of projections $\{E_k \mid k = 1, 2, 3, \cdots\}$ in M. But now $\{F_k = E_k - E_{k+1} \mid k = 1, \cdots\}$ is an infinite set of orthogonal projections in M and Corollary 2 to Lemma 1 yields a projection F_s on which TF_s is bounded-contradicting total unboundedness. Thus each nonzero projection in M contains a minimal projection.

Now pick a maximal family $\{F_\gamma \mid \gamma \in \Gamma\}$ of orthogonal minimal projections in M. Clearly, Γ is finite. If $\Sigma_{\gamma\in\Gamma} F_\gamma = P \neq I$, then $P^\perp$ contains a minimal projection F orthogonal to the F_γ contradicting maximality of $\{F_\gamma \mid \gamma \in \Gamma\}$.

Thus $I = \Sigma_{\gamma\in\Gamma} F_\gamma = \Sigma_{k\in_K} P_\kappa$, where the P_k are the central projections obtained by adding up the (finite) families of equivalent minimal projections in Γ. It is clear that $M = \Sigma_k MP_k$ is a direct sum decomposition of M into a *finite* number of *finite factors* of type I. It is also clear that $M' = \Sigma_k M'P_k$ is a direct sum decomposition of M' into factors of type I. Further, if one of the $M'P_k$ is of finite type, the fact that MP_k is a finite factor of type I leads to P_kH being finite dimensional in H and to TP_k being necessarily bounded-contradicting total unboundedness of T. Hence each $M'P_k$ is of infinite type I.

The next theorems are corollaries of Theorems 1 and 2. In each, T is everywhere defined and commutes with a ring of operators M.

THEOREM 3. *If M has no type I factor part, then T is bounded.*

THEOREM 4. *T is bounded if and only if it is bounded on each minimal projection in M.*

THEOREM 5. *If the coupling operator for the pair of rings M', M is essentially bounded then T is bounded. In fact, $M' = \{T\colon H$ into $H \mid T$ linear and commutes with $M\}$. (see* [2], p. 497, Def. 3.2).

THEOREM 6. *In order that every linear mapping from H into H which commutes with M be bounded, it is necessary and sufficient that the finite central part of M should contain no minimal projection whose range is an infinite subspace of H.*

Proof. Sufficiency is clear. In case a minimal projection E has infinite dimension in H, let an unbounded operator with domain equal to EH be selected leaving EH invariant; and extend it to an orthogonal family of minimal projections (whose union is the central cover of E) by means of partial isometries. On parts orthogonal to the central cover define the mapping to be zero. If T is this operator, it is clear that T is unbounded, everywhere defined, and commutes with M — thus contradicting our hypothesis.

THEOREM 7. *If M' is finite, then T is bounded.*

THEOREM 8. (*See Ringrose* [4], *Lemma* 2.12) *If $M = M'$, T is bounded.* (*This is a corollary to Theorems* 5 *and* 7.)

5. We now consider the *well-known theorem*: If T is an every-

where defined linear operator on H, then T is *bounded* if and only if T is *closed*. This is usually deduced from the closed graph theorem, but we shall give here a proof along the lines of the first sections of this paper.

By a theorem of von Neumann ([3]) a closed operator with dense domain $\mathscr{D}_T$ has a polar decomposition VS with $S \geqq 0$, $\mathscr{D}_S = \mathscr{D}_T$, $\| V \| \leqq 1$, so that it suffices to restrict ourselves to the case $T \geqq 0$. We assume T unbounded.

Since T is now self-adjoint, we apply the spectral theorem to obtain a sequence of orthogonal projections $\{E_k \in M \mid k = 1, 2, \cdots\}$ ($M =$ ring generated by spectral family of T) with $\| TE_k \|$ unbounded. But now the reasoning of Lemma 1 proves that T cannot be defined on all of H.

References

1. J. Dixmier, *Les algèbres d'opérateurs dans l'espace hilbertien*, Paris, 1957.
2. E. L. Griffin, *Some contributions to the theory of rings of operators*, Trans. Amer. Math. Soc. **75** (1953), 471-504.
3. J. von Neumann, *Über adjungierte Funktionaloperatoren*, Ann. of Math, **33** (1932), 294-310.
4. J. R. Ringrose, *On some algebras of operators II* (unpublished).

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