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CONDITIONS IMPLYING NORMALITY IN HILBERT SPACE

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The problem with which this paper is concerned is that of finding new conditions which imply the normality of an operator on a complete inner product space S. Each such condition, presented in this paper, involves the commutativity of certain operators, associated with a given operator A. Theorem 1 states the equivalence of the following conditions: (i) A is normal, (ii) each of AA^* and A^*A commutes with Re A, (iii) AA^* commutes with Re A and A^*A commutes with Im A. Theorem 2 states that A is normal if AA^* and A^*A commute and Re A is nonnegative definite. Finally, Theorem 3 states that if AA^* commutes with each of A^*A and Re A, then AA^* commutes with A. In this case, if A is reversible, then A is normal.

The notation and terminology used will be as follows. S is a complex, linear space and Q is an inner product for S, such that S is complete with respect to the norm N, induced by Q. T is the space of linear operators on S to S, continuous with respect to N. If A is in T, A^* is the adjoint of A with respect to Q, Re $A = (A + A^*)/2$, and Im $A = (A - A^*)/2i$. An element A of T is nonnegative definite if $Q(Ax, x) \geqq 0$ for each x in S, Hermitian if $A = A^*$, normal if $AA^* = A^*A$, reversible if A is one-to-one, and invertible if A is one-to-one and onto.

The following special notation will be used throughout the paper. Let $B^2 = AA^*$ and $C^2 = A^*A$, where B and C are nonnegative definite. b and c will denote the spectral resolutions of B^2 and C^2, respectively (1, pp. 114–116). These spectral resolutions will be taken to be continuous from the right at each point.

One can see from the following example that relatively strong hypotheses on operators associated with A are necessary in order that A be normal. Let A be the operator on l_2, defined by $A = \{a_{i,j}\}_{i,j=1}^{\infty}$ where $a_{i,i+1} = 1$ and $a_{i,j} = 0$ for $j \neq i + 1$. Then $B = 1$ and $C = P$, where P is a certain projection not equal to 1 or 0. Since $B = 1$, then B commutes with C, Re A, Im A, and even with A itself. However, A is not normal.

2. Commutativity relations concerning B and C.

THEOREM 1. *The following are equivalent*:

(i) *A is normal*,

(ii) *each of* B *and* C *commutes with* Re A,
(iii) B *commutes with* Re A *and* C *commutes with* Im A.

Proof. That (i) implies (ii) and (iii) is obvious. Let $H = \mathrm{Re}\, A$ and $K = \mathrm{Im}\, A$.

(ii) $\Rightarrow$ (i). If $HB^2 = B^2H$ and $HC^2 = C^2H$, then one has

$$A(B^2 - C^2) = (B^2 - C^2)A^* \tag{1}$$

$$\text{and} \quad A^*(B^2 - C^2) = (B^2 - C^2)A \, . \tag{2}$$

Multiplying (1) on the left by A^* and using (2), one finds that

$$C^2(B^2 - C^2) = (B^2 - C^2)B^2 \, . \tag{3}$$

Multiplying (2) on the left by A and using (1), one has

$$B^2(B^2 - C^2) = (B^2 - C^2)C^2 \, . \tag{4}$$

Subtracting (4) from (3), one sees that $(B^2 - C^2)^2 = -(B^2 - C^2)^2$. Therefore, $B^2 = C^2$, and A is normal.

(iii) $\Rightarrow$ (i). If $KC^2 = C^2K$, then

$$(B^2 - C^2)A = -A^*(B^2 - C^2) \, . \tag{5}$$

Multiplying (5) on the left by A and using (1), one has $-B^2(B^2 - C^2) = (B^2 - C^2)C^2$. Therefore, $B^4 = C^4$, and $B^2 = C^2$ (2, p. 262).

LEMMA 2.1. (i) $A\left[\int f(t)dc\right] = \left[\int f(t)db\right]A$, *for each continuous complex-valued function on the real line.*

(ii) $AC^n = B^nA$, *for each positive integer* n.
(iii) $Ac(t) = b(t)A$, *for each value of* t.

Proof. (i) By definition of B^2 and C^2, $AC^{2n} = B^{2n}A$ for each positive integer n. Therefore, $A\left[\int t^n \, dc\right] = \left[\int t^n \, db\right]A$ for each non-negative integer n. The desired result follows by use of the Weierstrass approximation theorem. (ii) and (iii) are both special cases of (i).

THEOREM 2. *If* $BC = CB$ *and* Re A *is nonnegative definite, then* A *is normal.*

Proof. Let t be a real number and let $H = \mathrm{Re}\, A$ and $K = \mathrm{Im}\, A$. Define $k(t) = [1 - c(t)]\, Ac(t)$ and $n(t) = c(t)A[1 - c(t)]$. Then, using Lemma 2.1, one finds that $Ak(t)^*(S) \subset k(t)\,(s)$ and $An(t)^*(S) \subset n(t)\,(S)$. Since $k(t)^2 = 0$ and $n(t)^2 = 0$, $k(t)Ak(t)^* = 0$ and $n(t)An(t)^* = 0$.

Therefore, $k(t)Hk(t)^* = 0$ and $n(t)Hn(t)^* = 0$. Since H is nonnegative definite, then $Hk(t)^* = Hn(t)^* = 0$. Substituting for $k(t)$ and $n(t)$, one sees that

$$(1) \qquad H[1 - c(t)]A^*c(t) = 0 \quad \text{and}$$

$$(2) \qquad Hc(t)A^*[1 - c(t)] = 0 \,.$$

Subtracting (2) from (1) gives

$$H[A^*c(t) - c(t)A^*] = 0, \text{ so that}$$

$$(3) \qquad HA^*[c(t) - b(t)] = 0 \text{ by Lemma 2.1.}$$

In an analogous fashion, using $p(t) = [1 - b(t)]A^*b(t)$ and $q(t) = b(t)A^*[1 - b(t)]$, one arrives at

$$(4) \qquad HA[b(t) - c(t)] = 0 \,.$$

Combining (3) and (4), one finds that $HK[b(t) - c(t)] = 0$ and $H^2[b(t) - c(t)] = 0$. Then $H[b(t) - c(t)] = 0$. A simple calculation shows that $B^2 - C^2 = 2i(KH - HK)$. Combining these last three equations, one has $(B^2 - C^2)(b(t) - c(t)) = 0$. Since t was arbitrary, then $(B^2 - C^2)^2 = B^2 - C^2 = 0$ and A is normal.

THEOREM 3. *If B commutes with each of C and* Re A, *then B commutes with A. Moreover, in this case, if A is reversible, then A is normal.*

Indication of proof. The final conclusion follows easily from Lemma 2.1. Again let $H = \operatorname{Re} A$ and $K = \operatorname{Im} A$. By use of the hypotheses, Lemma 2.1, and certain algebraic manipulations, one can show the following:

$$(1) \qquad (B - C)CH = 0$$

$$(2) \qquad (B - C)H(B - C) = 0$$

$$(3) \qquad C(CH - HC)C = 0$$

$$(4) \qquad AHA^*B = BAHA^* \quad \text{and}$$

$$(5) \qquad A(B^2 - C^2)B^2 = AC^2(B^2 - C^2) = 0 \,.$$

This final equation then implies that $A(B^2 - C^2) = 0$. Therefore, by Lemma 2.1, $AB^2 = B^2A$. Since B^2 commutes with A, so does B (2, p. 260).

In concluding this paper, I should like to note that the proofs of Theorems 2 and 3 can be made much simpler algebraically, if it is assumed that A is invertible. However, it seemed reasonable to make the extra effort to prove the theorems without this added hypothesis.

I should also like to note that Lemma 2.1 appeared in my doctoral thesis at the University of North Carolina. Theorems 2 and 3 appeared in the same thesis with the added hypothesis of invertibility of A. Again I would like to thank Dr. J. S. Mac Nerney of the Department of Mathematics of the University of North Carolina for the direction of my doctoral thesis.

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