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ON SETS REPRESENTED BY THE SAME FORMULA IN DISTINCT CONSISTENT AXIOMATIZABLE ROSSER THEORIES

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In this note a theorem is proved which includes the following: if T is a consistent, axiomatizable Rosser theory in which all recursive functions of one argument are definable and S is any sentence undecidable in T, then given any pair (d_1, d_2) of *re* (recursively enumerable) degrees, there is a formula F which represents a set of degree d_1 in T and of degree d_2 in $T' = T(S)$, the theory obtained from T by adjoining S as a new axiom.

For the theory of recursive functions, we follow [1]. If T is a theory and S a sentence undecidable in T, we write $T(S)$ for the theory obtained by adding S to T as a new axiom.

THEOREM. *If T is a consistent, axiomatizable theory in which all recursive functions of one argument are definable, and in which some EI (effectively inseparable) pair of re sets is separable, and S is any sentence undecidable in T, then if (A, B) is any pair of re sets with $A \subset R \subset B$, where R is recursive, there is a formula which represents A in T and B in $T(S)$.*

Proof. The quite simple proof proceeds by way of two lemmas.

LEMMA 1. *If T and S are as in the theorem, A is an re set and R is a recursive subset of A, there is a formula which represents R in T and A in $T(S)$.*

Proof. We take formulas $F(x)$ and $G(x)$ such that $F(x)$ represents A in $T(S)$ and G *defines* R in T and hence in $T(S)$. The formula $H(x) = (F(x) \wedge S) \vee G(x)$ represents R in T and A in $T(S)$.

LEMMA 2. *If T and S are as above and A is any re set, there is a formula which represents A in T and the set I of nonnegative integers in $T(S)$.*

Proof. Consider an *re EI* pair (U_1, U_2) and a formula $F(x)$ which separates (U_1, U_2) in T. The formula $F(x) \vee S$ represents I in $T(S)$; it represents in T a superset of U_1 disjoint from U_2, and consequently represents a creative set C in T. Using a well-known theorem of Myhill, we take a recursive function f such that $A = f^{-1}(C)$. Using an argument similar to that of Lemma 1 of [2], we can find a formula $G(X)$ which represents $A = f^{-1}(C)$ in T and $I = f^{-1}(I)$ in $T(S)$. The lemma is proved.

To complete the proof of the theorem, we take $F(x)$ representing A in T and the set I in $T(S)$, by Lemma 2, and $G(x)$ representing R in T and B in $T(S)$. The formula $H(x) = F(x) \wedge G(x)$ represents A in T and B in $T(S)$.

If (d_1, d_2) is any pair of *re* degrees, we can find *re* sets A and B, with $A \subset R \subset B$, where R is recursive, such that A is of degree d_1 and B of degree d_2. We consequently have:

COROLLARY. *If T and S are as in the theorem and (d_1, d_2) is any pair of re degrees, there is a formula F which represents a set of degree d_1 in T and of degree d_2 in $T(S)$.*

Thus, with regard to the consequences of adding sentences S undecidable in a theory T as new axioms, we see that one undecidable sentence is as good as another insofar as representation of sets of distinct degree of unsolvability by the same formula is concerned.

REFERENCES

1. M. Davis, *Computability and Unsolvability*, McGraw Hill, 1958.
2. H. Putnam and R. Smullyan, *Exact separation of recursively enumerable sets within theories*, Proc. Amer. Math. Soc. **11** (1960), 574-577.

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