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HERMITIAN AND ANTI-HERMITIAN PROPERTIES OF GREEN'S MATRICES

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In this paper hermitian and anti-hermitian properties of the components of Green's matrices of related boundary value problems are studied. Necessary and sufficient conditions, depending only on the matrices defining the boundary conditions, for the components of the Green's matrix of one problem to be hermitian or anti-hermitian with respect to certain components of the kernel matrix of a related problem, are found. It is also shown-for a wide class of problems-that some components of these Green's matrices cannot be hermitian (anti-hermitian).

The techniques used depend on

(i) A construction of Green's matrices due to J. W. Neuberger [2, Theorem B],

(ii) A new vector-matrix formulation of ordinary linear differential equations developed in [3], and

(iii) Chapter 11 of [1].

Denote by $[a, b]$ a finite interval, each of A, B, P, Q a $k \times k$ complex matrix, and $F = (f_{ij})$ a $k \times k$ matrix of continuous complex valued functions on $[a, b]$ such that

$$f_{ij} = 0 \text{ if } i + j \text{ is even.} \tag{1}$$

Let

$$H = (\bar{f}_{k+1-j,k+1-i}), \quad T = ((-1)^i \delta_{i,k+1-j}) .$$

Consider the following vector matrix equations with boundary conditions:

$$Y' = FY, AY(a) + BY(b) = 0, \tag{2}$$

$$X' = HX, PX(a) + QX(b) = 0. \tag{3}$$

Assume that each problem has only the trivial solution. Let M and N be the unique functions such that

$$M(t, u) = I + \int_u^t F(s)M(s, u)ds ,$$

and

$$N(t, u) = I + \int_u^t H(s)N(s, u)ds ,$$

for all t, u in $[a, b]$.

Define K and L by:

$$(4)\qquad K(t, y) = \begin{cases} M(t, a)UM(a, y), & a \leqq t < y \leqq b, \\ M(t, a)UM(a, y) + M(t, y), & a \leqq y < t \leqq b, \\ M(t, a)UM(a, y) + \dfrac{1}{2} M(t, y), & a \leqq y = t \leqq b, \end{cases}$$

$$(5)\qquad L(t, y) = \begin{cases} N(t, a)DN(a, y), & a \leqq t < y \leqq b, \\ N(t, a)DN(a, y) + N(t, y), & a \leqq y < t \leqq b, \\ N(t, a)DN(a, y) + \dfrac{1}{2} N(t, y), & a \leqq y = t \leqq b. \end{cases}$$

where $U = -[A + BM(b, a)]^{-1}BM(b, a)$ and $D = -[P + QN(b, a)]^{-1}QN(b, a)$. The existence of $[A + BM(b, a)]^{-1}$ and $[P + QN(b, a)]^{-1}$ follows from the assumption that problems (2) and (3), respectively, have only the trivial solution-(see [2]-Theorem A). The Green's functions K and L are modified versions of kernels as constructed by Neuberger in [2]. This modification is obtained with the help of the following property of M(or N):

$$(6)\qquad M(t, u)M(u, v) = M(t, v), \quad t, u, v \in [a, b] .$$

This property is an immediate consequence of the fact that for any fundamental matrix Φ of $Y' = FY$, $M(t, u) = \Phi(t)\Phi^{-1}(u)$.

The main theorems of this paper are:

THEOREM 1. *Suppose-using the notation of* [1, p. 286]*-the rank of* $(A\colon B)$ *and* $(P\colon Q)$ *are both* k. *Then* $L(t, y) = -T^{-1}K^*(y, t)T$ *for all* $t, y \in [a, b]$ *if and only if* $PT^{-1}A^* = QT^{-1}B^*$.

THEOREM 2. *If* $U = D$, *there exist* $t, y \in [a, b]$ *such that* $L(t, y) \neq T^{-1}K^*(y, t)T$.

We first establish some lemmas.

LEMMA 1. $N(t, u) = T^{-1}M^*(u, t)T, t, u \in [a, b]$.

For a proof see [4, Theorem 3.3]; it is a consequence of the fact that $H = -T^{-1}F^*T$.

LEMMA 2. $L(t, y) = -T^{-1}K^*(y, t)T$ *for all* $t, y \in [a, b]$ *if and only if* $U = -[T^{-1}D^*T + I]$.

Lemma 2 follows from Lemma 1, (4), (5) and (6).

Proof of Theorem 2. Just as in Lemma 2 one can show that $L(t, y) = T^{-1}K^*(y, t)T$ for all $t, y \in [a, b]$ if and only if

$$U = T^{-1}D^*T + I . \tag{7}$$

But for $U = D$ equation (7) implies $u_{11} = \bar{u}_{kk} + 1$ and $u_{kk} = \bar{u}_{11} + 1$. Hence $1 = u_{11} - \bar{u}_{kk} = \bar{u}_{11} - u_{kk} = -1$. This contradiction completes the proof.

Proof of Theorem 1. K is the Green's matrix for

$$\mathfrak{L}Y = Y' - FY = G, AY(a) + BY(b) = 0 \tag{8}$$

i.e., for each continuous G the unique solution Y of (8) is given by

$$Y(t) = \int_a^b K(t, s)G(s)ds, \ t \in [a, b] ;$$

whereas L is the Green's function of

$$X' - HX = G, \quad PX(a) + QX(b) = 0 . \tag{9}$$

Now X satisfies (9) if and only if $Z = TX$ satisfies

$$\mathfrak{L}^+Z = -Z' - F^*Z = \tilde{G}, \tilde{P}Z(a) + \tilde{Q}Z(b) = 0 , \tag{10}$$

where $\tilde{G} = -TG, \tilde{P} = PT^{-1}, \tilde{Q} = QT^{-1}$.

Problem (10) is adjoint to problem (8) if and only if

$$\tilde{P}A^* = \tilde{Q}B^* . \tag{11}$$

The argument for this is entirely similar to that for Theorem 3.1, p. 289 in [1]. By problem 6, p. 297 of [1], $\tilde{L}(t, y) = K^*(y, t)$ for $t, y \in [a, b]$, if and only if (11) holds, where $\tilde{L}$ is the Green's matrix for (10). From the definition of $\tilde{L}$, we have $\tilde{L} = -TLT^{-1}$. Hence $L(t, y) = -T^{-1}K^*(y, t)T$ if and only if $PT^{-1}A^* = QT^{-1}B^*$.

Note. Some special cases of Theorem 1 were obtained in [4]-Theorems 6.13 and 7.1.

THEOREM 3. *If, in addition, F satisfies $f_{ij} = 0$ if $j > i + 1$ and $f_{i,i+1}(t) \neq 0$ for t in $[a, b]$, $i = 1, \cdots, k-1$, then $L_{1k}(t, y) = (-1)^k \bar{K}_{1k}(y, t)$ for all $t, y \in [a, b]$ implies $L(t, y) = -T^{-1}K^*(y, t)T$ for all $t, y \in [a, b]$.*

Proof. Observe that for any y in $[a, b]$ the functions $m_{1i}(t, y)$, $i = 1, \cdots, k$, are linearly independent functions of t. Assume $L_{1k}(t, y) = (-1)^k \bar{K}_{1k}(y, t)$ for all t, y in $[a, b]$. Using (4), (5) and (6) we can show that the conditions of Lemma 2 are satisfied.

Comments.

1. In [3] it is shown that

$$Ly = y^{(k)} + \sum_{i=1}^{k-1} p_i y^{(i)} = 0 \ ,$$

with p_i continuous, is equivalent to a vector-matrix equation $Y' = FY$ with F satisfying the hypothesis of Theorem 3.

2. The $(1, k)$ component of K, K_{1k}, corresponds to the Green's function of an ordinary boundary value problem. An important special case occurs when F satisfies the hypothesis of Theorem 3 and $F = H$, $A = P, B = Q$. For this case Theorems 1 and 3 yield necessary and sufficient conditions, expressed solely in terms of the matrices defining the boundary conditions, for the ordinary Green's function K_{1k} of even and odd order problems to be hermitian or anti-hermitian, respectively. Furthermore, by Theorem 2, K_{1k} cannot be anti-hermitian for k even and hermitian for k odd.

3. An interesting way of getting examples of matrices A, B, P, Q satisfying the conditions of Theorem 1 is as follows: for any continuous $k \times k$ matrix F satisfying (1) define M and N as above, then for arbitrary B, Q, t, and u, let $A = BM(t, u)$ and $P = QN(t, u)$.

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