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# ON THE $\Gamma$-RINGS OF NOBUSAWA

WILFRED E. BARNES

**N. Nobusawa recently introduced the notion of a $\Gamma$-ring, more general than a ring, and obtained analogues of the Wedderburn theorems for $\Gamma$-rings with minimum condition on left ideals. In this paper the notions of $\Gamma$-homomorphism, prime and (right) primary ideals, $m$-systems, and the radical of an ideal are extended to $\Gamma$-rings, where the defining conditions for a $\Gamma$-ring have been slightly weakened to permit defining residue class $\Gamma$-rings. The radical $R$ of a $\Gamma$-ring $M$ is shown to be an ideal of $M$, and the radical of $M/R$ to be zero, by methods similar to those of McCoy. If $M$ satisfies the maximum condition for ideals, the radical of a primary ideal is shown to be prime, and the ideal $Q \neq M$ is $P$-primary if and only if $P^n \subseteqq Q$ for some $n$, and $AB \subseteqq Q$, $A \nsubseteqq P$ implies $B \subseteqq Q$. Finally, in $\Gamma$-rings with maximum condition, if an ideal has a primary representation, then the usual uniqueness theorems are shown to hold by methods similar to those of Murdoch.**

2. Preliminary definitions. If $M = \{a, b, c, \cdots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \cdots\}$ are additive abelian groups, and for all $a, b, c$ in $M$ and all $\alpha, \beta$ in $\Gamma$, the following conditions are satisfied

(0) $a\alpha b$ is an element of $M$,

(1) $(a + b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha + \beta)b = a\alpha b + a\beta b$, $a\alpha(b + c) = a\alpha b + a\alpha c$,

(2) $(a\alpha b)\beta c = a\alpha(b\beta c)$,

then $M$ is called a $\Gamma$-ring. If these conditions are strengthened to

(0′) $a\alpha b$ is an element of $M$, $\alpha a\beta$ is an element of $\Gamma$,

(1′) same as (1),

(2′) $(a\alpha b)\beta c = a(\alpha b\beta)c = a\alpha(b\beta c)$,

(3′) $a\alpha b = 0$ for all $a, b$ in $M$ implies $\alpha = 0$,

we then have a $\Gamma$-ring in the sense of Nobusawa [3]. As indicated in [3], an example of a $\Gamma$-ring is obtained by letting $X$ and $Y$ be abelian groups, $M = \mathrm{Hom}\,(X, Y)$, $\Gamma = \mathrm{Hom}\,(Y, X)$, and $a\alpha b$ the usual composite map. (While Nobusawa does not explicitly require that $M$ and $\Gamma$ be abelian groups, it appears clear that this is intended.) We may note that it follows from (0)–(2) that $0\alpha b = a0b = a\alpha 0 = 0$ for all $a$ and $b$ in $M$ and all $\alpha$ in $\Gamma$.

A subset $A$ of the $\Gamma$-ring $M$ is a right (left) *ideal* of $M$ if $A$ is an additive subgroup of $M$ and $A\Gamma M = \{a\alpha c : a \in A, \alpha \in \Gamma, c \in M\}$ $(M\Gamma A)$ is contained in $A$. If $A$ is both a left and a right ideal, then $A$ is a two-sided ideal, or simply an ideal of $M$.

If $a \in M$, then the *principal ideal* generated by $a$, denoted by $(a)$, is the intersection of all ideals containing $a$, and is the set of all finite sums of elements of the form $na + x\alpha a + a\beta y + u\gamma a\delta v$, where $n$ is an integer, $x, y, u$ and $v$ are elements of $M$, and $\alpha, \beta, \gamma, \delta$ are elements of $\Gamma$.

If $A$ and $B$ are both right (resp. left, two-sided) ideals of $M$, then $A + B = \{a + b : a \in A, b \in B\}$ is clearly also a right (resp. left, two-sided) ideal, called the sum of $A$ and $B$.

It is also clear that the intersection of any number of right (resp. left, two-sided) ideals of $M$ is again a right (resp. left, two-sided) ideal of $M$.

If $A$ is a right ideal of $M$, $B$ a left ideal of $M$, and $S$ is any nonempty subset of $M$, then the set $SA = \{\sum_{i=1}^{n} s_i\alpha_i a_i : s_i \in S, \alpha_i \in \Gamma, a_i \in A, n$ any positive integer$\}$ is a right ideal of $M$, $BS$ is a left ideal of $M$, and $BA$ is a two-sided ideal of $M$.

When $A$ is a (two-sided) ideal of $M$, then $M/A = \{x + A : x \in M\}$, the set of cosets of $A$, is again a $\Gamma$-ring with respect to the operations $(x + A) + (y + A) = (x + y) + A$ and $(x + A)\alpha(y + A) = x\alpha y + A$, as may be verified by a straightforward computation. (We note that if Nobusawa's conditions $(0')$–$(3')$ are imposed on a $\Gamma$-ring, then this residue class $\Gamma$-ring could not be defined, as one would have no way of unambiguously defining $\alpha(x + A)\beta$, and moreover $(x + A)\alpha(y + A) = 0 + A$ for all $x$ and $y$ need not imply $\alpha = 0$, as may be seen by taking $A$ to be $M^2 = MM$.)

Let $M$ and $N$ both be $\Gamma$-rings, and $\theta$ a map of $M$ into $N$. Then $\theta$ is a $\Gamma$-homomorphism if and only if $(x + y)\theta = x\theta + y\theta$ and $(x\alpha y)\theta = (x\theta)\alpha(y\theta)$ for all $x, y$ and $\alpha$. If $\theta$ is also one-to-one and onto then $\theta$ is a $\Gamma$-isomorphism.

If $\theta$ is a $\Gamma$-homomorphism of $M$ into $N$, then the kernel of $\theta$, $0\theta^{-1} = \{x \in M : x\theta = 0\}$, is immediately seen to be an ideal of $M$. More generally, if $B$ is a right (resp. left, two-sided) ideal of $N$ then $B\theta^{-1} = \{x \in M : x\theta \in B\}$ is also a right (resp. left, two-sided) ideal of $M$. Similarly, if $\theta$ is a $\Gamma$-homomorphism of $M$ onto $N$ and $A$ is any right (resp. left, two-sided) ideal of $M$, then $A\theta = \{a\theta : a \in A\}$ is a right (resp. left, two-sided) ideal of $N$.

The proofs of the following three theorems are minor modifications of the proofs of the corresponding theorems in ordinary ring theory, and will be omitted.

THEOREM 1. *Let $A$ be an ideal of the $\Gamma$-ring $M$ and $\theta$ the natural mapping $x \rightarrow x + A$ of $M$ onto $M/A$. Then $\theta$ is a $\Gamma$-homomorphism of $M$ onto $M/A$ with kernel $A$. Conversely, if $\theta$ is a $\Gamma$-homomorphism of $M$ onto a $\Gamma$-ring $N$ and $A$ is the kernel of $\theta$, then $M/A$ is $\Gamma$-isomorphic to $N$.*

THEOREM 2. *Let $\theta$ be a $\Gamma$-homomorphism of a $\Gamma$-ring $M$ onto a $\Gamma$-ring $N$ with kernel $A$. Then $B'$ is an ideal of $N$ if and only if $B'\theta^{-1} = B$ is an ideal of $M$ containing $A$. In this case we have $M/B, N/B'$ and $(M/A)/(B/A)$ are all $\Gamma$-isomorphic.*

THEOREM 3. *Let $A$ and $B$ be ideals of the $\Gamma$-ring $M$ and $\theta : M \to M/B$ the canonical $\Gamma$-homomorphism. Then $A + B = (A\theta)\theta^{-1}$ and $(A + B)/B$ is $\Gamma$-isomorphic to $A/(A \cap B)$.*

**3. Prime ideals and the radical.** An ideal $P$ of the $\Gamma$-ring $M$ is said to be *prime* if for any ideals $A$ and $B$ of $M$, $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.

THEOREM 4. *An ideal $P$ of $M$ is prime if and only if $(a)(b) \subseteq P$ implies $a \in P$ or $b \in P$.*

*Proof.* $P$ a prime ideal clearly implies that if $(a)(b) \subseteq P$ then $a \in P$ or $b \in P$. Conversely, suppose $(a)(b) \subseteq P$ implies $a \in P$ or $b \in P$, and that $A$ and $B$ are ideals such that $AB \subseteq P$ but $A \not\subseteq P$. Then there exists $a \in A$ such that $a \notin P$, and for any $b \in B$ we have $(a)(b) \subseteq AB \subseteq P$, whence $b \in P$. Thus $B \subseteq P$ and $P$ is prime.

THEOREM 5. *Let $M$ be a $\Gamma$-ring satisfying Nobusawa's conditions* (0′)-(3′), *and $P$ an ideal of $M$. Then $P$ is prime if and only if $a\Gamma b \subseteq P$ implies $a \in P$ or $b \in P$, i.e. if and only if $a \notin P$ and $b \notin P$ implies there exists $\alpha \in \Gamma$ such that $a\alpha b \notin P$.*

*Proof.* Suppose $P$ is prime, and $a\Gamma b \subseteq P$ with $a \notin P$. Then if $x$ is any element of $(a)(b)$, $x$ is a finite sum of elements of the form $(na + c\alpha a + a\beta d + e\gamma a\delta f)\rho(mb + g\mu b + b\nu h + j\xi b\eta k)$, where $m$ and $n$ are integers, $c, d, e, f, g, h, j$ and $k$ are in $M$, and $\alpha, \beta, \gamma, \delta, \rho, \mu, \nu, \xi, \eta$ are in $\Gamma$. But every element in such a product is in $P$ by conditions (0′)-(3′) and the assumption that $a\Gamma b \subseteq P$. For example, $(c\alpha a)\rho(g\mu b) = c\alpha(a\rho(g\mu b)) = c\alpha(a(\rho g\mu)b) = c\alpha(a\sigma b)$, some $\sigma \in \Gamma$, hence is in $c\alpha P \subseteq P$. Thus $(a)(b) \subseteq P$ and $a \notin P$ then implies $b \in P$ by Theorem 4.

Conversely, suppose $a\Gamma b \subseteq P$ implies $a \in P$ or $b \in P$ and that $A$ and $B$ are ideals with $AB \subseteq P$ but $A \not\subseteq P$. Then for some $a \in A$, $a \notin P$, hence for any $b \in B, a\Gamma b \subseteq AB \subseteq P$ implies $b \in P$. Thus $B \subseteq P$ and $P$ is prime.

THEOREM 6. *If $A$ and $P$ are ideals of $M$, $A \subseteq P$ and $P$ prime, then $P/A$ is prime in $M/A$. Conversely, if $P'$ is a prime ideal of*

*$M/A$, $\theta$ the canonical homomorphism of $M$ onto $M/A$, then $P'\theta^{-1} = P$ is a prime ideal of $M$.*

*Proof.* Assume $A \subseteqq P$, $P$ prime, and $(B/A)(C/A) \subseteqq P/A$. Then $BC \subseteqq P$, hence $B \subseteqq P$ or $C \subseteqq P$, i.e. $B/A \subseteqq P/A$ or $C/A \subseteqq P/A$, and $P/A$ is prime. Conversely, suppose $P'$ is a prime ideal of $M/A$, and $BC \subseteqq P = P'\theta^{-1}$. Then $(B\theta)(C\theta) \subseteqq P\theta = P'$, and $B\theta \subseteqq P'$ or $C\theta \subseteqq P'$. But then $B \subseteqq (B\theta)\theta^{-1} \subseteqq P'\theta^{-1} = P$ or $C \subseteqq (C\theta)\theta^{-1} \subseteqq P'\theta^{-1} = P$, so that $P$ is prime.

In order to define the radical of an ideal $A$ in a $\Gamma$-ring $M$, we proceed in a manner analogous to that of McCoy [1], first defining the notion of an $m$-system, corresponding to McCoy's $m$-system in that an ideal $P$ will be prime if and only if its complement $P^c$ is an $m$-system, and essentially the same as the $m$-system of van der Walt [4]. If $S$ is any subset of the $\Gamma$-ring $M$, we call $S$ an $m$-system if $S = \varnothing$ or if $a$ and $b$ in $S$ implies that $(a)(b) \cap S \neq \varnothing$. We note that the ideal $P$ is a prime divisor of the ideal $A$ if and only if $P^c$ is an $m$-system disjoint from $A$.

For any ideal $A$ of $M$, we then define the *radical of* $A$, $r(A)$, to be the set of all elements $x$ of $M$ such that every $m$-system containing $x$ contains an element of $A$. It is immediate that $A \subseteqq r(A)$, and that if $P$ is a prime ideal divisor of $A$ then $r(A) \subseteqq P$ since $P^c$ is an $m$-system disjoint from $A$. The radical of the ring $M$ is defined to be $r(0)$. In order to establish that $r(A)$ is the intersection of all prime ideal divisors of $A$ and hence is an ideal, we proceed by means of lemmas analogous to those of McCoy [1] or Murdoch [2].

LEMMA 1. *Let $A$ be an ideal disjoint from the $m$-system $S$. Then there exists an $m$-system $T \supseteqq S$ which is maximal in the class of $m$-systems disjoint from $A$.*

*Proof.* Zorn's lemma applied to the class of $m$-systems disjoint from $A$.

LEMMA 2. *Let $A$ and $S$ be as in Lemma 1. Then there exists an ideal $P \supseteqq A$ which is maximal in the class of ideals containing $A$ and disjoint from $S$. Moreover, $P$ is necessarily prime.*

*Proof.* The ideal $P$ exists by Zorn's lemma applied to the class of ideals containing $A$ and disjoint from $S$. Now suppose that both $b$ and $c$ are elements of $M$ not in $P$. Then $P + (b) \neq P$ and $P + (c) \neq P$, hence by the maximality of $P$ there exist elements $p_1$ and $p_2$ in $P$,

$b_1$ in $(b)$ and $c_1$ in $(c)$ such that $s_1 = p_1 + b_1$ and $s_2 = p_2 + c_1$ are both in $S$. Since $S$ is an $m$-system, we then have that $(s_1)(s_2) \cap S \neq \varnothing$. But $(s_1)(s_2) \subseteqq (P + (b_1))(P + (c_1)) \subseteqq (P + (b))(P + (c)) \subseteqq P + (b)(c)$. Thus $(b)(c) \nsubseteqq P$, so that $b$ and $c$ not in $P$ implies $(b)(c) \nsubseteqq P$, and $P$ is prime.

LEMMA 3. *A nonempty subset $P$ of the $\Gamma$-ring $M$ is a minimal prime ideal containing an ideal $A$ (i.e. $P$ is a minimal prime of $A$) if and only if $P^c$ is a maximal $m$-system disjoint from $A$.*

*Proof.* Suppose $P^c$ is a maximal $m$-system disjoint from $A$, and let $P_1$ be the prime ideal of Lemma 2, maximal in the class of ideals containing $A$ and disjoint from $P^c$, so that $P_1^c \supseteqq P^c$. Now $P_1^c$ is an $m$-system disjoint from $A$, hence the maximality of $P^c$ implies $P_1^c = P^c$, or $P_1 = P$, so that $P$ is a prime ideal. Clearly, the maximality of $P^c$ then implies that $P$ is a minimal prime of $A$.

Conversely, suppose that $P$ is a minimal prime of $A$. Then $P^c$ in an $m$-system disjoint from $A$, and by Lemma 1 there exists a maximal $m$-system $S$ containing $P^c$ and disjoint from $A$. By that part of this lemma just proved, $S^c$ is a minimal prime of $A$. But $S^c \subseteqq P$ and the minimality of $P$ implies that $S^c = P$, or $S = P^c$ as required.

COROLLARY. *If $P$ is a prime ideal divisor of $A$, then $P$ contains a minimal prime of $A$.*

*Proof.* $P^c$ is an $m$-system disjoint from $A$, hence $P^c \subseteqq S$ for $S$ a maximal $m$-system disjoint from $A$, by Lemma 1. But then $S^c \subseteqq P$ and $S^c$ is a minimal prime of $A$.

THEOREM 7. *If $A$ is any ideal of the $\Gamma$-ring $M$, then $r(A)$ is the intersection of the minimal primes of $A$.*

*Proof.* Clearly $r(A)$ is contained in this intersection. So suppose that $x$ is an element not in $r(A)$. Then, by definition of $r(A)$, there exists an $m$-system $S$ containing $x$ but disjoint from $A$. Then by Lemma 1 there exists $S_1 \supseteqq S$ such that $S_1$ is a maximal $m$-system disjoint from $A$. By Lemma 3, $S_1^c = P$ is a minimal prime of $A$, and $x \in S \subseteqq S_1$ implies $x \notin P$. Thus an element not in $r(A)$ is not in the intersection of the minimal primes of $A$.

COROLLARY 1. *The radical of an ideal is an ideal.*

COROLLARY 2. $r(r(A)) = r(A)$.

COROLLARY 3. *If* $R = r(0)$, *then* $r(M/R) = 0$.

*Proof.* Suppose $x + R \neq 0 + R$. Then $x \notin R$ and there exists some prime ideal $P \supseteqq R$ such that $x \notin P$. Then $P/R$ is a prime ideal of $M/R$ and $x + R \notin P/R$, hence $x + R \notin r(M/R)$.

COROLLARY 4. *For any ideals* $A$ *and* $B$, $r(A \cap B) = r(A) \cap r(B)$.

*Proof.* Since every prime divisor of $A$ or of $B$ is also a prime divisor of $A \cap B$, we have $r(A \cap B) \subseteqq r(A) \cap r(B)$. Conversely, if a prime $P \supseteqq A \cap B$, then $P \supseteqq AB$ and $P$ contains either $A$ or $B$, hence $r(A) \cap r(B) \subseteqq r(A \cap B)$.

We may note that this last corollary evidently extends to arbitrary finite intersections of ideals.

**4. Primary ideals.** We define an ideal $Q$ in the $\Gamma$-ring $M$ to be *right primary* if for any ideals $A$ and $B$, $AB \subseteqq Q$ implies $A \subseteqq r(Q)$ or $B \subseteqq Q$. A similar definition holds for left primary ideals. Since throughout this paper we shall deal only with right primary ideals, we shall refer to a right primary ideal as simply a primary ideal. Similar results would of course hold for left primary ideals.

THEOREM 8. *An ideal* $Q$ *of the* $\Gamma$*-ring* $M$ *is primary if and only if* $(a)(b) \subseteqq Q$ *implies* $a \in r(Q)$ *or* $b \in Q$.

*Proof.* The only if part follows trivially from the definition of a primary ideal. So suppose that $(a)(b) \subseteqq Q$ implies $a \in r(Q)$ or $b \in Q$, and let $AB \subseteqq Q$ for $B \not\subseteqq Q$. Then there exists $b \in B \cap Q^c$, and for any $a \in A$ we have $(a)(b) \subseteqq AB \subseteqq Q$, hence $a \in r(Q)$, $A \subseteqq r(Q)$ and $Q$ is primary.

THEOREM 9. *An ideal* $Q$ *of a* $\Gamma$*-ring* $M$ *satisfying Nobusawa's conditions* $(0')$–$(3')$ *is primary if and only if* $a\Gamma b \subseteqq Q$ *implies* $a \in r(Q)$ *or* $b \in Q$ *(i.e. if and only if* $a \notin r(Q)$ *and* $b \notin Q$ *implies* $a\Gamma b \cap Q^c \neq \varnothing$*).*

*Proof.* Suppose $Q$ is primary, $a\Gamma b \subseteqq Q$ and $a \notin r(Q)$. Then any element of $(a)(b)$ is a finite sum of elements of the form

$$(na + c\alpha a + a\beta d + e\gamma a\delta f)\rho(mb + g\mu b + b\nu h + j\xi b\eta k)\ ,$$

each of which is in $Q$, hence $(a)(b) \subseteqq Q$ and $b \in Q$ by Theorem 8.

Conversely, suppose $a\Gamma b \subseteqq Q$ implies $a \in r(Q)$ or $b \in Q$, and $AB \subseteqq Q$

with $B \nsubseteq Q$. Then there exists $b \in B \cap Q^c$, and for any $a \in A$ we have $a\Gamma b \subseteqq AB \subseteqq Q$, hence $a \in r(Q)$, $A \subseteqq r(Q)$, and $Q$ is primary.

THEOREM 10. *Let $Q_1$ and $Q_2$ be primary and $r(Q_1) = r(Q_2) = C$. Then $Q_1 \cap Q_2$ is primary with radical $C$.*

*Proof.* $r(Q_1 \cap Q_2) = C$ by Theorem 7, Corollary 4. Hence if $AB \subseteqq Q_1 \cap Q_2$ but $B \nsubseteq Q_1 \cap Q_2$ we may assume that $B \nsubseteq Q_1$, and $AB \subseteqq Q_1$ then implies $A \subseteqq r(Q_1) = C$. Hence $Q_1 \cap Q_2$ is primary.

COROLLARY. *Let $Q_1$ be primary with radical $C$, $i = 1, 2, \cdots, n$. Then $Q_1 \cap Q_2 \cap \cdots \cap Q_n$ is primary with radical $C$.*

**5. $\Gamma$-rings with ascending chain condition.** Throughout the rest of this paper it will be assumed that $M$ is a $\Gamma$-ring satisfying the ascending chain condition on ideals.

THEOREM 11. *Any ideal of $M$ contains a finite product of its minimal primes.*

*Proof.* If not, let $A$ be an ideal maximal in the set of ideals not meeting the stated condition. Then $A$ is not itself prime, and there exist ideals $B$ and $C$ properly containing $A$ such that $BC \subseteqq A$. Then $B$ and $C$ must each meet the stated condition, hence $A$ contains some finite product $P_1P_2 \cdots P_n$ of prime ideal divisors. Since any prime ideal divisor of $A$ contains a minimal prime of $A$, any $P_i$ which is not a minimal prime of $A$ may clearly be replaced by a minimal prime of $A$. Hence $A$ does meet the condition, contrary to assumption, and the theorem is proved.

COROLLARY 1. *Any ideal $A$ of $M$ has only a finite number of minimal primes.*

*Proof.* Let $P_1P_2 \cdots P_n \subseteqq A$ and $P$ be a minimal prime of $A$. Then $A \subseteqq P$ implies some $P_i \subseteqq P$ and hence $P_i = P$ by the minimality of $P$.

COROLLARY 2. $(r(A))^n \subseteqq A$ *for some positive integer $n$.*

*Proof.* $P_1P_2 \cdots P_n \subseteqq A$ and $r(A) \subseteqq P_i$ for all $i$ together imply $(r(A))^n \subseteqq A$.

THEOREM 12. *If $Q$ is a primary ideal, then $r(Q) = P$ is prime.*

*Proof.* Suppose $Q$ is primary and $BC \subseteqq r(Q)$ with $B \not\subseteqq r(Q)$. Then for some $n$, $(BC)^n \subseteqq Q$ and we may assume $n$ to be the least positive integer with this property. If $n = 1$, then $BC \subseteqq Q$ implies $C \subseteqq r(Q)$. If $n > 1$, then $B(C(BC)^{n-1}) \subseteqq Q$ implies $C(BC)^{n-1} \subseteqq Q$, and since $(BC)^{n-1} \not\subseteqq Q$ we again have $C \subseteqq r(Q)$. Hence $r(Q)$ is prime.

COROLLARY 1. *If $Q$ is a primary ideal, then $Q$ has a unique minimal prime $r(Q) = P$.*

COROLLARY 2. *An ideal $Q$ is primary if and only if either*
(i) *$AB \subseteqq Q$ implies $A^n \subseteqq Q$ for some $n$ or $B \subseteqq Q$, or*
(ii) *$(a)(b) \subseteqq Q$ implies $(a)^n \subseteqq Q$ for some $n$ or $(b) \subseteqq Q$.*

We shall say that the ideal $Q$ is $P$-primary, or is primary for $P$, if $Q$ is primary and its prime radical is $P$.

THEOREM 13. *An ideal $Q \neq M$ is $P$-primary if and only if either*
(1) *$P^n \subseteqq Q$ for some $n$, and $AB \subseteqq Q$ implies $A \subseteqq P$ or $B \subseteqq Q$, or*
(2) *$P^n \subseteqq Q$ for some $n$, and $(a)(b) \subseteqq Q$ implies $(a) \subseteqq P$ or $(b) \subseteqq Q$.*
*If either condition* (1) *or* (2) *holds, then $P$ is prime.*

*Proof.* If $Q$ is $P$-primary, then by definition $Q$ is primary and $P = r(Q)$. $P$ is prime by Theorem 12, and $P^n \subseteqq Q$ for some $n$ by Theorem 11, Corollary 2. Hence if $Q$ is $P$-primary, then both (1) and (2) hold, and $P$ is prime.

Now suppose (1) holds. Then from $QM \subseteqq Q$ with $M \not\subseteqq Q$ we have $Q \subseteqq P$, and from $P^n \subseteqq Q$ follows $P \subseteqq r(Q)$. If $AB \subseteqq P$ then $(AB)^n \subseteqq Q$ and hence either $A \subseteqq P$ or $B(AB)^{n-1} \subseteqq Q$. In the latter case either $B \subseteqq P$ or $(AB)^{n-1} \subseteqq Q$. Continuing, we have either $A \subseteqq P$ or $B \subseteqq P$ so that $P$ is prime. Thus $P = r(Q)$, and (1) then says $Q$ is $P$-primary.

Finally, suppose (2) holds. Then if $AB \subseteqq Q$, but $B \not\subseteqq Q$, there must exist $b \in B \cap Q^c$. Hence for every $a \in A$, $(a)(b) \subseteqq AB \subseteqq Q$ implies $(a) \subseteqq P$, or $A \subseteqq P$. Thus (2) implies (1) and the proof is complete.

**6. Primary representations of ideals in $\Gamma$-rings.** In this section we obtain analogues of the classical Noether-Lasker theorems concerning primary representations of ideals. Since $\Gamma$-rings include non-commutative rings as special cases, even the ascending chain condition on ideals will not assure that every ideal has a representation as a finite intersection of primary ideals. But for ideals having such representations, the usual uniqueness theorems hold. Recall that we are still assuming the ascending chain condition for ideals.

We call $A = Q_1 \cap Q_2 \cap \cdots \cap Q_n$ an *irredundant primary representation* of $A$, if each $Q_1$ is primary and none can be omitted. If in addition the prime radicals $P_1, P_2, \cdots, P_n$ are all distinct the representation is called *short*.

THEOREM 14. *An irredundant intersection $A = Q_1 \cap Q_2 \cap \cdots \cap Q_n$ of primary ideals is again primary if and only if $P_1 = P_2 = \cdots = P_n$, where $P_i = r(Q_i)$.*

*Proof.* If $P = P_1 = \cdots = P_n$, then $A$ is primary for $P$ by the corollary to Theorem 10. Conversely, suppose that the $P_i$ are not all equal, say $P_1 \not\subseteq P_2$. Now $P_1^n \subseteq Q_1$ for some $n$, hence $P_1^n(Q_2 \cap \cdots \cap Q_n) \subseteq A$. But $P_1^n \not\subseteq P_2$ since $P_2$ is prime, hence $P_1^n \not\subseteq r(A)$. Since the intersection is irredundant, $Q_2 \cap \cdots \cap Q_n \not\subseteq A$, hence $A$ is not primary if the $P_i$ are not all equal.

COROLLARY. *If the ideal $A$ has a representation as the intersection of a finite number of primary ideals, then $A$ has a short primary representation.*

*Proof.* Any redundant intersectants can be deleted to obtain an irredundant representation. By the theorem, the intersection of those intersectants having the same radical is again primary with the same radical, and the representation so obtained is still irredundant, hence short.

THEOREM 15. *Let $A = Q_1 \cap Q_2 \cap \cdots \cap Q_n$ be a primary representation of $A$, and $r(Q_i) = P_i$. Then the minimal primes of $A$ are the minimal elements of $\{P_1, P_2, \cdots, P_n\}$.*

*Proof.* $r(A) = P_1 \cap P_2 \cap \cdots \cap P_n \subseteq P$ for any minimal prime $P$ of $A$. Then $P_1P_2 \cdots P_n \subseteq P$ implies $P_i \subseteq P$ for some $i$. But $P$ a minimal prime of $A$ then implies $P_i = P$. Thus the minimal primes of $A$ are among $P_1, P_2, \cdots, P_n$. If $P_i$ is minimal in $\{P_1, P_2, \cdots, P_n\}$ then $P_i$ cannot properly contain any other prime ideal divisor $P$ of $A$, since $P$ must contain some $P_j$, hence $P_i$ is a minimal prime of $A$.

If $A$ is any ideal of the $\Gamma$-ring $M$, and $P \neq M$ is a prime ideal, then we define $A_P$, the *$P$-(isolated) component* of $A$, to be the union of all ideals $C$ such that for each $C$ there exists some ideal $B \not\subseteq P$ with $BC \subseteq A$. For convenience $A_M$ is defined to be $A$.

$A_P$ is well-defined. For by the ascending chain condition there exists at least one maximal such ideal $C$, with $BC \subseteq A$, $B \not\subseteq P$. If also $B'C' \subseteq A$, $B' \not\subseteq P$, then $BB' \not\subseteq P$ since $P$ is prime, and $BB'(C + C') \subseteq A$, so $C + C'$ is again such an ideal. The maximality of $C$ then implies $C' \subseteqq C$. Thus $A_P$ is the unique maximal ideal such that $BA_P \subseteqq A$ for some $B \nsubseteq P$. It is clear that $A \subseteqq A_P$, and that if $A \nsubseteq P$ then $A_P = M$. We collect some other simple properties of $P$-components in the following.

LEMMA. (1) *If* $P_1 \supseteqq P_2$, *then* $A_{P_1} \subseteqq A_{P_2}$.

(2) *If* $A_1 \subseteqq A_2$, *then* $(A_1)_P \subseteqq (A_2)_P$.

(3) $(A_1 \cap A_2)_P = (A_1)_P \cap (A_2)_P$.

(4) $Q$ *is* $P$-*primary if and only if* $Q_P = Q$ *and* $P^m \subseteqq Q$ *for some* $m$.

*Proof.* (1) $BA_{P_1} \subseteqq A$ for some $B \nsubseteq P_1$, hence $B \nsubseteq P_2$ and $A_{P_1} \subseteqq A_{P_2}$.

(2) $B(A_1)_P \subseteqq A_1 \subseteqq A_2$ for some $B \nsubseteq P$, hence $(A_1)_P \subseteqq (A_2)_P$.

(3) $B(A_1 \cap A_2)_P \subseteqq A_1 \cap A_2 \subseteqq A_1$ for some $B \nsubseteq P$, hence $(A_1 \cap A_2)_P \subseteqq (A_1)_P$, and similarly $(A_1 \cap A_2)_P \subseteqq (A_2)_P$, whence $(A_1 \cap A_2)_P \subseteqq (A_1)_P \cap (A_2)_P$. Now if $x$ is in $(A_1)_P \cap (A_2)_P$ then there exist ideals $B_1 \nsubseteq P$ and $B_2 \nsubseteq P$ such that $B_1(x) \subseteqq A_1$ and $B_2(x) \subseteqq A_2$, hence $B_1B_2(x) \subseteqq A_1 \cap A_2$. Then $B_1B_2 \nsubseteq P$ implies $x \in (A_1 \cap A_2)_P$.

(4) Assume $Q$ is $P$-primary. $Q_P \supseteqq Q$ trivially. If $B \nsubseteq P$, then $BC \subseteqq Q$ implies $C \subseteqq Q$ since $Q$ is $P$-primary, hence $Q_P = Q$. Conversely, if $Q_P = Q$ then $BC \subseteqq Q$ and $C \nsubseteq Q$ implies $B \subseteqq P$, which together with $P^m \subseteqq Q$ implies $Q$ is $P$-primary.

THEOREM 16. *Let* $A = Q_1 \cap Q_2 \cap \cdots \cap Q_n$, $Q_i$ *primary for* $P_i$, *and* $P$ *be a prime ideal containing* $P_i$ *if and only if* $1 \leqq i \leqq m \leqq n$. *Then* $A_P = Q_1 \cap Q_2 \cap \cdots \cap Q_m$. *If* $P$ *contains no* $P_i$, *then* $A_P = M$.

*Proof.* If $P$ contains no $P_i$ then since $A$ contains a product of its minimal primes, i.e. a product of the $P_i$, $P$ does not contain $A$ and $A_P = M$. So suppose $P \supseteqq P_i$ if and only if $1 \leqq i \leqq m \leqq n$. Then $A_P \subseteqq A_{P_i} \subseteqq (Q_i)_{P_i} = Q_i$ for $1 \leqq i \leqq m$, and $A_P \subseteqq Q_1 \cap Q_2 \cap \cdots \cap Q_m$. If $m = n$ then $A_P \subseteqq A \subseteqq A_P$, whence $A_P = A = Q_1 \cap Q_2 \cap \cdots \cap Q_n$. If $m < n$ then for $m < i \leqq n$, $Q_i \nsubseteq P$ since $P_i \nsubseteq P$, hence $Q_{m+1}Q_{m+2} \cdots Q_n \nsubseteq P$. Then $(Q_{m+1}Q_{m+2} \cdots Q_n)(Q_1 \cap Q_2 \cap \cdots \cap Q_m) \subseteqq A$, whence $Q_1 \cap Q_2 \cap \cdots \cap Q_m \subseteqq A_P \subseteqq Q_1 \cap Q_2 \cap \cdots \cap Q_m$, and the theorem is proved.

If $A = Q_1 \cap Q_2 \cap \cdots \cap Q_n$, $P_i = r(Q_i)$, is a short primary representation of $A$, a set $S = \{P_{i_1}, \cdots, P_{i_k}\}$ of associated primes is an *isolated set* of associated primes if $P_i \in S$ and $P_j \subseteqq P_i$ implies $P_j \in S$. For $S$ an isolated set of primes, $A_S = \bigcap Q_i$, $P_i \in S$, is an *isolated component* of $A$. We then have the following.

COROLLARY.[1] *Let* $A = Q_1 \cap Q_2 \cap \cdots \cap Q_n$, $P_i = r(Q_i)$, *be a short primary representation of* $A$ *and* $S$ *an isolated set of associated primes of* $A$. *Then the isolated component* $A_S$ *depends only on* $A$ *and* $S$, *and not on the particular primary representation of* $A$. *In particular, if* $P$ *is a minimal prime of* $A$, *then* $A_P = Q_i$ *for some* $i$, *hence the isolated component corresponding to a minimal prime* $P$ *is* $P$*-primary and occurs in every short primary representation of* $A$.

*Proof*. For $S$ an isolated set, $A_S = \bigcap_{P_i \in S} Q_i = \bigcap_{P_i \in S} A_{P_i}$ and the result follows at once.

THEOREM 17. *Let* $A = Q_1 \cap Q_2 \cap \cdots \cap Q_n$, $P_i = r(Q_i)$, *be a short primary representation of* $A$. *Then a prime ideal* $P$ *containing* $A$ *is one of the* $P_i$ *if and only if* $PB \subseteqq A_P$ *for some ideal* $B \not\subseteqq A_P$.

*Proof*. Suppose $P = P_m$ and the indexing is so chosen that $P_i \subseteqq P$ if and only if $i \leqq m$. Then $A_P = Q_1 \cap Q_2 \cap \cdots \cap Q_m$ is short, hence $C = Q_1 \cap Q_2 \cap \cdots \cap Q_{m-1} \not\subseteqq A_P$ ($C = M$ if $m = 1$). Since $P^k \subseteqq Q_m$ for some $k$, $P^k C \subseteqq A_P$, and we may choose $r$ minimal such that $P^r C \subseteqq A_P$. ($r \geqq 1$ since $C \not\subseteqq A_P$). If $r = 1$ then $PC \subseteqq A_P$ and we may take $B = C$. If $r > 1$ then $P(P^{r-1}C) \subseteqq A_P$ but $P^{r-1}C \not\subseteqq A_P$ so we may take $B = P^{r-1}C$.

Conversely, suppose $P \supseteqq A$ and $PB \subseteqq A_P$ with $B \not\subseteqq A_P$. We may choose the indexing so that $A_P = Q_1 \cap Q_2 \cap \cdots \cap Q_m$ (i.e. $P \supseteqq P_i$ if and only if $1 \leqq i \leqq m$), which is a short representation of $A_P$. Then $PB \subseteqq Q_i$ for $1 \leqq i \leqq m$, but $B \not\subseteqq Q_j$ for some $j \leqq m$. $Q_j$ primary for $P_j$ then implies that $P \subseteqq P_j$, hence $P = P_j$.

COROLLARY. *If* $A = Q_1 \cap Q_2 \cap \cdots \cap Q_n = Q'_1 \cap Q'_2 \cap \cdots \cap Q'_m$ *are two short primary representations of* $A$, *then* $m = n$ *and the two sets of radicals*, $\{P_1, P_2, \cdots, P_n\}$ *and* $\{P'_1, P'_2, \cdots, P'_m\}$, *are the same*.

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[1] I thank the referee for suggesting the inclusion of this corollary, and for improving the proof of Theorem 13.

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WASHINGTON STATE UNIVERSITY
Present address: Iowa State University