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SAMPLE FUNCTION REGULARITY FOR GAUSSIAN PROCESSES WITH THE PARAMETER IN A HILBERT SPACE

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In this note, Gaussian processes $\{\xi_t; t \in H\}$ where H is the Hilbert space l_2 are considered. It is shown that if T is a compact subset of a set of the form $\{(t_1, t_2, \cdots, t_n, \cdots):$ $a_n \leqq t_n \leqq a_n + 1/2^n$, $(a_1, a_2, \cdots a_n, \cdots) \in H\}$ (thus including all compact subsets of N-dimensional Eulidean space), and there exists constants $\delta > 0$ and $K > 0$ such that

$$E(|\xi_t - \xi_s|^2) \leqq \frac{K}{|\log ||t-s|||^{4+\delta}}$$

for t, s in H, then almost all sample functions of the process are continuous on T. Furthermore, if there are constants $\alpha > 0$ and K such that

$$E(|\xi_t - \xi_s|^2) \leqq K||t-s||^\alpha$$

for all t, s in H, then "almost all" sample functions of the process are Lipschitz-β continuous on T for $0 < \beta < \alpha/2$. The phrase "almost all" is used in the sense that the process defines a probability measure μ on the space C_T of continuous or Lipschitz-β continuous functions on T, such that for any k points $t^1, t^2, \cdots t^k$ in T and any Borel set A in k-dimensional Euclidean space R^k

$$\mu\{x \in C_T: (x(t^1), \cdots x(t^k)) \in A\} = P^{t^1, \cdots t^k}(A)$$

where $P^{t^1, \cdots t^k}$ is the probability measure defined by the random vector $\{\xi_{t^1}, \cdots \xi_{t^k}\}$. In the case where the process $\{\xi_t: t \in H\}$ is separable and is separated by the set of dyadic numbers in H, then the phrase "almost all" as defined here takes on the usual meaning.

In application, it is shown that the Brownian process in a Hilbert space defined by Paul Levy satisfies the latter condition for $\alpha = 1$. Thus almost all sample functions are Lipschitz-β continuous on T for $0 < \beta < 1/2$ if T is a compact set of the form described above. Furthermore, it is shown that Levy's result that almost all sample functions of this process are discontinuous in the Hilbert sphere may be extended to arbitrary noncompact subsets of the form $T = \{(t_1, t_2, \cdots, t_n, \cdots): a_n \leqq t_n \leqq b_n\}$.

We sufficient condition for Lipschitz-β continuity of sample functions of Gaussian processes with the parameter in a Hilbert space. We use the following notations. $\{\xi_t; t \in H\}$ denotes a real valued process where the parameter space H is the Hilbert space l_2. R^N is

N-dimensional Euclidean space. X is the space of real valued functions defined on H or subsets T of H, depending on the context. C_T is the space of continuous (or Lipschitz-β continuous) functions defined on T. P is the complete probability measure on X defined by the process $\{\xi_t; t \in H\}$. For any $t^1, \cdots t^k$ in H, $p^{t^1, \cdots t^k}$ is the probability measure defined by the random vector $\{\xi_{t^1}, \cdots \xi_{t^k}\}$.

LEMMA 1. *Let T be a compact subset of H and D a dense subset of T. If $\{\xi_t; t \in H\}$ satisfies*

(1) $E(|\xi_t - \xi_s|^\alpha) \to 0$ as $\|t - s\| \to 0$ *for some constant* $\alpha > 0$, *and*

(2) $P\{x \in X$: *x is continuous (Lipschitz-β continuous) in* $D\} = 1$, *then there is a measure μ on the space C_T such that*

$$\mu\{x \in C_T: (x(t^1), \cdots x(t^k)) \in A\} = P^{t^1, \cdots t^k}(A)$$

for all finite sets $t^1, \cdots t^k$ in T and Borel sets A in R^k.

REMARK. If the process $\{\xi_t: t \in H\}$ is separable and is separated by D, then the conclusion of Lemma 1 is that $P\{x \in X: x$ is continuous in $T\} = 1$, or in other words, almost all sample functions are continuous in the usual sense of the words "almost all".

Proof. Let $\bar{X} = \{x \in X: x$ is continuous (Lipschitz-β continuous) in $D\}$. Extend each x in $\bar{X}$ by closure to a function x^* which is continuous (Lipschitz-β continuous) in T. Define map $\Pi: X \to C_T$ as follows:

$$\Pi(x) = \begin{cases} x^* & \text{if } x \in \bar{X} \\ 0 & \text{if } x \notin \bar{X} \end{cases}$$

We will show for fixed t that $P\{x \in X: x^*(t) = x(t)\} = 1$. This is clear if t is in D. If t is not in D, let $t^1, \cdots t^j, \cdots$ be a sequence in D such that $t^j \to t$ as $j \to \infty$. Furthermore let $t^1, \cdots t^j, \cdots$ be chosen so that

$$\sum_{j=1}^{\infty} P\{|x(t^j) - x(t)| > \varepsilon_j\} \leqq \sum_{j=1}^{\infty} \frac{E(|\xi_{t^j} - \xi_t|^\alpha)}{\varepsilon_j^\alpha} < \infty$$

where $\varepsilon_j \to 0$ as $j \to \infty$.

Apply the Borel-Cantelli lemma to conclude that for almost all x, $x(t^j) \to x(t)$. Thus we have $x(t^j) \to x(t)$ for almost all x, $x^*(t^j) = x(t^j)$ for all t^j, and $x^*(t^j) \to x^*(t)$, for all x; hence $x^*(t) = x(t)$ for almost all x in X, or $P\{x \in X: x^*(t) = x(t)\} = 1$.

It follows that for any finite set $t^1, \cdots t^k$, $P\{x: (x(t^1), \cdots x(t^k)) = (x^*(t^1), \cdots x^*(t^k)\} = 1$.

We will next show that Π is a measurable map of $X \to C_T$. Let

E be a Borel cylinder in C_T;

$$E = \{x \in C_T\colon (x(t^1), \cdots x(t^k)) \in A\}$$

where A is a Borel set in R^k. If the set A does not contain the point $(0, 0, \cdots 0)$, then

$$\begin{aligned}
&\Pi^{-1}(E)\\
&= \{x \in X\colon (x^*(t^1), \cdots x^*(t^k)) \in A\}\\
&= \{x \in X\colon (x^*(t^1), \cdots x^*(t^k)) \in A \text{ and } (x^*(t^1), \cdots x^*(t^k)) = (x(t^1), \cdots x(t^k))\}\\
&\quad \cup \{\ '' \qquad '' \qquad '' \qquad '' \qquad \neq \qquad '' \qquad \}\\
&= \{x \in X\colon (x(t^1), \cdots x(t^k)) \in A\}\\
&\quad - \{x \in X\colon (x(t^1), \cdots x(t^k)) \in A, (x^*(t^1), \cdots x^*(t^k) \neq (x(t) \cdots x(t^k))\}\\
&\quad \cup \{x \in X\colon (x^*(t^1), \cdots x^*(t^k)) \in A \text{ and } (x^*(t^1), \cdots x^*(t^k)) \neq (x(t^1), \cdots x(t^k))\}\,.
\end{aligned}$$

The first set on the right side of the last $=$ sign is a Borel cylinder in X and is therefore measurable. The second and third sets have probability measure 0 and hence are also measurable sets. Thus $\Pi^{-1}(E)$ is a measurable set. On the other hand, if the set A does contain the point $(0, 0, \cdots 0)$, then

$$\begin{aligned}
\Pi^{-1}(E) &= \{x \in X\colon (x^*(t^1), \cdots x^*(t^k)) \in A\}\\
&\quad \cup \{x \in X\colon x \text{ is discontinuous in } D\}\,.
\end{aligned}$$

The probability of the second set on the right of this equality is zero, therefore again $\Pi^{-1}(E)$ is a measurable set. Consequently, Π is a measurable map.

For Borel sets $E = \{x \in C_T\colon (x(t^1), \cdots x(t^k)) \in A\}$, define $\mu(E) = P\{\Pi^{-1}(E)\}$. Thus

$$\begin{aligned}
\mu(E) &= \mu\{x \in C_T\colon (x(t^1), \cdots x(t^k)) \in A\}\\
&= P\{x \in X\colon (x^*(t^1), \cdots x^*(t^k)) \in A\}\\
&= P\{x \in X\colon (x(t^1), \cdots x(t^k)) \in A\}\\
&= P^{t^1, \cdots t^k}(A)\,.
\end{aligned}$$

THEOREM 1. *Let $\{\xi_t; t \in H\}$ be a Gaussian process, $E(\xi_t) = 0$, and T a compact subset of the set $\{t\colon a_n \leqq t_n \leqq a_n + 1/2^n\}$. If there are constants $\alpha > 0$ and k such that*

$$E(|\,\xi_t - \xi_s\,|^2) \leqq K\|\,t - s\,\|^\alpha$$

for t, s in H, then almost all sample functions of the process are Lipschitz-β continuous in T for $0 < \beta < \alpha/2$.

Proof. Since continuity in a compact set implies continuity in any compact subset of the set, we may assume without loss of generality that $T = \{t: 0 \leqq t_m \leqq 1/2^m, m = 1, 2, \cdots\}$. Let D be the subset of all dyadic numbers in T, i.e., t in D implies that $t = (t_1, t_2, \cdots, t_m, \cdots)$ where each t_m is of the form $t_m = k/2^n$. Write $h_n = 1/2^n$; $\vec{k} = (k_1, k_2, \cdots, k_m, \cdots)$, $\vec{j} = (j_1, j_2, \cdots, j_m, \cdots)$ where the k_m and j_m are positive integers. Write $\vec{k}h_n = (k_1h_n, k_2h_n, \cdots, k_mh_n, \cdots)$. Let $I_{nk} = \{t: k_mh_n \leqq t_m \leqq (k_m + 1)h_n\}$. Observe that every dyadic number t in $T \cap I_{nk}$ is of the form

$$(1) \qquad k_mh_n + \sum_{r=1}^{M_m} \theta_r h_{n+r-1} \text{ where } \theta_r = 0 \text{ or } 1 \text{ for } m \leqq n$$
$$0 + \sum_{r=1}^{M_m} \theta_r h_{n+r-1} \text{ where } \theta_r = 0 \text{ or } 1 \text{ for } m > n\ .$$

Note that t in T implies that $0 \leqq t_m \leqq 1/2^m = h_m$. Use the notation $\varepsilon_m = (0, 0, \cdots, \underset{\substack{\uparrow \\ m\text{th place}}}{1}, 0, 0, \cdots)$.

Once we have chosen β such that $0 < \beta < \alpha/2$, choose θ such that

$$0 < \beta < \theta < \alpha/2\ .$$

Then for any k, ε_m, h_n and

$$\sigma_n^2 = E(|\,\xi_{(\vec{k}+\varepsilon_m)h_n} - \xi_{\vec{k}h_n}\,|^2) \leqq kh_n^\alpha$$

we have

$$\begin{aligned}
&P\{|\,\xi_{(\vec{k}+\varepsilon_m)h_n} - \xi_{\vec{k}h_n}\,| \geqq \|\,(\vec{k} + \varepsilon_m)h_n - \vec{k}h_n\,\|^\theta\} \\
&= P\{|\,\xi_{(\vec{k}+\varepsilon_m)h_n} - \xi_{\vec{k}h_n}\,| \geqq h_n^\theta\} = 2\int_{h_n^\theta}^\infty \frac{1}{\sqrt{2\pi\sigma_n^2}} \exp\left(-\frac{u^2}{2\sigma_n^2}\right) du \\
&= 2\int_{\frac{h_n^\theta}{\sigma_n}}^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{w^2}{2}\right) dw \quad \text{setting } w = \frac{u}{\sigma_n} \\
&\leqq 2\int_{\frac{h_n^{\theta-(\alpha/2)}}{\sqrt{k}}}^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{w^2}{2}\right) dw \leqq \frac{2\sqrt{k}}{\sqrt{2\pi}\, h_n^{\theta-(\alpha/2)}} \exp\left(-\frac{1}{2k}\,(h_n^{\theta-(\alpha/2)})^2\right).
\end{aligned}$$

Let $\gamma = ((\alpha/2) - \theta) > 0$, then we have

$$\begin{aligned}
&P\{|\,\xi_{(\vec{k}+\varepsilon_m)h_n} - \xi_{\vec{k}h_n}\,| \geqq h_n^\theta\} \\
&\qquad \leqq \frac{2\sqrt{k}}{\sqrt{2\pi}\, h_n^{-\gamma}} \exp\left(-\frac{1}{2k}\,(h_n^{-\gamma})^2\right) = \frac{2\sqrt{k}}{\sqrt{2\pi}\, 2^{n\gamma} \exp\left(\dfrac{1}{2k}\, 2^{2n\gamma}\right)}\,.
\end{aligned}$$

Hence,

$$\sum_n P\left\{\max_{\vec{k},m,(\vec{k}+\varepsilon_m)h_n \in D, \vec{k}h_n \in D} |\,\xi_{(\vec{k}+\varepsilon_m)h_n} - \xi_{\vec{k}h_n}\,| \geqq \frac{1}{2^{n\theta}}\right\}$$

$$\leqq \sum_n (2^{n(n-1)/2})\, n \left(\frac{2\sqrt{k}}{\sqrt{2\pi}\, 2^{n\gamma} \exp\left(\frac{1}{2k}\, 2^{2n\gamma}\right)} \right) < \infty\ .$$

To prove this, simply observe that

$$\left(2^{n(n-1)/2} \Big/ \exp \frac{1}{2k}\, 2^{2n\gamma}\right) < 1$$

for n sufficiently large, and $\sum_n (n/2^{n\gamma}) < \infty$. Also, observe that the number $(2^{n(n-1)/2}) \cdot n$ represents the maximum number of possible ways of choosing $\vec{k}$ and m such that $(\vec{k} + \varepsilon_m)h_n$ and $\vec{k}h_n$ are in D; i.e., there are n ways of choosing m and for each of these there are 2^{n-1} ways of choosing k_1, 2^{n-2} ways of choosing $k_2, \cdots 1$ way of choosing k_n, hence $n(2^{n-1} \cdot 2^{n-2} \cdots 2^1) = (2^{n(n-1)/2}) \cdot n$.

Now we may apply the Borel-Cantelli lemma to conclude that for almost all sample functions x, there is an integer N (dependent on x) such that $n \geqq N$ implies that

$$| x((\vec{k} + \varepsilon_m)h_n) - x(\vec{k}h_n) | < \frac{1}{2^{n\theta}} \tag{2}$$

for all $\vec{k}$, m such that $(\vec{k} + \varepsilon_m)h_n$ and $\vec{k}h_n$ are in D.

Let x be a sample function for which (2) holds whenever $n \geqq N$. Let t be in $I_{nk} \cap D$, $n \geqq N$. Then we may apply (1) and by repeated application of the triangle inequality

$$\begin{aligned} &| x(t) - x(\vec{k}h_n) | \\ &\quad \leqq \sum_{r=1}^{M_1} \left(\frac{1}{2^{n+r-1}}\right)^\theta + \sum_{r=1}^{M_2} \left(\frac{1}{2^{n+r-1}}\right)^\theta + \cdots \sum_{r=1}^{M_n} \left(\frac{1}{2^{n+r-1}}\right)^\theta \\ &\quad + \sum_{r=1}^{M_{n+1}} \left(\frac{1}{2^{n+r}}\right)^\theta + \sum_{r=1}^{M_{n+2}} \left(\frac{1}{2^{n+1+r}}\right)^\theta + \cdots \\ &\quad \leqq n \sum_{r=1}^{\infty} \left(\frac{1}{2^{n+r-1}}\right)^\theta + \sum_{r=1}^{\infty} r \left(\frac{1}{2^{n+r}}\right)^\theta \leqq (n+1) \sum_{r=1}^{\infty} r \left(\frac{1}{2^{n+r-1}}\right)^\theta \\ &\quad = \frac{n+1}{2^{n\theta}} \sum_{r=1}^{\infty} \frac{r}{2^{(r-1)\theta}} = \frac{n+1}{2^{n\theta}} M \text{ where } M = \sum_{r=1}^{\infty} \frac{r}{2^{(r-1)\theta}} < \infty\ . \end{aligned} \tag{3}$$

Now observe that we have chosen θ such that $0 < \beta < \theta < \alpha/2$. This means that $\theta - \beta = \varepsilon > 0$ so that

$$\frac{n+1}{2^{n\theta}} = \frac{n+1}{2^{n\beta + n\varepsilon}} \leqq \frac{1}{2^{n\beta}} \text{ for } n \text{ sufficiently large}\ . \tag{4}$$

Let x be a sample function for which (2), (3), and (4) holds for $n \geqq N$. Let s in D be such that $\| s \| \leqq (1/2^N)$. We can find $n \geqq N$ such that $(1/2^n) \leqq \| s \| \leqq (1/2^{n-1})$. Then for each t in D there are $\vec{k}$ and $\vec{j}$ such that

$$k_m h_n \leqq t_m \leqq (k_m + 1)h_n$$
$$(k_m + j_m)h_n \leqq t_m + s_m \leqq (k_m + j_m + 1)h_n$$

where $j_m = 0$ or 1 for $m \leqq n$, and $j_m = 0$ for $m > n$, $k_m = n$ for $m > n$. Hence it follows that

$$\begin{aligned} |x(t+s) - x(t)| &\leqq |x(t+s) - x((\vec{k} + \vec{j})h_n)| + |x(\vec{k}h_n) - x(t)| \\ &\leqq \frac{(n+1)M}{2^{n\theta}} + \frac{n}{2^{n\theta}} + \frac{(n+1)M}{2^{n\theta}} \leqq \frac{3M}{2^{n\beta}} \leqq \tilde{A}\|s\|^\beta . \end{aligned} \tag{5}$$

Hence we have shown that $\|s\| \leqq 1/2^N$ implies that $|x(t+s) - x(t)| \leqq \tilde{A}\|s\|^\beta$ for all t in D. Now, since s in D implies that $\|s\|^2 \leqq \sum_{m=1}^{\infty}(1/2^m)^2 < \infty$ it is clear that there is a constant A such that $|x(t+s) - x(t)| \leqq A\|s\|^\beta$ for all t, s in D. Apply Lemma 1 to terminate the proof.

Corollary 1. *Let $\{\xi_t; t \in R^N\}$ be a Gaussian process, $E(\xi_t) = 0$, and T a compact subset of R^N. If there are constants $\alpha > 0$ and K such that*

$$E(|\xi_t - \xi_s|^2) \leqq K\|t - s\|^\alpha$$

for t, s in R^N, then almost all sample functions of the process are Lipschitz-β continuous in T for $0 < \beta < \alpha/2$.

Remark. Theorem 1 is an extension of a result of Z. Ciesielski for the 1-dimensional case. [2]

2. A sufficient condition for continuity of sample functions of Gaussian processes with the parameter in a Hilbert space. The following lemma is a formulation of well known results [1] in analytic geometry and topology. It is used in the proof of Theorem 2.

Lemma 2. (a) *Let a K-dimensional simplex be specified by the $K+1$ vertices $p_0, \cdots p_k$. Then every point t in the simplex may be uniquely expressed as $t = \sum_{i=0}^{K} a_i p_i$ where $\sum_{i=0}^{K} a_i = 1$, $a_i \geqq 0$.*

(b) *A K-dimensional parallelopiped can be divided into $K!$ simplexes such that:*

(1) *they are disjoint except at the surfaces*
(2) *their union is the parallelopiped*
(3) *the vertices of the simplex are points that are vertices of the parallelopiped.*

Theorem 2. *Let $\{\xi_t; t \in H\}$ be a Gaussian process, $E(\xi_t) = 0$, and T a compact subset of the set $\{t: a_n \leqq t_n \leqq a_n + 1/2^n\}$. If there are* *constants* $\delta > 0$ *and* K *such that*

$$E(|\xi_t - \xi_s|^2) \leqq \frac{K}{|\log||t-s|||^{4+\delta}}$$

for t, s *in* H, *then almost all sample functions of the process are continuous in* T.

Proof. As in Theorem 1, assume that $T = \{(t_1, \cdots t_n, \cdots)\colon 0 \leqq t_n \leqq 1/2^n\}$. Let $T_n = \{(t_1, t_2, \cdots t_n)$: there is $t = (t_1, \cdots t_n, t_{n+1}, \cdots)$ in $T\}$. For each n, let $G_n = \{(a_1, a_2, \cdots a_n)\colon (a_1, \cdots a_n)$ in T_n and $a_i = j/2^n$, $j = 0, 1, 2, \cdots 2^n\}$. By Lemma 2, G_n consists of the vertices of less than $2^{nn}\, n!$ simplexes in n-dimensional space.

For each integer n, and sample function $x(t)$ we shall define a continuous function $(\Pi_n(x))(t)$ as follows: Consider an arbitrary but fixed point $t = (t_1, t_2, \cdots)$ in T. Write $(t_1, \cdots t_n) = p^{(n)}$. (We will also let $p^{(n)} = (t_1, \cdots t_n, 0, 0, \cdots)$ whenever appropriate.) Let n be fixed. Then either $p^{(n)}$ is on the surface of a simplex defined by points in G_n, or else it is interior to a simplex with vertices say $(p_0, p_1, \cdots p_n)$ where each p_i is in G_n. In either case, $p^{(n)}$ may be uniquely expressed as

$$p^{(n)} = \sum_{i=1}^{n} a_i p_i \text{ where } \sum_{i=0}^{n} a_i = 1;\ a_i \geqq 0\ .$$

For each sample function $x(\cdot)$, we may define a function $\Pi_n(x)$ as follows.

$$(\Pi_n(x))\,(t) = \sum_{i=0}^{n} a_i x(p_i)$$

where the points p_i and the coefficients a_i are chosen as described above. $\Pi_n(x)$ is clearly a continuous function of t for t in T.

We next show that for almost all x, $\{\Pi_n(x)\}$ forms a Cauchy sequence in the complete space of continuous functions f on T with norm defined by $||f|| = \max_{t \in T} |f(t)|$.

In estimating $||\Pi_n(x) - \Pi_{n-1}(x)||$, for a fixed $t = (t_1, t_2, \cdots t_n, t_{n+1}, t_{n+2}, \cdots)$ in T, write $p = (t_1, \cdots t_{n-1}) \overset{\text{or}}{=} (t_1, \cdots t_{n-1}, 0, 0, \cdots)$ and $q = (t_1, \cdots t_{n-1}, t_n) \overset{\text{or}}{=} (t_1, \cdots t_{n-1}, t_n, 0, 0, \cdots)$. Then according to the above discussion, we can express p and q as $p = \sum_{i=0}^{n-1} a_i p_i$; $q = \sum_{i=0}^{n} b_i q_i$ where $\sum_{i=0}^{n-1} a_i = 1$, $\sum_{i=0}^{n} b_i = 1$, $a_i \geqq 0$, $b_i \geqq 0$, and where $(p_0, \cdots p_{n-1})$ and $(q_0, \cdots q_n)$ are the vertices of the simplexes determined by p and q respectively in G_{n-1} and G_n.

$$(\Pi_n(x))\,(t) = \sum_{i=1}^{n} b_i x(q_i);\ (\Pi_{n-1}(x))(t) = \sum_{i=0}^{n-1} a_i x(p_i)$$

$$| (\Pi_n(x))(t) - (\Pi_{n-1}(x))(t) | = \left| \sum_{i=0}^{n} b_i x(q_i) - \sum_{i=0}^{n-1} a_j x(p_j) \right|$$
$$\leqq \max_{i,j} | x(q_i) - x(p_j) | .$$

This last inequality is proved as follows:

$$\left| \sum_{i=0}^{n} b_i x(q_i) - \sum_{j=0}^{n-1} a_j x(p_j) \right| = \left| \sum_i b_i \left(x(q_i) - \sum_j a_j x(p_j) \right) \right|$$
$$\leqq \max_i \left| x(q_i) - \sum_j a_j x(p_j) \right|$$
$$= \max_i \left| \sum_j a_j (x(q_i) - x(p_j)) \right|$$
$$\leqq \max_i \max_j | x(q_i) - x(p_j)$$
$$= \max_{i,j} | x(q_i) - x(p_j) | .$$

Observe also that because of the way the p_j and q_i are chosen, $| q_i - p_j | \leqq (\sqrt{n}/2^{n-1})$.

Hence,

$$|| \Pi_n(x) - \Pi_{n-1}(x) || = \max_{t \in T} | (\Pi_n(x))(t) - (\Pi_{n-1}(x))(t) |$$
$$\leqq \max_S \{ \max_{i,j} | x(q_i) - x(p_j) | \}$$

where S is the following set. Let $(p_0, \cdots p_{n-1})$ denote a simplex in G_{n-1} with vertices $p_0, \cdots p_{n-1}$. Let $(q_0, \cdots q_{n-1}, q_n)$ denote a simplex in G_n with vertices $q_0, \cdots q_{n-1}, q_n$. Then S is the set of all pairs of simplexes $(p_0, \cdots p_{n-1})$ and $(q_0, \cdots q_{n-1}, q_n)$ such that

$$| q_i - p_j | \leqq \frac{\sqrt{n}}{2^{n-1}} , \quad i = 1, \cdots n, \; j = 1, \cdots n - 1 .$$

Since the process is Gaussian, we have

$$P\{x : | x(q_i) - x(p_j) | > a_n\}$$
$$= 2\int_{a_n}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma_n} \exp\left(- \frac{u^2}{2\sigma_n^2} \right) du = 2\int_{\frac{a_n}{\sigma_n}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(- \frac{w^2}{2} \right) dw$$

where

$$\sigma_n^2 = E(| \xi_{q_i} - \xi_{p_j} |^2) \leqq \frac{K}{| \log | q_i - p_j ||^{4+\delta}}$$
$$\leqq \frac{2}{\sqrt{2\pi}\,\dfrac{a_n}{\sigma_n}} \exp\left(- \frac{1}{2} \left(\frac{a_n}{\sigma_n} \right)^2 \right) .$$

Hence

$$P\{x: \| \Pi_n(x) - \Pi_{n-1}(x) \| > a_n\} \leqq \frac{(2^{n\cdot n} n!)\,(n+1)^2 2}{\sqrt{2\pi}\,\dfrac{a_n}{\sigma_n}\exp\left(\dfrac{1}{2}\left(\dfrac{a_n}{\sigma_n}\right)^2\right)}\,.$$

Let

$$a_n = \frac{\sqrt{K}\,n^{1+\varepsilon}}{(n-1)^{2+\delta/2}}\left(\left|\log 2 - \frac{\log\sqrt{n}}{\mathrm{n}-1}\right|\right)^{2+\delta/2};\ \frac{a_n}{\sigma_n} \geqq n^{1+\varepsilon}$$

where $0 < \varepsilon < \delta/2$ so that $\sum a_n < \infty$. Since $|\,q_i - p_j\,| \leqq (\sqrt{n}/2^{n-1})$,

$$\sigma_n \leqq \frac{\sqrt{K}}{|\log|\,q_i - p_j\,||^{4+\delta}} \leqq \frac{\sqrt{K}}{|\log\sqrt{n}/2^{n-1}\,|^{4+\delta}}$$

$$= \frac{\sqrt{K}}{(n-1)^{2+\delta/2}\left|\dfrac{\log\sqrt{n}}{n-1} - \log 2\right|^{2+\delta/2}}\,.$$

Thus

$$\sum_n P\{x: \| \Pi_n(x) - \Pi_{n-1}(x) \| > a_n\}$$

$$\leqq \sum_n \frac{(2^{n\cdot n} n!)\,(n+1)^2 2}{\sqrt{2\pi}\,\dfrac{a_n}{\sigma_n}\exp\left(\dfrac{1}{2}\left(\dfrac{a_n}{\sigma_n}\right)^2\right)}$$

$$\leqq \sum_n \frac{(2^{n\cdot n} n!)\,(n+1)^2 2}{\sqrt{2\pi}\,n^{1+\varepsilon}\exp\left(\dfrac{1}{2}\,(n^{1+\varepsilon})^2\right)}$$

$$= \sum_n \frac{2^{n\cdot n} n!\,(n+1)^2 2}{\sqrt{2\pi}\,n^{1+\varepsilon}\exp\left(\dfrac{1}{2}\,n(2+2\varepsilon)\right)}$$

$$< \sum_n \frac{2^{n\cdot n} n^n (n+1)^2\cdot 2}{\sqrt{2\pi}\,n^{1+\varepsilon}\exp\left(\dfrac{1}{2}\,n(2+2\varepsilon)\right)}\,.$$

Observe that

$$\frac{2^{n\cdot n} n^n (n+1)^2}{\exp\left(\dfrac{1}{2}\,n(2+2\varepsilon)\right)} < \frac{2^{n\cdot n}\cdot 2^{n\cdot n}\cdot 2^{n\cdot n}}{\exp\left(\dfrac{1}{2}\,n^2 n^{2\varepsilon}\right)} < 1$$

for n sufficiently large. Hence

$$\sum_n P\{x: \| \Pi_n(x) - \Pi_{n-1}(x) \| > a_n\}$$

$$< \sum_n \frac{2^{n\cdot n} n^n (n+1)^2\cdot 2}{\sqrt{2\pi}\,n^{1+\varepsilon}\,e^{n^{2+2\varepsilon}}} < M + \sum_{n=N}^{\infty} \frac{e}{n^{1+\varepsilon}} < \infty\,.$$

Hence, by the Borel-Cantelli lemma, for almost all sample functions x,

$$|| \Pi_n(x) - \Pi_{n-1}(x) || \leqq a_n$$

for n sufficiently large (n dependent on x). Thus,

$$\begin{aligned} || \Pi_{n+r} - \Pi_n(x) || &\leqq \sum_{j+1}^{r} || \Pi_{n+j}(x) - \Pi_{n+j-1}(x) || \\ &\leqq \sum_{j=1}^{r} a_{n+j} \longrightarrow 0 \text{ as } n \longrightarrow \infty . \end{aligned}$$

$\{\Pi_n(x)\}$ is therefore a Cauchy sequence for almost all x, and hence converges to a continuous function on T for almost all x.

Let D be the collection of elements of T of the form $t = (t_1, t_2, \cdots t_n, 0, 0, \cdots)$ where $(t_1, \cdots t_n)$ is in G_n for some n. Clearly D is dense in T. The limit function continuous on T for almost every x coincides with x on D. Thus almost all x are continuous in D and Lemma 1 applies.

REMARK. Theorem 2 is an extension of Kolmogorov's well known theorem to the Hilbert space. [7]

3. Extension of results to general processes with the parameter in a Hilbert space.

THEOREM 3. *Let $\{\xi_t; t \in H\}$ be a stochastic process with parameter space H. If there are constants $\delta > 0$, $K > 0$ such that for each positive integer n there is a Q_n such that $(1/2)\delta Q_{n-1} > (Q_n - Q_{n-1})$ and $E(| \xi_t - \xi_s |^{Q_n}) \leqq K^n || t - s ||^{n+n\delta}$ for all t, s, in H, then almost all sample functions of the process are continuous in compact subsets of $\{t: a_n \leqq t_n \leqq a_n + 1/2^n\}$.*

Proof. The proof is exactly that of Theorem 2 if we replace the estimate

$$P\{x: | x(q_i) - x(p_j) | > a_n\} \leqq \frac{2}{\sqrt{2\pi}\left(\dfrac{a_n}{\sigma_n}\right)} \exp\left(-\frac{1}{2}\left(\frac{a_n}{\sigma_n}\right)^2\right)$$

valid for Gaussian processes, by the weaker generalized Chebyshev estimate

$$P\{x: | x(q_i) - x(p_j) | > a_n\} \leqq \frac{E(| \xi_t - \xi_s |^{Q_n})}{a_n^{Q_n}} \leqq \frac{K^n || t - s ||^{n+n\delta}}{a_n^{Q_n}}$$

which is valid for general processes.

THEOREM 4. *Let* $\{\xi_t : t \in T\}$ *be a process with* $T = \{(t_1, t_2, \cdots t_Q) : 0 \leqq t_i \leqq 1,\ i = 1, 2, \cdots Q\}$ *and* $E(\xi_t) = 0$. *If there are constants* $\lambda > 0$, $\alpha > 0$ *and* K *such that*

$$E(|\xi_t - \xi_s|^\lambda) \leqq k\|t - s\|^{Q+\alpha}$$

for all t, s *in* T, *then almost all sample functions of the process are Lipschitz-*β *continuous for* $0 < \beta < \alpha/\lambda$.

Proof. The proof is the Q-dimensional analogue of Theorem 1 with the following modifications.

Choose θ such that $0 < \beta < \theta < \alpha/\lambda$.

Observe that in the case here of general processes we may use the generalized Chebyshev inequality to obtain the following estimate:

$$\begin{aligned} &P\{x : |x((\vec{k} + \varepsilon_i)h_n - x(\vec{k}h_n)| \geqq h_n^\theta\} \\ &\leqq \frac{E(|x((\vec{k} + \varepsilon_i)h_n) - x(\vec{k}h_n)|^\gamma)}{(h_n^\theta)^\lambda} \leqq \frac{kh_n^{Q+\alpha}}{h_n^{\theta\lambda}} = \frac{k}{2^{n(Q+\alpha-\theta\lambda)}} . \end{aligned}$$

Hence

$$\begin{aligned} &\sum_n P\left\{\max_{\vec{k},\vec{i},(\vec{k}+\varepsilon_i)h_n \in D, \vec{k}h_n \in D} |x((\vec{k} + \varepsilon_i)h_n) - x(\vec{k}h_n)| \geqq \frac{1}{2^{n\theta}}\right\} \\ &\leqq \sum_n (2^n)^Q Q\left(\frac{k}{2^{n(Q+\alpha-\theta\lambda)}}\right) = \sum_n \frac{Qk}{2^{n(\alpha-\theta\lambda)}} < \infty \text{ since } \alpha - \theta\lambda > 0 . \end{aligned}$$

Now apply the Borel-Cantelli lemma to conclude that for almost all sample functions x, there is an integer N (dependent on x) such that $n \geqq N$ implies that

$$(6) \qquad |x((\vec{k} + \varepsilon_i)h_n) - x(\vec{k}\,h_n)| < \frac{1}{2^{n\theta}}$$

for all $\vec{k}$, i such that $(\vec{k} - \varepsilon_i)h_n$ and $\vec{k}h_n$ are in D.

Observe that inequality (6) here is identical to inequality (2) in the proof of Theorem 1. From this point onward, the two proofs are identical except $\alpha/2$ should be replaced by α/λ, and ∞-dimensional computations replaced by Q-dimensional computations.

4. Levy's Brownian process. Paul Levy defined a Gaussian process $\{\xi_t ; t \in H\}$ where the parameter space H is the Hilbert space l_2 by specifying that

(1) $E(\xi_t) = 0$ for all $t \in H$

(2) $E(\xi_t\xi_s) = (1/2)\{\|t\| + \|s\| - \|t - s\|\}$.

He showed that this process has the property that almost all sample functions are discontinuous in the Hilbert sphere. [3]

In the following we shall describe the process further by showing that in every compact subset T of $\{t: a_n \leqq t_n \leqq a_n + 1/2^n\}$ almost all sample functions are Lipschitz-β continuous for $0 < \beta < 1/2$ whereas in every noncompact subset T of H of the form $T = \{(t_1, t_2, \cdots t_n, \cdots):$ $a_n \leqq t_n \leqq b_n\}$, almost all sample functions are discontinuous.

LEMMA 3 [6]. *Let* $T = \{t = (t_1, t_2, \cdots t_n, \cdots): 0 \leqq t_n \leqq a_n\}$, $a_n \neq \infty$. *T is compact if and only if* $\sum_{n=1}^{\infty} a_n^2 < \infty$.

In the next lemma, $\{A_n\}$ is any sequence in l_2 with the property that $\| A_n \| = 1$ and $\| A_i - A_j \| = \sqrt{2}$, $i \neq j$.

LEMMA 4. *Let X be the Brownian process defined by the covariance function* $r(s, t) = 1/2\{\| s \| + \| t \| - \| s - t \|\}$ *and* $E(X_t) = 0$. *Set* $X_n = X(A_n)$. If $X_1, X_2, \cdots X_n$ *are known then*

$$X_{n+1} = \mu_n + \sigma_n \tilde{\xi}_n$$

where

$$\mu_n = \frac{1}{n}(X_1 + X_2 \cdots + X_n)$$

$$\sigma_n^2 = \frac{1}{\sqrt{2}} + \frac{1}{n\sqrt{2}}$$

$\tilde{\xi}_n =$ *Gaussian random variable with mean* 0, *variance* 1.

Proof. Using the formula of Paul Levy [5], we have

$$X(B) - X(A) = \xi\sqrt{\| A - B \|}$$

where ξ is a Gaussian random variable with mean 0 and variance 1. Hence we have

$$(7) \qquad \begin{cases} X(A_{n+1}) - X(A_1) = X_{n+1} - X_n = \xi_1\sqrt{\sqrt{2}} \\ \qquad \vdots \\ X(A_{n+1}) - X(A_n) = X_{n+1} - X_n = \xi_n\sqrt{\sqrt{2}} \end{cases}$$

where $\xi_1, \xi_2, \cdots \xi_n$ are Gaussian random variables with mean 0 and variance 1.

Summing equations (7) and dividing by n, we have

$$\begin{aligned} X_{n+1} &= \frac{X_1 + X_2 + \cdots + X_n}{n} + \frac{\sqrt{\sqrt{2}}}{n}\{\xi_1 + \xi_2 \cdots + \xi_n\} \\ &= \mu_n + \sigma_n \tilde{\xi}_n \end{aligned}$$

where

$$\mu_n = \frac{1}{n}(X_1 + X_2 \cdots + X_n)$$

and

$$\sigma_n^2 = E\left\{\left[\frac{\sqrt{\sqrt{2}}}{n}(\xi_1 + \xi_2 + \cdots + \xi_n)\right]^2\right\}$$
$$= \frac{\sqrt{2}}{n^2} E\{(\xi_1 + \xi_2 + \cdots + \xi_n)^2\}$$
$$= \frac{\sqrt{2}}{n^2}(n) E\{\xi_1^2\} + \frac{\sqrt{2}}{n^2} n(n-1) E\{\xi_1\xi_2\}$$

when we use the fact that $E(\xi_1^2) = E(\xi_j^2)$ for all $j = 1, \cdots n$; and $E(\xi_1\xi_2) = E(\xi_i\xi_j)$ for all $i \neq j$. $E(\xi_1^2) = 1$;

$$E(\xi_1\xi_2)$$
$$= E\left\{\frac{1}{\sqrt{\sqrt{2}}}(X_{n+1} - X_1)\frac{1}{\sqrt{\sqrt{2}}}(X_{n+1} - X_2)\right\}$$
$$= \frac{1}{\sqrt{2}} E\{X_{n+1}^2 - X_1X_{n+1} - X_2X_{n+1} + X_1X_2\}$$
$$= \frac{1}{\sqrt{2}}\left\{1 - \frac{1}{2}(1 + 1 - \sqrt{2}) - \frac{1}{2}(1 + 1 - \sqrt{2}) + \frac{1}{2}(1 + 1 - \sqrt{2})\right\}$$
$$= \frac{1}{\sqrt{2}}\left\{1 - \frac{1}{2}(2 - \sqrt{2})\right\} = \frac{1}{\sqrt{2}}\left\{1 - 1 + \frac{\sqrt{2}}{2}\right\} = \frac{1}{2}.$$

Hence

$$\sigma_n^2 = \left(\frac{\sqrt{2}}{n^2}\right)n + \left(\frac{\sqrt{2}}{n^2}\right)n(n-1)\frac{1}{2} = \frac{\sqrt{2}}{n} + \frac{\sqrt{2}}{2n}(n-1)$$
$$= \frac{\sqrt{2}}{n} + \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2n} = \frac{1}{\sqrt{2}} + \frac{1}{n\sqrt{2}}.$$

THEOREM 5. *Let* $\{\xi_t; t \in H\}$ *be the Brownian process defined by the covariance function* $r(s, t) = 1/2\{\|s\| + \|t\| - \|s - t\|\}$ *and* $E(\xi_t) = 0$. *Let* T *be a subset of* H.

(1) *Almost all sample functions of the process are Lipschitz-*β *continuous,* $0 < \beta < (1/2)$, *in* T *if* T *is a compact subset of a set of the form* $\{(t_1, t_2, \cdots t_n, \cdots): a_n \leqq t_n \leqq a_n + (1/2^n)\}$.

(2) *Almost all sample functions of the process are discontinuous in* T *if* T *is not compact, and of the form* $T = \{(t_1, t_2, \cdots t_n, \cdots): a_n \leqq t_n \leqq b_n\}$.

Proof. Part (1) follows from Theorem 1 since

$$\begin{aligned} E(|\xi_t - \xi_s|^2) &= r(t, t) - 2r(t, s) - r(s, s) \\ &= \| t \| - \{\| s \| + \| t \| - \| s - t \|\} + \| s \| \\ &= \| t - s \| \end{aligned}$$

so that conditions of the theorem are satisfied for $\alpha = 1$. Hence we need only prove part (2).

Without loss of generality, we may assume that T is of the form $T = \{t = (t_1, \cdots t_n, \cdots): 0 \leqq t_n \leqq a_n\}$. By Lemma 3, we have $\sum a_n^2 = \infty$ and hence there is a sequence $\{A_n\}$ in T such that $\| A_n \| = 1$, $\| A_i - A_j \| = \sqrt{2}$. Using the notation of Lemma 4, we have

$$X_{n+1} = \mu_n + \sigma_n \tilde{\xi}_n$$

where

$$\mu_n = \frac{1}{n}(X_1 + X_2 \cdots + X_n)$$

$$\sigma_n^2 = \frac{1}{\sqrt{2}} + \frac{1}{n\sqrt{2}}$$

$\tilde{\xi}_n =$ Gaussian random variable with mean 0 and variance 1.

Let M be an arbitrarily large real number. If we are given $|X_1| < M$, $|X_2| < M, \cdots |X_n| < M$, then

$$\mu_n = \frac{1}{n}(X_1 + X_2 \cdots + X_n) < M$$

so that the conditional probability

$$\begin{aligned} P\{&|X_{n+1}| < M / |X_1| < M, |X_2| < M, \cdots |X_n| < M\} \\ &= P\{|\mu_n + \sigma_n \tilde{\xi}_n| < M\} \\ &\leqq P\{|\sigma_n \tilde{\xi}_n| < M + |\mu_n|\} \leqq P\{|\sigma_n \tilde{\xi}_n| < 2M\} \\ &\leqq P\left\{|\tilde{\xi}_n| < \frac{2M}{\sigma_n}\right\} \leqq P\left\{|\tilde{\xi}_n| < \frac{2M}{1/\sqrt{\sqrt{2}}}\right\} \\ &= K_M < 1 \end{aligned}$$

since

$$\sigma_n < \frac{1}{\sqrt{\sqrt{2}}}.$$

Hence

$$\begin{aligned} P\{&|X_1| < M, |X_2| < M, \cdots |X_n| < M, \cdots\} \\ &= P\{|X_1| < M\} \cdot P\{|X_2| < M / |X_1| < M\} \cdots \\ &\quad \times P\{|X_{n+1}| < M / |X_1| < M, \cdots |X_n| < M\} \cdots \end{aligned}$$

$$\leqq \lim_{n\to\infty} (k_M)^n = 0 \text{ for any } M.$$

Hence, the sequence $\{X_n\}$ is almost surely not bounded, which is to say that almost all sample functions of the process are discontinuous in T.

The proof of part 2 of Theorem 5 incorporates many of the steps in Paul Levy's proof that almost all sample functions of the Brownian process are discontinuous in the unit sphere.

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