PACIFIC JOURNAL OF MATHEMATICS
Vol. 19, No. 1, 1966

AN ASYMPTOTIC PROPERTY OF THE EULER FUNCTION

R. J. MIECH

Let $\varphi(n)$ denote the Euler function. The starting point of this paper is the simple observation that if p is a prime then p and $\varphi(p)+1=p$ have a common divisor which is greater than 1; its conclusion is: if $\{m_k\}$ is the sequence of positive square free integers which have k prime factors, where $k \geqq 2$, then the number of integers m_k not exceeding x such that m_k and $\varphi(m_k)+1$ have a common divisor other than 1 is asymptotic to

$$\lambda_k \frac{x}{\log x} (\log \log x)^{k-2} ,$$

where λ_k is a positive constant that depends on k.

The source of the problem under consideration was a question raised by Gordon in the course of his investigations of Hajós factorization of abelian groups. The question was: are there integers n, other than primes and their doubles, such that $\varphi(n)+1$ divides n. This is still an open problem. However, if we relax our demands, as we have done above, it is possible to prove the asymptotic relation stated there.

One of the main results needed to establish the first assertion of this paper is:

LEMMA 1. *Let a be a positive integer and $b_1, \cdots, b_k$ be a set of integers such that $0 < b_i < a$ and $(b_i, a) = 1$ for $i = 1, 2, \cdots, k$. Let $b'_1, \cdots, b'_q$ denote the distinct integers which appear in the sequence $b_1, \cdots, b_k$ and suppose that b'_j appears r_j times for $j = 1, 2, \cdots, q$. Let $\pi(x; a, b_1, \cdots, b_k)$ denote the number of square free integers n not exceeding x such that $n = p_1 p_2 \cdots p_k$ where p_i is a prime and $p_i \equiv b_i \bmod a$ for $i = 1, 2, \cdots, k$. Then for $k \geqq 2$ we have*

$$\pi(x; a, b_1, \cdots, b_k) = \frac{1}{r_1! \cdots r_q!} \frac{k}{\varphi^k(a)} \frac{\log_2^{k-1} x}{\log x} \left[1 + \mathcal{O}\left(\frac{\log_3 x}{\log_2 x}\right)\right]$$

uniformly for $a \leqq \log_{k+1} x$, where $\log_j x$ is the jth iterated logarithm of x. The constant implied by the $\mathcal{O}$-term depends on k.

The proof of Lemma 1 is based on a generalization of one of Wright's ideas [2]. We begin this proof by listing several known results about primes in arithmetic progressions.

LEMMA 2. *Let $\pi(x; a, b)$ denote the number of primes not exceeding x which are congruent to b modulo a, where $0 < b < a$ and $(a, b) = 1$. Then if*

$$a \leqq \exp [c_1 \log x/\log_2 x] ,$$

where c_1 is an absolute constant, we have

$$\pi(x; a, b) = \frac{1}{\varphi(a)} \frac{x}{\log x} \left[1 + \mathcal{O}\left(\frac{1}{\log x}\right)\right]$$

except, possibly, for a set of integers $\{a'\}$ all of which are multiples of a single integer a'' which, in turn, is greater than $\log^A x$, where A is any fixed positive constant. The constant implied by the $\mathcal{O}$-term depends on A.

See Chapter 9, Theorem 2.3 of [1] for a proof.

Frequent use will be made of the following form of this lemma: if $\log x \leqq u$ and $a \leqq \log_2 x$ then

$$\pi(u; a, b) = \frac{1}{\varphi(a)} \frac{u}{\log u} \left[1 + \mathcal{O}\left(\frac{1}{\log u}\right)\right],$$

where the constant implied by the $\mathcal{O}$-term is independent of u and x.

We shall also employ:

LEMMA 3. *If $a < x$ then there is an absolute constant c_2 such that*

$$\pi(x; a, b) \leqq \frac{c_2}{\varphi(a)} \frac{x}{\log (x/a)} .$$

See [1], Chapter 2, Theorem 4.1 for a proof.

The next lemma is a straightforward consequence of Lemmas 2 and 3.

LEMMA 4. *If $x \geqq 27$, $0 < b < a$ and $(a, b) = 1$ then*

$$\sum_{\substack{p \leqq x \\ p \equiv b \bmod a}} \frac{1}{p} = \frac{1}{\varphi(a)} \log_2 x + \mathcal{O}\left(\frac{\log_3 x}{\varphi(a)}\right)$$

uniformly for all integers $a \leqq 2 \log_3 x$.

The balance of this section deals with the proof of Lemma 1. As for notation, $a, b_1, \cdots,$ and b_k will be the integers defined in Lemma 1, any prime p_i which occurs will be congruent to b_i modulo a for $i = 1, 2, \cdots, k$, and a prime on a summation symbol (as in $\sum'$) will indicate that any prime p_i appearing in the index of summation is congruent to b_i modulo a. The symbols $c_1, c_2, \cdots$ will denote constants that depend, at most, on k. We also assume that $k \geqq 2$.

LEMMA 5. *Let*

$$L(x; a, b_1, \cdots, b_k) = \sum_{p_1 \cdots p_k \leqq x}' \frac{1}{p_1 \cdots p_k}$$

where the primes p_i run independently through the residue classes b_i. Then if $x \geqq c_3$ we have

$$L(x; a, b_1, \cdots, b_k) = \frac{1}{\varphi^k(a)} \log_2^k x \left[1 + \mathcal{O}\left(\frac{\log_3 x}{\log_2 x}\right)\right]$$

uniformly for $a \leqq \log_3 x$.

Proof. Since

$$\prod_{i=1}^k \left(\sum_{p_i \leqq x^{1/k}}' \frac{1}{p_i}\right) \leqq L(x; a, b_1, \cdots, b_k) \leqq \prod_{i=1}^k \left(\sum_{p_i \leqq x}' \frac{1}{p_i}\right)$$

and since a is chosen so that

$$a \leqq \log_3 x \leqq 2 \log_3 x^{1/k} ,$$

Lemma 5 follows from Lemma 4.

LEMMA 6. *Let*

$$\vartheta(x; a, b_1, \cdots b_k) = \sum_{p_1 p \cdots_k \leqq x}' \log p_1 \cdots p_k$$

where the primes p_i run independently through the residue classes b_i. Then if $x > c_4$ and $a \leqq \log_{k+1} x$ we have

$$\vartheta(x; a, b_1, \cdots, b_k) = \frac{k}{\varphi^k(a)} x \log_2^{k-1} x \left[1 + \mathcal{O}\left(\frac{\log_3 x}{\log_2 x}\right)\right] .$$

The proof is an inductive one. For $k = 2$ we have

$$\vartheta(x; a, b_1, b_2) = \sum_{p_2 \leqq x}' \sum_{p_1 \leqq x/p_2}' \log p_1 + \sum_{p_2 \leqq x}' \sum_{p_1 \leqq x/p_1}' \log p_2 .$$

Now, the first double sum on the right hand side of this equation can be expressed as the sum of three double sums, $\sum_1, \sum_2, \sum_3$ whose indices of summation are

$$p_2 \leqq x/\log x \ p_1 \leqq \log x, \quad p_2 \leqq x/\log x \quad \log x < p_1 \leqq x/p_2$$
$$x/\log x < p_2 \leqq x \quad p_1 \leqq x/p_2 ,$$

respectively. Let us consider $\sum_2$ first. We have, by Lemma 2

$$\sum_{\log x < p_1 \leqq x/p_2}' \log p_1$$
$$= \frac{1}{\varphi(a)}\left[\frac{x}{p_2} - \log x + \mathcal{O}\left(\frac{x}{p_2}\frac{1}{\log_2 x}\right)\right].$$

Moreover, since $2 \log_3(x/\log x) \geqq \log_3 x \geqq a$, Lemma 4 can be applied. Doing so we find that

$$\sum_{p_2 \leqq x/\log x}' \sum_{\log x < p_1 \leqq x/p_2}' \log p_1 = \frac{x}{\varphi^2(a)} \log_2 x\left[1 + \mathcal{O}\left(\frac{\log_3 x}{\log_2 x}\right)\right].$$

A straightforward application of Lemma 3 will lead us to tolerable bounds for $\sum_1$ and $\sum_3$. If we do this and then apply the same argument to the second double sum that appears in the equation for $\vartheta(x; a, b_1, b_2)$ we will have our result for $k = 2$.

Let us go on to the induction. Set

$$\begin{aligned} f(x/p_i) &= \vartheta(x/p_i; a, b_1, \cdots, b_{i-1}, b_{i+1}, \cdots, b_{k+1}) \quad \text{for} \quad 1 \leqq i \leqq k+1, \\ g(x/p_i) &= L(x/p_i; a, b_1, \cdots, b_{i-1}, b_{i+1}, \cdots, b_k) \quad \text{for} \quad 1 \leqq i \leqq k \\ g(x/p_{k+1}) &= L(x/p_{k+1}; a, b_2, \cdots, b_k) \end{aligned}$$

and

$$h(x/p_i) = f(x/p_i) - (k/\varphi(a))(x/p_i)g(x/p_i) \qquad \text{for} \quad 1 \leqq i \leqq k+1.$$

Then, since

$$k\vartheta(x; a, b_1, \cdots, b_{k+1}) = \sum_{i=1}^{k+1} \sum_{p_i \leqq x}' f(x/p_i)$$

and since for $1 \leqq i \leqq k$,

$$L(x; a, b_1, \cdots, b_k) = \sum_{p_i \leqq x}' (1/p_i)g(x/p_i) ,$$

it follows that

$$\begin{aligned} k\vartheta(x; a, b_1, \cdots, b_{k+1}) - \frac{k^2}{\varphi(a)} xL(x; a, b_1, \cdots, b_k) \\ - \frac{k}{\varphi(a)} xL(a, b_2, \cdots, b_{k+1}) \\ = \sum_{i=1}^{k+1} \sum_{p_i \leqq x}' h(x/p_i) . \end{aligned}$$

Now, if $p_i \leqq x/\log x$ then $x/p_i \geqq \log x$ and $\log_{k+1}(x/p_i) \geqq \log_{k+2} x \geqq a$. Thus, by the inductive hypothesis and Lemma 5, we have

$$\sum_{p_i \leqq x/\log x}' h(x/p_i) = \mathcal{O}\left[\frac{x}{\varphi^{k+1}(a)} (\log_3 x) \log_2^{k-1} x\right].$$

If, on the other hand, we have $x/\log x < p_i \leqq x$ then $x/p_i \leqq \log x$, and we can show, by referring to the definitions of the quantities involved, that

$$\sum_{x/\log x<p_i\leqq x}' f(x/p_i) = \mathcal{O}\left[\frac{x}{\varphi^{k+1}(a)}(\log_3 x)\log_2^{k-1} x\right]$$

and

$$\sum_{x/\log x<p_i\leqq x}' \frac{k}{\varphi(a)}\frac{x}{p_i} g(x/p_i) = \mathcal{O}\left[\frac{x}{\varphi^{k+1}(a)}(\log_3 x)\log_2^{k-1} x\right].$$

In short, we have

$$\begin{aligned} k\vartheta(x; a, b_1, \cdots, b_{k+1}) &= \frac{k^2}{\varphi(a)} xL(x; a, b_1, \cdots, b_k) \\ &+ \frac{kx}{\varphi(a)} L(x; a, b_2, \cdots, b_{k+1}) \\ &+ \mathcal{O}\left[\frac{k}{\varphi^{k+1}(a)} \log_3 x \log_2^{k-1} x\right]. \end{aligned}$$

Lemma 6 follows from this formula and Lemma 5.

LEMMA 7. *Let $d(n; a, b_1, \cdots, b_k)$ be the number of representations of the integer n of the form $n = p_1 \cdots p_k$ where $p_i \equiv b_i \bmod a$ for $i = 1, \cdots, k$ and the primes p_i run independently through the residue classes b_i. Then we have*

$$\sum_{n\leqq x} d(n; a, b_1, \cdots, b_k) = \frac{k}{\varphi^k(a)} x \frac{\log_2^{k-1} x}{\log x}\left[1 + \mathcal{O}\left(\frac{\log_3 x}{\log_2 x}\right)\right]$$

uniformly for $a \leqq \log_{k+1} x$.

Proof. Set $d_k(n) = d(n; a, b_1, \cdots, b_k)$. Then we have

$$\begin{aligned} \vartheta(x; a, b_1, \cdots, b_k) &= \sum_{n\leqq x} d_k(n)\log n = (\sum_{n\leqq x} d_k(n))\log x \\ &- \int_2^x (\sum_{n\leqq u} d_k(n)) d(\log u). \end{aligned}$$

Since $d_k(n) \leqq k!$ and since $d_k(n)$ is positive only if $n \equiv b_1 \cdots b_k \bmod a$ it follows that

$$\sum_{n\leqq u} d_k(n) \leqq k!\left(\frac{u}{\varphi(a)} + 1\right).$$

Thus

$$\int_2^x (\sum_{n\leqq u} d_k(n)) d(\log u) \leqq \frac{c_5 x}{\varphi(a)} \leqq \frac{c_6 x}{\varphi^k(a)}(\log_3 x)(\log_2^{k-2} x).$$

These results, along with Lemma 6, give us Lemma 7.

We are now in a position to prove Lemma 1. Set

$$e(n; a, b_1, \cdots, b_k) = \mu^2(n) d(n; a, b_1, \cdots, b_k),$$

where $\mu(n)$ is the Möbius function. Then

$$0 \leqq \sum_{n \leqq x} d(n; a, b_1, \cdots, b_k) - \sum_{n \leqq x} e(n; a, b_1, \cdots, b_k)$$
$$\leqq \sum_{i=1}^{k} \sum_{n \leqq x} d(n; a, b_1, \cdots, b_{i-1}, b_{i+1}, \cdots, b_k) \leqq \frac{c_6 x}{\varphi^{k-1}(a)} \frac{\log_2^{k-2} x}{\log x} .$$

Consequently

$$\sum_{n \leqq x} e(n; a, b_1, \cdots, b_k) = \frac{kx}{\varphi^k(a)} \frac{\log_2^{k-1} x}{\log x} \left[1 + \mathcal{O}\left(\frac{\log_3 x}{\log_2 x}\right)\right] .$$

This completes the proof of Lemma 1 since

$$\pi(x; a, b_1, \cdots, b_k) = \frac{1}{r_1! \cdots r_q!} \sum_{n \leqq x} e(n; a, b_1, \cdots, b_k) .$$

2. In this section we shall prove the assertion made in the introduction of this paper. To that end, let $\Phi'(k, x)$ denote the number of integers in the set

$$\{m : 1 < m \leqq x, m = p_1, \cdots, p_k, \mu^2(m) = 1, (m, \varphi(m) + 1) > 1\} ,$$

and let $\Phi(k, x)$ be the number of *odd* integers counted by $\Phi'(k, x)$. Let n_j be a generic symbol for an odd positive square free integer which has j prime factors, for $j = 1, \cdots, k$.

Our first goal is formula (3) below. Suppose we have $n = n_k = p_1 \cdots p_k$, $d \mid (n, \varphi(n) + 1)$, and $d > 1$; then d is a square free integer which has j prime factors where $1 \leqq j \leqq k - 1$. Thus if we set

$$A(n_j) = \{n_{k-j} : n_j n_{k-j} \leqq x, (n_j, n_{k-j}) = 1, \varphi(n_j n_{k-j}) + 1 \equiv 0 \bmod n_j\}$$

an elementary combinatorial argument will yield the equation

$$\Phi(k, x) = \sum_{j=1}^{k-1} (-1)^{j+1} \sum_{n_j \leqq x} \sum_{n_{k-j} \in A(n_j)} 1 .$$

Consider next the quantity

$$(1) \qquad \sum_{\log_{k+1} x < n_j \leqq x} \sum_{n_{k-j} \in A(n_j)} 1 .$$

Since we have $n_j = p_1 \cdots p_j$ at least one of the prime factors of n_j, say p_1, must be greater than $(\log_{k+1} x)^{1/j} = z(j, x)$. Moreover, if n_j and n_{k-j} are relatively prime integers such that

$$\varphi(n_j n_{k-j}) + 1 \equiv 0 \bmod n_j$$

we then have $n_j n_{k-j} = p_1 n_{k-1}$ and

$$\varphi(p_1 n_{k-1}) + 1 \equiv 0 \bmod p_1 .$$

Consequently the quantity (1) is bounded above by

$$\binom{k}{j} \sum_{p_1 > z(j,x)} \sum_{n_{k-1} \in A(p_1)} 1 , \tag{2}$$

for any integer $n_j n_{k-j}$ which appears in (1) will appear at most $\binom{k}{j}$ times in (1) and at least once in the double sum of (2). If we set $z = z(k, x) = (\log_{k+1} x)^{1/k}$ we have

$$\Phi(k, x) = \sum_{j=1}^{k-1} (-1)^{j+1} \sum_{n_j \leqq \log_{k+1} x} \sum_{n_{k-j} \in A(n_j)} 1 + \mathcal{O}(S) , \tag{3}$$

where

$$S = \sum_{p_1 > z} \sum_{n_{k-1} \in A(n_1)} 1 .$$

We shall now show that if $k \geqq 2$ then

$$\begin{aligned} &\sum_{j=1}^{k-1} (-1)^{j+1} \sum_{n_j \leqq \log_{k+1} x} \sum_{n_{k-j} \in A(n_j)} 1 \\ &\quad = \alpha_k x \frac{\log_2^{k-2}}{\log x} \left[1 + \mathcal{O}\left(\frac{1}{\log_{k+1} x}\right)\right] , \end{aligned} \tag{4}$$

where α_k is a constant that depends on k.

Consider any fixed n_j which appears in (4). If $n_{k-j} \in A(n_j)$ then $(n_{k-j}, n_j) = 1$ and $\varphi(n_j)\varphi(n_{k-j}) + 1 \equiv 0 \bmod n_j$. Thus, if $(n_j, \varphi(n_j)) > 1$ the set $A(n_j)$ is empty. On the other hand if $(\varphi(n_j), n_j) = 1$, and $n_{k-j} = p_{j+1} \cdots p_k$, we have the congruence

$$(p_{j+1} - 1) \cdots (p_k - 1) \equiv l(n_j) \bmod n_j \tag{5}$$

where $l(n_j)$ is chosen so that $l(n_j)\varphi(n_j) \equiv -1 \bmod n_j$. Furthermore, if $p_{j+1} \cdots p_k$ is a set of primes that satisfies (5) then there is a set of integers $l_{j+1}, \cdots, l_k$ such that

$$l_{j+1} \cdots l_k \equiv l(n_j) \bmod n_j \tag{6}$$

$$(1 + l_i, n_j) = 1 \quad \text{for} \quad i = j + 1, \cdots, k , \tag{7}$$

for we need only take l_i so that $p_i \equiv 1 + l_i \bmod n_j$. Conversely, if $l_{j+1}, \cdots,$ and l_k are integers which satisfy (6) and (7) then there are primes $p_{j+1}, \cdots, p_k$ which satisfy (5). Note also that the number of distinct solutions of (6), where two solutions, $l_{j+1}, \cdots, l_k$ and $l'_{j+1}, \cdots, l'_k$, are said to be the same if and only if both contain the same integers modulo a to the same multiplicity, obviously does not exceed $\varphi^{k-j-1}(a)$; thus the number of solutions of (6) which also satisfy (7) is bounded above by $\varphi^{k-j-1}(a)$.

Now, suppose that $(n_j, \varphi(n_j)) = 1$, let $l_{j+1}, \cdots, l_k$ be a set of integers that satisfies (6) and (7), and let $b_i = 1 + l_i$ for $i = j + 1, \cdots, k$.

Then we have

$$\lambda(n_j) = \sum_{n_{k-j} \in A(n_j)} 1 = \sum_{\{b_{j+1}, \cdots, b_k\}} \pi\Big(\frac{x}{n_j}\,; n_j, b_{j+1}, \cdots, b_k\Big)$$

where $\{b_{j+1}, \cdots, b_k\}$ runs over the sets of integers we get when $\{l_{j+1}, \cdots, l_k\}$ runs over the distinct solutions of (6) which satisfy (7). Lemma 1 will be applicable here if

$$\log_{k-j+1}(x/n_j) \geqq n_j\ ,$$

but this is the case if $n_j \leqq \log_{k+1} x$ since

$$\log_{k-j+1}(x/n_j) \geqq \log_{k-j+1}(x/\log_{k+1} x) \geqq \log_k(x/\log_{k+1} x) \geqq \log_{k+1} x \geqq n_j$$

for $x \geqq c_1$, c_1 being a constant that depends on k. Consequently if $j \leqq k-2$ and $n_j \leqq \log_{k+1} x$ then

$$\lambda(n_j) = \frac{a(n_j)}{\varphi^{k-j}(n_j)} \frac{x}{n_j} \frac{\log_2^{k-j-1} x}{\log x}\Big[1 + \mathcal{O}\Big(\frac{\log_3 x}{\log_2 x}\Big)\Big] \tag{8}$$

where $a(n_j)$ is an integer such that $a(n_j) \leqq k\varphi^{k-j-1}(n_j)$. Lemma 2 implies that (8) also holds if $j = k-1$.

If we take $j = 1$ we have, by (8)

$$\begin{aligned}\sum_{n_1 \leqq \log_{k+1} x} \ \sum_{n_{k-1} \in A(n_1)} 1 \\ = \sum_{n_1 \leqq \log_{k+1} x} \frac{a(n_1)}{\varphi^{k-1}(n_1)} \frac{x}{n_1} \frac{\log_2^{k-2} x}{\log x}\Big[1 + \mathcal{O}\Big(\frac{\log_3 x}{\log_2 x}\Big)\Big]\ .\end{aligned} \tag{9}$$

Set

$$\alpha_k = \sum_{n_1} \frac{a(n_1)}{\varphi^{k-1}(n_1) n_1}\ .$$

Since $a(n_1) \leqq k\varphi^{k-2}(n_1)$ this infinite series converges. Furthermore, $\alpha_k \neq 0$. For, since n_1 is a prime, say $n_1 = p_1$, $a(p_1)$ is the number of solutions of the congruence $l_2 \cdots l_k \equiv 1 \bmod p_1$ such that $(1 + l_i, p_1) = 1$ for $i = 2, \cdots, k$. Since the set of values $l_2 = l_3 = \cdots = l_k = 1$ satisfies these conditions we have $a(n_1) = a(p_1) > 0$ for $p_1 \geqq 3$, i.e. $\alpha_k \neq 0$. In short, the left hand side of (9) is equal to

$$\frac{\alpha_k x \log_2^{k-2} x}{\log x}\Big[1 + \mathcal{O}\Big(\frac{1}{\log_{k+1} x}\Big)\Big]\,,$$

where α_k is a positive constant.

If $2 \leqq j \leqq k-1$ then we have, by (8)

$$\sum_{n_j \leqq \log_{k+1} x} \ \sum_{n_{k-j} \in A(n_j)} 1 = \mathcal{O}\Big[\frac{x \log_2^{k-3} x}{\log x}\Big]\ .$$

Formula (4) follows from this and preceding result.

The main problem that remains is that of finding a reasonable bound for the quantity S where

$$S = \sum_{p_1 > z} \sum_{n_{k-1} \in A(p_1)} 1$$

and $z = (\log_{k+1} x)^{1/k}$.

To that end, fix p_1 and define $B(p_1, i)$ to be the set of integers $p_2 \cdots p_k$ in $A(p_1)$ such that $i - 1$ of the prime factors of $p_2 \cdots p_k$ are less than p_1 and $k - i$ are greater than p_1, for $i = 1, \cdots, k$. Then

$$S = \sum_{p_1 > z} \sum_{i=1}^{k} \sum_{n_{k-1} \in B(p_1, i)} 1 \, . \tag{10}$$

If we fix p_1 and i, where $2 \leqq i \leqq k - 1$, we have

$$\sum_{n_{k-1} \in B(p_1, i)} 1 = \sum_{p_2 \cdots p_i} \sum_{p_{i+1} \cdots p_k \in C} 1$$

where each of the prime factors of $p_2 \cdots p_i$ is less than p_1 and C, which depends on $p_1, \cdots,$ and p_i, is the set of integers $p_{i+1} \cdots p_k$ such that $p_j > p_1$ for $j > i$, $p_1 \cdots p_i p_{i+1} \cdots p_k$ is square free and less than or equal to x, and $\varphi(p_1 \cdots p_k) + 1 \equiv 0 \bmod p_1$. Similar statements can be made when $i = 1$. If we fix $p_1, \cdots,$ and p_i our problem then is that of finding primes $p_{i+1}, \cdots,$ and p_k such that

$$(p_{i+1} - 1) \cdots (p_k - 1) \equiv l \bmod p_1$$

where l is an integer, relatively prime to p_1, that depends on $p_1, \cdots,$ and p_i. Let D be the set of $(k - i)$-tuples

$$\{(b_{i+1}, \cdots, b_k) : (b_{i+1} - 1) \cdots (b_k - 1) \equiv l \bmod p_1, 0 \leqq b_j \leqq p_1\} \, .$$

Then, holding $p_1, p_2, \cdots,$ and p_i fixed we have

$$\sum_{p_{i+1} \cdots p_k \in C} 1 \leqq \sum_{(b_{i+1}, \cdots, b_k) \in D} \sum_{\substack{p_1 \cdots p_k \leqq x \\ p_1 < p_j \equiv b_j \bmod p_1, j > i}} 1 \, .$$

If we fix $(b_{i+1}, \cdots, b_k)$ and let $(b'_{i+1}, \cdots, b'_k)$ run over the $(k - i)$-tuples we get by permuting the integers $b_{i+1}, \cdots,$ and b_k we have

$$\sum_{\substack{p_1 \cdots p_k \leqq x \\ p_1 < p_j \equiv b_j \bmod p_1, i > i}} 1 \leqq \sum_{(b'_{i+1}, \cdots, b'_k)} \sum_{p_{i+1} \cdots p_k \in E} 1$$

where E is the set of integers

$$\{p_{i+1} \cdots p_k : p_1 \cdots p_i \cdots p_k \leqq x, p_1 < p_{i+1} < \cdots < p_k, p_j \equiv b'_j \bmod p_1, j > i\} \, .$$

Now, fix $(b'_{i+1}, \cdots, b'_k)$. If $p_{i+1} \cdots p_k$ is in E it follows, by induction, that

$$p_{k-r} \leqq (x/p_1 \cdots p_{k-r-1})^{\frac{1}{r+1}} = t(r)$$

since $p_1 \cdots p_{k-1}p_k \leqq x$ and $p_{k-(r+1)} \leqq p_{k-r}$ for $r = 0, 1, \cdots, k-(i+1)$. Consequently we have

$$\sum_{p_{i+1}\cdots p_k \in E} 1 \leqq \sum_{p_{i+1}}' \cdots \sum_{p_{k-r}}' \cdots \sum_{p_{k-1}}' \pi\left(\frac{x}{p_1 \cdots p_{k-1}}; p_1, b_k'\right) \tag{11}$$

where the prime on the summation symbol indicates that $p_1 < p_{k-r} \leqq t(r)$ and $p_{k-r} \equiv b_{k-r}' \bmod p_1$ for $r = 1, \cdots, k-(i+1)$.

Split the quantity on the right hand side of (11) into two sums, $\sum_1$ and $\sum_2$. The index of summation of $\sum_1$ will be those integers $p_1 \cdots p_{k-1}$ such that $p_1^2 p_2 \cdots p_{k-1} \leqq x^{1-\delta}$, where δ is a positive number that will be chosen later; the index of summation of $\sum_2$ will be those $p_1 \cdots p_{k-1}$ such that $p_1^2 p_2 \cdots p_{k-1} > x^{1-\delta}$.

We have, by Lemmas 3 and 4,

$$\begin{aligned}\sum\nolimits_1 &\leqq \sum_{p_{i+1}}' \cdots \sum_{p_{k-1}}' \frac{c_2 x}{\varphi(p_1)p_1 \cdots p_{k-1}} \frac{1}{\log(x/p_1^2 p_2 \cdots p_{k-1})} \\ &\leqq \frac{c_3}{\varphi(p_1)p_1 \cdots p_i} \frac{x}{\log x} \prod_{j=i+1}^{k-1}\Biggl(\sum_{\substack{p_j \leqq x \\ p_j \equiv b_i \bmod p_1}} \frac{1}{p_j}\Biggr) \\ &\leqq \frac{c_4}{\varphi^{k-i}(p_1)p_1 \cdots p_i} \frac{x}{\log x} \log_2^{k-i-1} x\,,\end{aligned}$$

where $c_3, c_4, \cdots$ are constants that depend on k and δ. If we sum on $(b_{i+1}', \cdots, b_k')$ we have

$$\sum\nolimits_3 = \sum_{(b_{i+1}', \cdots, b_k')} \sum\nolimits_1 \leqq \frac{c_5}{\varphi^{k-i}(p_1)p_1 \cdots p_i} \frac{x \log_2^{k-i-1} x}{\log x}$$

since there are at most $(k-i)!$ permutations of $b_{i+1}, \cdots, b_k$. Summing on $(b_{i+1}, \cdots, b_k)$ yields

$$\sum\nolimits_4 = \sum_{(b_{j+1}, \cdots, b_k) \in D} \sum\nolimits_3 \leqq \frac{c_6}{\varphi(p_1)p_1 \cdots p_i}\, x \frac{\log_2^{k-i-1} x}{\log x}\,,$$

for there are at most $\varphi^{k-i-1}(p_1)$ $(k-i)$-tuples in the set D. Since $p_j < p_1$ for $j = 2, \cdots, i$ we have

$$\begin{aligned}\sum\nolimits_5 = \sum_{p_2 \cdots p_i} \sum\nolimits_4 &\leqq \frac{c_6}{\varphi(p_1)p_1}\left(\sum_{p<p_1} \frac{1}{p}\right)^{i-1} x \frac{\log_2^{k-i-1} x}{\log x} \\ &\leqq \frac{c_7 \log_2^{i-1} p_1}{\varphi(p_1)p_1} \frac{x \log_2^{k-i-1} x}{\log x}\,.\end{aligned}$$

Since we have the restriction $i \leqq k-1$, we have

$$\sum\nolimits_6 = \sum_{i=1}^{k-1} \sum\nolimits_5 \leqq c_8 \frac{\log_2^{k-2} p_1}{p_1 \varphi(p_1)} \frac{x \log_2^{k-2} x}{\log x}\,.$$

Finally, since $p_1 > z$, it follows that

$$\sum_{p_1>z} \sum\nolimits_{16} \leqq c_9 \frac{x}{\log x} \frac{\log_2^{k-2} x}{\sqrt{z}} .$$

In short, we have

$$\sum_{p_1>z} \sum_{i=1}^{k-1} \sum_{n_{k-1} \in B(p_1, i)} 1 \leqq c_9 \frac{x}{\log x} \frac{\log_2^{k-2} x}{\sqrt{z}} , \tag{12}$$

where the summation is restricted to those integers p_1 and $n_{k-1} = p_2 \cdots p_k$ for which $p_1^2 p_2 \cdots p_{k-1} \leqq x^{1-\delta}$.

Let us return to (11) and deal with $\sum_{12}$, i.e. with those integers $p_1 \cdots p_k$ where $p_1^2 p_2 \cdots p_{k-1} > x^{1-\delta}$. Under these circumstances we have

$$\pi\left(\frac{x}{p_1 \cdots p_{k-1}} ; p_1, b_k'\right) \leqq \pi(p_1 x^\delta; p_1, b_k') \leqq \frac{c_2 x^\delta}{\log x^\delta} \leqq c_3 x^\delta .$$

Since we also have, for $0 \leqq \alpha < 1$,

$$\sum_{\substack{p_1 < p \leqq y \\ p \equiv b \bmod p_1}} \frac{1}{p^\alpha} \leqq c_4 \frac{y^{1-\alpha}}{p_1}$$

we can prove, by induction, that

$$\sum\nolimits_{12} \leqq c_5 \sum_{p_{i+1}}{}' \cdots \sum_{p_{k-r}}{}' \frac{x^{\delta+1-\frac{1}{r}}}{p_1^{r-1}(p_1 \cdots p_{k-r})^{1-(1/r)}} .$$

Thus if follows that

$$\sum\nolimits_{12} \leqq \frac{c_6}{p_1^{k-i-1}} \frac{x^{\delta+\beta}}{(p_1 \cdots p_i)^\beta}$$

where $\beta = \beta(i) = 1 - (k-i)^{-1}$. If we now sum on $(b_{i+1}', \cdots, b_k')$ and (b_{i+1}, b_k) we obtain a quantity that is bounded above by

$$c_7 x^{\delta+\beta}/(p_1 \cdots p_i)^\beta .$$

Omit the summation on i for the moment, and divide the sum

$$\sum_{p_1} \sum_{p_2 \cdots p_i} x^{\delta+\beta}/(p_1 \cdots p_i)^\beta$$

into two parts, the first, $\sum_7$, being that part where $p_1 \leqq x^\varepsilon$, the second, $\sum_8$, being that part where $p_1 > x^\varepsilon$, ε being a positive number that will be chosen later. We have, since $p_j < p_1$ for $j = 2, \cdots, i$,

$$\sum\nolimits_7 = \sum_{z < p_1 \leqq x^\varepsilon} \sum_{p_2 \cdots p_i} \frac{x^{\delta+\beta}}{(p_1 \cdots p_i)^\beta} \leqq \left(\sum_{p_1 \leqq x^\varepsilon} \frac{1}{p_1^\beta}\right)^i x^{\delta+\beta} \leqq c_8 x^\lambda$$

where $\lambda = i(1-\beta)\varepsilon + \delta + \beta$. Furthermore since

$$x \geqq p_1 \cdots p_i p_{i+1} \cdots p_k \geqq p_1^{k-i+1} p_2 \cdots p_i$$

it follows that

$$\begin{aligned}\textstyle\sum_8 &\leqq \sum_{p_1 > x^\varepsilon} \frac{1}{p_1^\beta} \sum_{p_2 \cdots p_i \leqq x/p_1^{k-i+1}} \frac{x^{\delta+\beta}}{(p_2 \cdots p_i)^\beta} \\ &\leqq c_9 \Big(\sum_{p_1 > x^\varepsilon} \frac{1}{p_1^2} \Big) x^{\delta+\beta+1-\beta} \leqq c_{10} x^{1+\delta-\varepsilon} .\end{aligned}$$

Set $\varepsilon = 1/(2k)$ and $\delta = 1/(4k)$. Then $1 + \delta - \varepsilon = 1 - (4k)^{-1}$, $\lambda \leqq 1 - (4k)^{-1}$, and

$$\textstyle\sum_7 + \sum_8 \leqq c_{11} x^{1-\frac{1}{4k}} .$$

A summation on the i's yields the inequality

$$\sum_{p_1 > z} \sum_{i=1}^{k-1} \sum_{n_{k-1} \in B(p_1, i)} 1 \leqq c_{12} x^{1-\frac{1}{4k}} , \tag{13}$$

where the summation is restricted to those integers p_1 and $n_{k-1} = p_2 \cdots p_j$ for which $p_1^2 p_2 \cdots p_{k-1} > x^{1-\delta}$.

If we return to (10) we see that we must find a bound for that part of S corresponding to $i = k$. If we have $n_{k-1} = p_2 \cdots\cdot p_k$ in $B(p_1, k)$ then, by definition, $p_j < p_1$ for $j = 2, \cdots, k$ and

$$(p_2 - 1) \cdots (p_k - 1) - 1 \equiv 0 \mod p_1 .$$

Once again we have a two way split. On one hand we have

$$\sum_{z < p_1 \leqq x^{1/k}} \sum_{n_{k-1} \in B(p_1, k)} 1 \leqq \sum_{p_1 \leqq x^{1/k}} \pi(p_1)^{k-2} \leqq c_1 x^{1-1/k} ,$$

for the obvious reasons. On the other hand, if $p_1 > x^{1/k}$ then $p_2 \cdots p_k \leqq x^{1-1/k}$ and

$$\sum_{p_1 > x^{1/k}} \sum_{n_{k-1} \in B(p_1, k)} 1 \leqq \sum_{p_2 \cdots p_k \leqq x^{1-1/k}} \sum_{\substack{p_1 > x^{1/k} \\ p_1 | (\varphi(p_2 \cdots p_k) - 1)}} 1 .$$

Since the number of prime divisors of $\varphi(p_2 \cdots p_k) - 1$ which are greater than $x^{1/k}$ does not exceed k this last double sum is bounded above by $kx^{1-1/k}$. Therefore, we can conclude that

$$\sum_{p_1 > z} \sum_{n_{k-1} \in B(p_1, k)} 1 \leqq c_2 x^{1-1/k} . \tag{14}$$

Let us assemble our results. Items (10), (12), (13), and (14) imply that

$$S = \mathcal{O}\Big[\frac{x}{\log x} \frac{\log_2^{k-2} x}{\sqrt{z}} \Big] . \tag{15}$$

Relations (3), (4) and (15) yield

$$\Phi(k, x) = \alpha_k x \frac{\log_2^{k-2} x}{\log x}\left[1 + \mathcal{O}\left(\frac{1}{\sqrt{z}}\right)\right],$$

where $z = (\log_{k+1} x)^{1/k}$. Since $\Phi'(k, x)$ was defined to be the number of positive square free integers m less than or equal to x which have k prime factors and which have a factor in common with $\varphi(m) + 1$, and since $\Phi(k, x)$ was defined to be the number of odd integers counted

by $\Phi'(k, x)$ we have, for $k > 2$,

$$\Phi'(k, x) = \Phi(k, x) + \Phi(k - 1, x/2) \sim (\alpha_k x \log_2^{k-2} x)/\log x .$$

If $k = 2$ then

$$\Phi'(2, x) = \Phi(2, x) + \pi(x/2) \sim (\alpha_2 + 1/2)x/\log x .$$

These are the results we set out to prove.

BIBLIOGRAPHY

1. K. Prachar, *Primzahlverteihung*, Springer, 1957.
2. E. M. Wright, *A simple proof of a theorem of Landau,* Proc. Edinburgh Math. Soc. (2) **9** (1954), 87-90.

Received August 25, 1965. This research was supported, in part, by funds received under NSF grant GP-3933.

UNIVERSITY OF CALIFORNIA, LOS ANGELES