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CONTRACTION SEMI-GROUPS IN A FUNCTION SPACE

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Using the concepts of a semi inner-product and a dissipative operator, it is proven that if X is a complex Banach space (under the supremum norm) of bounded complex valued functions on a set S, p is a bounded positive function on S which is bounded away from zero, $pX \subset X$, and A is the infinitesimal generator of a strongly continuous (class (C_0)) semi-group of contraction operators in X, then pA is also the infinitesimal generator of such a semi-group.

The notion of a *semi inner-product* was introduced by G. Lumer in [3].

DEFINITION 1. A *semi inner-product* for a complex (real) Banach space X is a function $[\cdot, \cdot]$ from $X \times X$ into the complex (real) numbers which satisfies

$$[\alpha x + \beta y, z] = \alpha[x, z] + \beta[y, z] ,$$
$$|[x, z]| \leqq \|x\| \cdot \|z\| ,$$

and

$$[x, x] = \|x\|^2 .$$

There is at least one semi inner-product for every Banach space X, because we can define $[x, y] = f(x)$, where f is a bounded linear functional on X such that $\|f\| = \|y\|$, and $|f(y)| = \|y\|^2$ (see [4]).

By an operator in a Banach space X, we mean a linear transformation (not necessarily bounded) from a subspace of X to a subspace of X. The notion of a *dissipative operator* in a Banach space is treated by G. Lumer and R. S. Phillips in [4].

DEFINITION 2. An operator A in a Banach space X is said to be *dissipative* (with respect to a given semi inner-product for X) if

$$\mathrm{re}\,[Ax, x] \leqq 0$$

for all x in the domain of A.

By a contraction semi-group in a Banach space X we mean a strongly continuous semi-group of contraction operators in X which is of class (C_0) (see [2]). A contraction operator in X is a bounded linear transformation T from X into X with $\|T\| \leqq 1$. Lumer and Phillips have given the following characterization [4, Theorem 3.1] of the infinitesimal generator of a contraction semi-group.

THEOREM (*Lumer and Phillips*). *Suppose A is an operator in a Banach space X, the domain of A is dense in X, and $[\cdot, \cdot]$ is a semi inner-product for X. Then A is the infinitesimal generator of a contraction semi-group in X if and only if A is dissipative with respect to $[\cdot, \cdot]$, and the range of $I—A$ is all of X, where I denotes the identity transformation on X.*

THEOREM. *Suppose S is a set, X is a complex Banach space (under the supremum norm) of bounded complex valued functions on S, p is a bounded positive function on S which is bounded away from zero, $pX \subset X$, and A is the infinitesimal generator of a contraction semi-group in X. Then pA is also the infinitesimal generator of a contraction semi-group in X.*

Proof. Let U denote the Banach algebra of all bounded complex valued functions on S, and let S_1 denote the set of all nonzero multiplicative linear functionals on S. It follows from [1, pp. 272–277], especially [1, Corollary 19, p. 276], that

(i) if m is in S_1, and q is a nonnegative function in U, then $m(q) \geqq 0$, and

(ii) if x is in U, then there is an m in S_1 such that $|m(x)| = ||x||$. For each x in X, let m_x denote an element m of S_1 such that $|m(x)| = ||x||$, and for each x, y in X, let

$$[x, y] = m_y(x)[m_y(y)]^* ,$$

where the $*$ denotes complex conjugation. Then $[\cdot, \cdot]$ is a semi inner-product for X; it is the only one to be used from this point on. A dissipative operator in X will mean one which is dissipative with respect to this semi inner-product.

If q is a bounded nonnegative function on S, and $qX \subset X$, then

$$\operatorname{re}[qAx, x] = m_x(q) \operatorname{re}[Ax, x] \leqq 0 ,$$

for all x in $\mathfrak{D}(A)$, the domain of A, since A is dissipative by [4, Theorem 3.1]. Therefore, qA is dissipative. Also, the domain of qA is $\mathfrak{D}(A)$, which is dense in X by [2, Theorem 12.3.1, p. 360]. If

$$\sup_{s \in S} |1 - q(s)| < 1/2 ,$$

then $||I - q||$, the operator norm of $I - q$, is less than 1/2, so that $I - qA$ is invertible, since

$$I - qA = I - A + (I - q)A = \{I + (I - q)AR(1, A)\}(I - A) ,$$

and

$$\| AR(1, A) \| = \| R(1, A) - I \| \leqq 2$$

by [2, Theorem 12.3.1, p. 360]. Thus the range of $I - qA$ is all of X, and qA generates a contraction semi-group in X by [4, Theorem 3.1].

Since $F(p)X \subset X$ for every polynomial F, and p is bounded and nonnegative, it follows from the classical Weierstrass theorem that $p^{(1/n)}X \subset X$ for every positive integer n. Choose n so that

$$\sup_{s \in S} | 1 - [p(s)]^{(1/n)} | < 1/2 ,$$

and let $r = p^{(1/n)}$. This is possible because the range of p is contained in a closed and bounded interval of positive numbers. By what was shown in the previous paragraph, rA generates a contraction semi-group in X. If $1 \leqq j < n$, and $r^j A$ generates a contraction semi-group in X, then $r^{j+1}A$ does also, for

$$r^{j+1}A = r(r^j A) ,$$

and we can substitute r for q and $r^j A$ for A in the argument given in the previous paragraph.

REMARK. An argument similar to the one given will establish the theorem if X is taken to be a real Banach space (under the supremum norm) of bounded real valued functions on S, and the rest of the hypothesis remains the same. Also, we could take A to be the generator of a class (C_0) semi-group $[T(t); 0 \leqq t < \infty]$ of operators in X such that for some $\omega > 0$,

$$\| T(t) \| \leqq e^{\omega t} \qquad \text{for } t \geqq 0 .$$

If

$$\tilde{T}(t) = e^{-\omega t} T(t) \qquad \text{for } t \geqq 0 ,$$

then $[\tilde{T}(t)]$ is a contraction semi-group in X and has the generator $\tilde{A} = A - \omega$.

If

$$V(t) = e^{\omega t p} \tilde{V}(t) \qquad \text{for } t \geqq 0 ,$$

where $[\tilde{V}(t); 0 \leqq t < \infty]$ is the contraction semi-group generated by $p\tilde{A}$, then $[V(t)]$ is a class (C_0) semi-group of operators in X,

$$\| V(t) \| \leqq e^{\omega t \|p\|} \qquad \text{for } t \geqq 0 ,$$

and $[V(t)]$ is generated by pA. The author wishes to express his thanks to the referee for his suggestions.

REFERENCES

1. N. Dunford and J. Schwartz, *Linear Operators, Part I*, Interscience, 1958.
2. E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloquium Publication, 31, 1957.
3. G. Lumer, *Semi inner-product spaces*, Trans. Amer. Math. Soc. **100** (1961), 29-43.
4. G. Lumer and R. S. Phillips, *Dissipative operators in a Banach space*, Pacific J. Math. **11** (1961), 679-698.

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