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SOME EXISTENCE THEOREMS IN THE CALCULUS OF VARIATIONS

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In this paper are discussed theorems of existence of a minimum for nonparametric integrals of the calculus of variations defined on an infinite interval, depending on an unknown function, its derivative, and on a convolution integral. The approach of the direct methods of the calculus of variations will be employed.

The author has shown previously that under the usual conditions of convexity on the integrand the class of functionals to be considered are lower semicontinuous with respect to uniform convergence on $-\infty < x < \infty$ but not with respect to uniform convergence on every compact set. Therefore additional hypotheses on the admissible class of functions and the integrand must be imposed to assure that a minimizing sequence of elements converges in the stronger sense.

The reason for the study of such functionals arose from a certain class of optimization problems in communication theory (see W.M. Brown and C. Palermo [1] for example). The author discussed existence of a minimum and lower semicontinuity of these functionals in [4], and the theorems given here represent an improvement over those previously given and consider a special linear case as well. Necessary conditions for an extremum are not discussed, but have been considered in [6].

Consider $F(x, y, d, p)$ a real-valued continuous function defined on E^4.

DEFINITION 1. The function $F(x, y, d, p)$ is said to be

(a) semiregular positive if the function $z(d, p) = F(x, y, d, p)$ is a convex function in E^2 for every (x, y), and

(b) semiregular positive seminormal if (a) holds and for no (x_0, y_0) is $z(d, p)$ a linear function.

In addition, let P be a given compact set contained in E^2.

DEFINITION 2. The class K of admissible functions is any nonempty collection of functions $y = y(x)$, $-\infty < x < \infty$, satisfying

1. The graphs of every $y \in K$ contains a point $(x, y(x)) \in P$,
2. $y(x)$ is absolutely continuous in every finite interval,
3. $y'(x) \in L^1(-\infty, \infty)$,
4. $F[x, y(x), y'(x), p(x)] \in L^1(-\infty, \infty)$ where $p(x)$ is one of the following convolution integrals:

$$y'^*y' = \int_{-\infty}^{\infty} y'(x-t)y'(t)dt ,$$

$$|y'|^*|y'| = \int_{-\infty}^{\infty} |y'(x-t)||y'(t)|dt ,$$

$$y'^*g = \int_{-\infty}^{\infty} y'(x-t)g(t)dt ,$$

where $g(x) \in L^1(-\infty, \infty)$ is given, or $|y'|^*|g| = \int_{-\infty}^{\infty} |y'(x-t)||g(t)|dt$ with $g(x)$ as above, and

5. K is closed with respect to uniform covergence on every compact set in $-\infty < x < \infty$.

2. Existence theorems.

THEOREM 1. *Consider* $I[y] = \int_{-\infty}^{\infty} F[x, y(x), y'(x), p(x)]dx$ *where* $p(x) = y'^*y'$ *or* $p(x) = |y'|^*|y'|$, *and suppose that for all* $(x, y) \in E^2$

(a) $F(x, y, d, p)$ *is semiregular positive,*

(b) $F(x, y, d, p) \geqq |d| \Omega(|d|)$ *for any value of* p *where* $\Omega(s)$ *is defined in* $0 \leqq s < \infty$, $\lim_{s\to\infty} \Omega(s) = \infty$, *and for all* s, $\Omega(s) \geqq k^2$, $k \neq 0$ *and constant, and*

(c) *For any value of* d *there exists a constant* $L > 0$ *such that*

$$|F(x, y, d, p_1) - F(x, y, d, p_2)| \leqq L|p_1 - p_2| .$$

Furthermore suppose that given a class K *of admissible functions, the following is satisfied:*

(d) *There exists at least one* $y \in K$ *and a constant* N *such that* $I[y] \leqq N$ *where* N *satisfies* $NL < k^4$.

Then $I[y]$ *possesses an absolute minimum in* K.

Proof. We assert that hypotheses (a), (b) and, (c) guarantee that

(i) $\inf_{y\in K} I[y] = i \geqq 0$,

(ii) $F(x, y, d, p)$ is semiregular positive seminormal,

(iii) given a minimizing sequence $y_n(x)$, $n = 1, 2, \cdots, -\infty < x < \infty$, of elements of K satisfying $I[y_n] \leqq i + 1/n$, they contain a subsequence which converges uniformly on every compact set in $-\infty < x < \infty$ to an element $y_0(x)$, $-\infty < x < \infty$, of K, and

(iv) $\int_{-\infty}^{\infty} |y_n'(x)|dx \leqq \left(i + \frac{1}{n}\right)/k^2$, $n = 1, 2, \cdots$,

$$\int_{-\infty}^{\infty} |y_0'(x)|dx \leqq (i+1)/k^2.$$

For a proof of this assertion only minor modifications of the proof of Theorem 4.1 of [4] are required. We will denote the convergent subsequence by $y_n(x)$, $n = 1, 2, \cdots, -\infty < x < \infty$, to avoid additional indices.

Since $y_0 \in K$, given $\varepsilon < 0$ we may choose $A > 0$ such that

$$\int_{|x| \geqq A/2} |y_0'(x)|\, dx < \varepsilon \quad \text{and} \quad \int_{|x| \geqq A/2} F[x, y_0(x), y_0'(x), p_0(x)]dx < \varepsilon\,.$$

It follows that

$$\frac{1}{n} = i + \frac{1}{n} - i \geqq \int_{-A/2}^{A/2} F[x, y_n(x), y_n'(x), p_n(x)]dx$$
$$- \int_{-A/2}^{A/2} F[x, y_0(x), y_0'(x), p_0(x)]dx + k^2 \int_{|x| \geqq A/2} |y_n'(x)|\, dx - \varepsilon\,.$$

Now define a new sequence of functions as follows:

$$\bar{y}_n(x) = \begin{cases} y_n(x) & \text{if } |x| \leqq A \\ y_0(x) + [y_n(\acute{A}) - y_0(A)] & \text{if } x > A \\ y_0(x) + [y_n(-A) - y_0(-\acute{A})] & \text{if } x < -\hat{A}\,,\quad n = 1, 2, \cdots\,. \end{cases}$$

Hence for $|t| \leqq A/2$ and $|x| \leqq A/2$ we have $|x - t| \leqq A$, and by hypothesis (c) the following estimate can be made:

$$\int_{-A/2}^{A/2} |p_n(x) - \bar{p}_n(x)|\, dx \leqq \left(i + \frac{1}{n}\right)k^2 \int_{|x| \geqq A/2} |y_n'(x)|\, dx + \varepsilon\left(\frac{i+1}{k^2} + \varepsilon\right).$$

This implies that

$$\int_{-A/2}^{A/2} F[x, y_n(x), y_n'(x), p_n(x)]dx \geqq \int_{-A/2}^{A/2} F[x, \bar{y}_n(x), \bar{y}_n'(x), \bar{p}_n(x)]dx$$
$$- \frac{L}{k^2}\left(i + \frac{1}{n}\right) \int_{|x| \geqq A/2} |y_n'(x)|\, dx - L\varepsilon(i + 1)/k^2$$

where the ε^2 term is ignored.

By definition, the $\bar{y}_n(x), n = 1, 2, \cdots$, are absolutely continuous on every finite interval, $\bar{y}_n'(x) \in L^1(-\infty, \infty)$, and $\lim_{n\to\infty} \bar{y}_n(x) = y_0(x)$ uniformly on $-\infty < x < \infty$. This implies, by the proof of lower semicontinuity in [4] that $I[y_0] \leqq \liminf_{n\to\infty} I[\bar{y}_n]$ where the interval of integration is restricted to $-A/2 \leqq x \leqq A/2$. Therefore, for n sufficiently large we have

$$\frac{1}{n}[1 + L(i + 1)/k^4] + \varepsilon[2 + L(i + 1)/k^2] \geqq (k^2 - Li/k^2) \int_{|x| \geqq A/2} |y_n'(x)|\, dx\,.$$

Hypothesis (d) implies that $k^2 - Li/k^2 > 0$, and since A depends only on y_0 and the choice of ε, it follows that $\lim_{n\to\infty} y_n(x) = y_0(x)$ uniformly on $-\infty < x < \infty$. A proof of this assertion can be found in [5]. By the previously established lower semicontinuity we then have that y_0 gives $I[y]$ the desired absolute minimum in K, and the theorem is proved.

EXAMPLE 1. Let $F(x, y, d, p) = |d|(|d|+1) + |p|/(4+x^2)$ and K the class of all functions satisfying 2, 3, and 4 of Definition 3 and such that $y(0) = 0$ and $y(1) = 1$ for all $y \in K$. Then $P = \{(0, 0)\}$, $\Omega(s) = s + 1$, $k^2 = 1$ and $L = 1/4$. Let $y(x) = 0$ if $x < 0$, $y(x) = x$ if $0 \leqq x \leqq 1$, $y(x) = 1$ otherwise, and then $I[y] < 3 = N$ so all the hypotheses of Theorem 1 are satisfied.

THEOREM 2. *Consider* $I[y] = \int_{-\infty}^{\infty} F[x, y(x), y'(x), p(x)]dx$ *where* $p(x) = y'^*g$ *or* $p(x) = |y'|^*|g|$ *where* $g = g(x)$, $-\infty < x < \infty$, *is given and in* $L^1(-\infty, \infty)$ *with* $\int_{-\infty}^{\infty} |g(x)|dx \neq 0$. *Suppose that hypotheses* (a), (b), *and* (c) *of Theorem* 1 *are satisfied and*

(d) *The constants* k^2 *and* L *satisfy* $k^2 - L\int_{-\infty}^{\infty} |g(x)|dx > 0$. *Then* $I[y]$ *possesses an absolute minimum in* K.

Proof. No major changes are required in the previous proof to show the existence of a minimizing subsequence $y_n(x)$, $n = 1, 2, \cdots$, $-\infty < x < \infty$, of elements of K converging uniformly on every compact set to an element $y_0 \in K$. Given $\varepsilon > 0$ choose $A > 0$ as above and in addition such that $\int_{|x| \geqq A/2} |g(x)|dx < \varepsilon$. The sequence of functions $\bar{y}_n(x)$, $n = 1, 2, \cdots$, $-\infty < x < \infty$ are defined as before, and we are led to the following inequality for sufficiently large n:

$$\frac{1}{n} \geqq \left[k^2 - L\int_{-\infty}^{\infty} |g(x)|dx\right]\int_{|x| \geqq A/2} |y_n'(x)|dx - \varepsilon[2 + L(i+1)/k^2] .$$

By hypothesis (d) the expression in the first bracket is positive and we may now proceed as in Theorem 1 to show that y_0 gives $I[y]$ an absolute minimum in K. Note conditions of the type (d) are found in a different setting in [2].

3. A special linear case. We wish to consider the special case in which $p(x) = |y'|^*|y'|$, or equivalently the admissible class K is restricted to monotonic increasing or decreasing functions, and $F(x, y, d, p)$ is linear in p. As will be seen slightly weaker conditions on the growth of the integrand with respect to d can then be given, but the class K must be restricted as follows.

DEFINITION 3. The class K of admissible functions is any non-empty collection of functions $y = y(x)$, $-\infty < x < \infty$, satisfying

Definition 2 and

6. The total variation of elements of K is uniformly bounded away from zero.

THEOREM 3. *Consider* $I[y] = \int_{-\infty}^{\infty} F[x, y(x), y'(x), p(x)]dx$ *where* $p(x) = |y'|^*|y'|$ *and suppose that*

(a) $F(x, y, d, p)$ *is semiregular positive, and*

(b) $F(x, y, d, p) = \phi(|d|) + Ap + \omega(x, y, d)$ *where*

(i) $\phi(s)$ *is a positive continuous function defined on* $0 < s < \infty$ *and satisfies* $\lim_{s\to\infty} \phi(s)/s = \infty$.

(ii) A *is a positive constant, and*

(iii) $\omega(x, y, d)$ *is a positive continuous function defined on* E^3 *and* $\omega[x, y(x), y'(x)] \in L^1(-\infty, \infty)$ *for any* $y \in K$.

Then $I[y]$ *possesses an absolute minimum in* K.

Proof. Since the integrand is positive we have $i = \inf_{y \in K} I[y] \geqq 0$, and by hypothesis (b) that $F(x, y, d, p)$ is semiregular positive semi-normal. Let $y_n(x), n = 1, 2, \cdots, -\infty < x < \infty$, be a minimizing sequence of elements of K satisfying $I[y_n] \leqq i + 1/n$. By the assumptions given for the class K there exists a positive constant Q such that for each $n = 1, 2, \cdots$, there exists an x_n for which $|x_n| \leqq Q$, $|y_n(x_n)| \leqq Q$, and $[x_n, y_n(x_n)] \in P$.

Furthermore since each $y'_n(x) \in L^1(-\infty, \infty)$ we have that

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} |y'_n(x-t)|\,|y'_n(t)|\,dt = \left(\int_{-\infty}^{\infty} |y'_n(x)|\,dx\right)^2 .$$

Hence for any x and $n = 1, 2, \cdots$,

$$\begin{aligned} |y_n(x)| - Q \leqq |y_n(x)| - |y_n(x_n)| &\leqq \int_{-\infty}^{\infty} |y'_n(x)|\,dx \\ &\leqq (I[y_n]/A)^{1/2} \leqq (i + 1/A)^{1/2}, \end{aligned}$$

and we may assert the $y_n(x), n = 1, 2, \cdots, -\infty < x < \infty$, are equi-bounded.

It has been shown previously by Cinquini [3] that hypothesis (b) is sufficient to insure that the sequence $y_n(x), n = 1, 2, \cdots, -\infty < x < \infty$, is equicontinuous. This is an extension of a fundamental result due to Tonelli [7]. Therefore there exists a subsequence which converges uniformly on every compact set in $-\infty < x < \infty$ to a function $y_0(x)$, $-\infty < x < \infty$, which is absolutely continuous on every finite interval. Furthermore $y_0 \in K$.

Denoting the subsequence by $y_n(x), n = 1, 2, \cdots, -\infty < x < \infty$, we have by the definition of the class K that there exists a positive number m such that $\int_{-\infty}^{\infty} |y'_n(x)|\,dx \geqq m$, $n = 0, 1, 2, \cdots$. Therefore

$$\frac{1}{n} \geqq i + \frac{1}{n} - i \geqq 2mA\left[\int_{-\infty}^{\infty} |\, y'_n(x) \,|\, dx - \int_{-\infty}^{\infty} |\, y'_0(x) \,|\, dx\right] + \int_{-\infty}^{\infty} G[x, y_n(x), y'_n(x)]dx - \int_{-\infty}^{\infty} G[x, y_0(x), y'_0(x)]dx$$

where $G(x, y, d) = \phi(|\, d \,|) + \omega(x, y, d)$. But this latter integrand is an ordinary type of the calculus of variations, and by hypothesis is also semiregular positive seminormal, i.e., strictly convex in d. By an extension of the result of Cinquini this implies the functional $\int_{-\infty}^{\infty} G[x, y(x), y'(x)]dx$ is lower semicontinuous with respect to uniform convergence on every compact set in $-\infty < x < \infty$.

Hence for n sufficiently large

$$\int_{-\infty}^{\infty} |\, y'_n(x) \,|\, dx - \int_{-\infty}^{\infty} |\, y'_0(x) \,|\, dx \leqq \left(\varepsilon + \frac{1}{n}\right)\Big/ 2mA$$

which implies that $\lim_{n \to \infty} y_n(x) = y_0(x)$ uniformly on $-\infty < x < \infty$. By lower semicontinuity it follows that y_0 gives $I[y]$ the desired absolute minimum in K which completes the proof.

EXAMPLE 2. Let $f(x, y, d, p) = d^2 + p$ and K as in Example 1. Then $P = (0, 0)$, $\phi(s) = s^2$, $A = 1$, $m = 1$, and $\omega = 0$. All the hypotheses of Theorem 3 are satisfied and it is not difficult to show that the function given in the previous example gives an absolute minimum to $I[y]$.

It should be remarked that a theorem similar to Theorem 3 can be stated for the case $p(x) = |\, y' \,|^* |\, g \,|$ where $g = g(x)$, $-\infty < x < \infty$, is given and in $L^1(-\infty, \infty)$ with $\int_{-\infty}^{\infty} |\, g(x) \,|\, dx \geqq 1$. But since

$$\int_{-\infty}^{\infty} p(x)dx = \int_{-\infty}^{\infty} |\, g(x) \,|\, dx \int_{-\infty}^{\infty} |\, y'(x) \,|\, dx$$

one is essentially considering an ordinary integrand of the calculus of variations and nothing is gained by requiring the stronger mode of convergence.

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